IMPA - Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ, Brazil

Introduction to Computational Manifolds and Applications

Part 1 - Foundations

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Surfaces

Parametric Surfaces

What is a surface?

A precise answer cannot really be given without introducing the concept of a manifold.

An informal answer is to say that a surface is a set of points in $\mathbb{E}^3$ such that, for every point p on the surface, there is a small neighborhood U of p that is continuously deformable into a little flat open disk.

Thus, a surface should really have some topology.

Also, locally, unless the point p is "singular", the surface looks like a plane.

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As in the case of curves, properties of surfaces can be classified into *local properties* and *global properties*.

Local properties are the properties that hold in a small neighborhood of a point on a surface.

Curvature is a local property.

Local properties can be studied more conveniently by assuming that the surface is parametrized locally.

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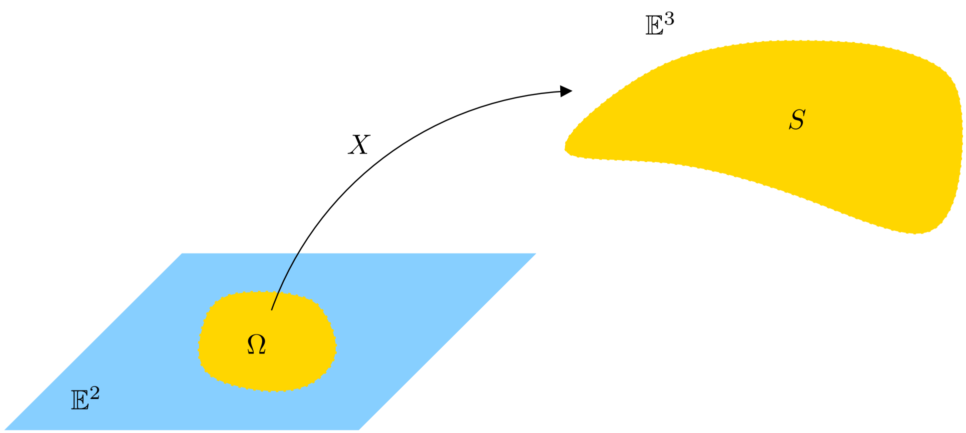
A parametric surface is a map

$$X : \Omega \to \mathbb{E}^3,$$

where Ω is some open subset of the plane $\mathbb{E}^2$, and X is at least C^3-continuous.



Actually, we will need to impose an extra condition on a surface X so that the tangent plane (and the normal) at any point is defined. Again, this leads us to consider curves on X.

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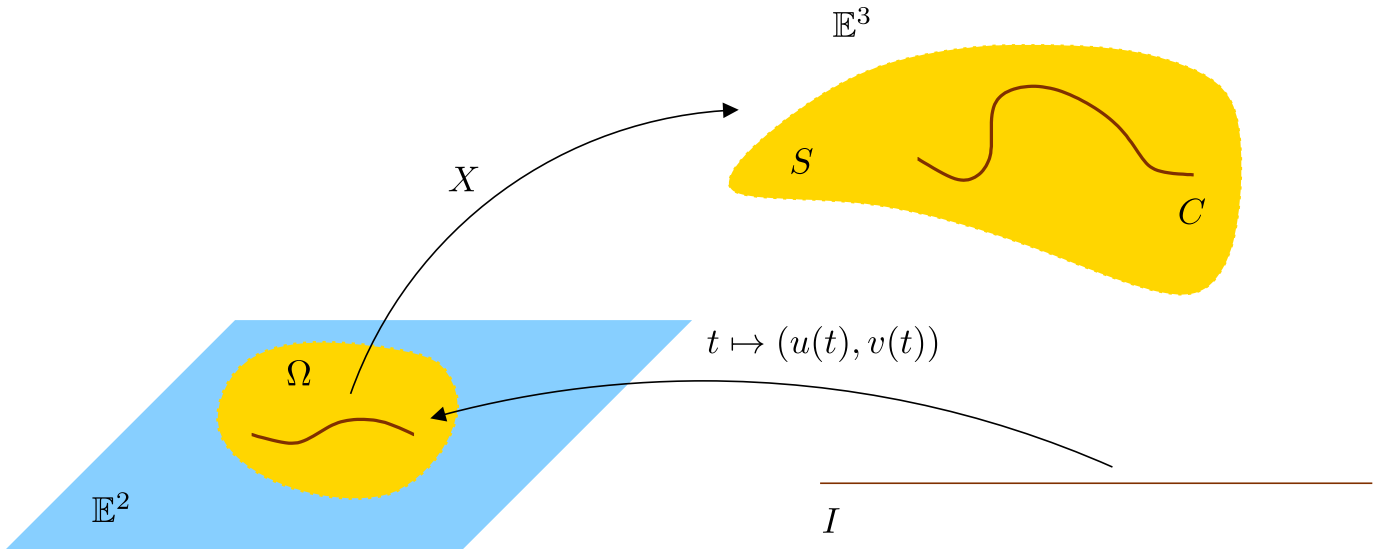
A curve C on X is defined as a map

$$C : t \mapsto X(u(t), v(t)) \,,$$

where u and v are continuous functions on some open interval I contained in Ω.



