Computational Manifolds and Applications–2011, IMPA

Homework 1

Due September 20, 2011

Problem 1. Plot the curve f defined by

$$f(t) = \begin{cases} (t,\, t^2 \sin(1/t)) & \text{if } t \neq 0; \\ (0,0) & \text{if } t = 0. \end{cases}$$

Show that $f'(0) = (1,0)$ and that $f'(t) = (1,\, 2t\sin(1/t) - \cos(1/t))$ for $t \neq 0$. Verify that f' is discontinuous at 0.

Problem 2. A cardioid is the curve defined in polar coordinates (ρ, θ) by

$$\rho = 1 + \cos\theta.$$

(a) Show that the cardioid is given by the parametric equations

$$\begin{aligned} x &= (1+\cos\theta)\cos\theta \\ y &= (1+\cos\theta)\sin\theta, \end{aligned}$$

and plot the cardioid. Prove that every point is regular except the origin (when $\theta = \pi$).

(b) Prove that the cardioid is also defined by the equations

$$\begin{aligned} x &= \frac{2(1-t^2)}{(1+t^2)^2} \\ y &= \frac{4t}{(1+t^2)^2}. \end{aligned}$$

Problem 3. The *Descartes folium* (more exactly a portion of this curve) is the parametric curve $\alpha\colon (-1,\infty) \to \mathbb{R}^2$ given by

$$\alpha(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3} \right).$$

(a) Plot this curve.

(b) Prove that α is injective and continuous but that its inverse is not continuous on $\alpha((-1,\infty))$.

Hint. Use a limit argument.

Problem 4. Given a circle C and a point P on C, consider the set of all lines Δ_Q such that if $Q \neq P$ is any point on C, the line Δ_Q is the line passing through P and forming an angle with the normal N_Q at Q equal to the angle of N_Q with PQ (in other words, Δ_Q is obtained by reflecting PQ about the normal N_Q at Q). When $Q = P$, the line Δ_Q is the diameter through P.

(a) Let O be the center of the circle and choose a coordinate system with origin O so that the coordinates of P are $(-1, 0)$ and the coordinates of Q are

$$(\cos\theta, \sin\theta).$$

Prove that the coordinates of the second point of intersection M of the line Δ_Q with the circle C are

$$(-\cos 2\theta, -\sin 2\theta).$$

(b) Prove that the equation of the line Δ_Q is

$$(\sin 2\theta + \sin\theta)x - (\cos 2\theta + \cos\theta)y - \sin\theta = 0.$$

Prove that the limit of the intersection of the line Δ_Q with the x-axis when θ goes to zero is the point $(1/3, 0)$.

(c) Make the change of variables $x = u + 1/3$, that is, translate the origin to the point $O' = (1/3, 0)$.

Now, if a line is given by an equation

$$ux + vy + w = 0,$$

where u, v, w are differentiable functions of a parameter θ, if this line is tangent to a curve Γ at some point (x, y), then (x, y) will be the limit of the intersection of the line of equation

$$ux + vy + w = 0$$

with the line of equation

$$u(\theta + \epsilon)x + v(\theta + \epsilon)y + w(\theta + \epsilon) = 0,$$

when $\epsilon \neq 0$ tends to zero. Using a Taylor expansion, prove that (x, y) is the solution of the two equations

$$\begin{aligned} ux + vy + w &= 0 \\ u'x + v'y + w' &= 0. \end{aligned}$$

Use the above to find the parametric equations of the envelope of the lines Δ_Q and prove that this is a cardioid with cusp O'.