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We also assume that the plane curve $t \mapsto (u(t), v(t))$ is regular, that is, that

$$\left(\frac{du}{dt}(t), \frac{dv}{dt}(t)\right) \neq (0, 0)$$

for all $t \in I$.

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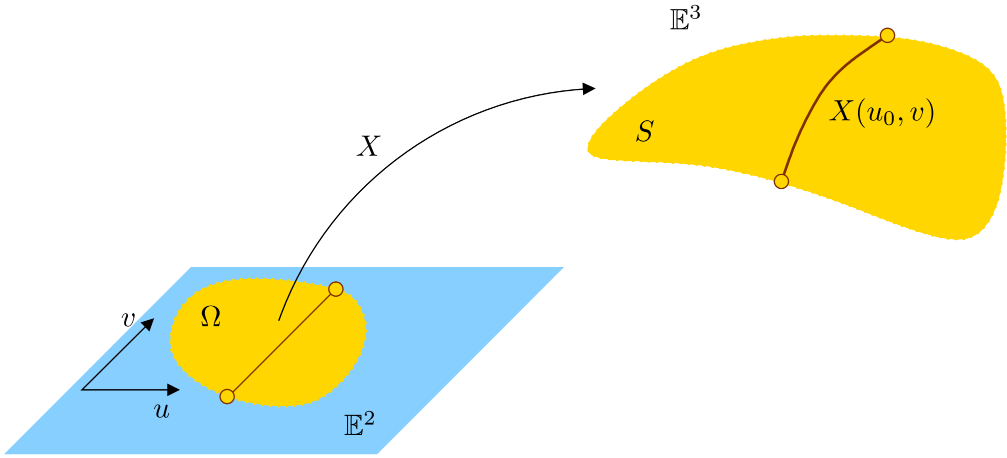
For example, the curves

$$v \mapsto X(u_0, v)$$

for some constant u_0 are called *u-curves*, and the curves

$$u \mapsto X(u, v_0)$$

for some constant v_0 are called *v-curves*. Such curves are also called the *coordinate curves*.



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The tangent vector, $\dfrac{dC}{dt}(t)$, to C at t can be computed using the chain rule:

$$\frac{dC}{dt}(t) = \frac{dX}{du}(u(t), v(t)) \cdot \frac{du}{dt}(t) + \frac{dX}{dv}(u(t), v(t)) \cdot \frac{dv}{dt}(t)\,.$$

Note that

$$\frac{dC}{dt}(t)\,, \quad \frac{dX}{du}(u(t), v(t))\,, \quad \text{and} \quad \frac{dX}{dv}(u(t), v(t))$$

are vectors, but for simplicity of notation, we omit the vector symbol in these expressions.

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It is customary to use the following abbreviations: the partial derivatives

$$\frac{dX}{du}(u(t), v(t)) \quad \text{and} \quad \frac{dX}{dv}(u(t), v(t))$$

are denoted as $\boldsymbol{X}_u(t)$ and $\boldsymbol{X}_v(t)$, or even as $\boldsymbol{X}_u$ and $\boldsymbol{X}_v$, and the derivatives

$$\frac{dC}{dt}(t), \quad \frac{du}{dt}(t), \quad \text{and} \quad \frac{dv}{dt}(t)$$

are denoted as $\dot{C}(t)$, $\dot{u}(t)$ and $\dot{v}(t)$, or even as $\dot{C}$, $\dot{u}$, and $\dot{v}$.

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When the curve C is parametrized by arc length s, we denote

$$\frac{dC}{ds}(s)\,, \quad \frac{du}{ds}(s)\,, \quad \text{and} \quad \frac{dv}{ds}(s)$$

as $C'(s)$, $u'(s)$, and $v'(s)$, or even as C', u', and v'. Thus, we reserve the prime notation to the case where the parametrization of C is by arc length.

Note that it is the curve

$$C : t \mapsto X(u(t), v(t))$$

which is parametrized by arc length, not the curve

$$t \mapsto (u(t), v(t))\,.$$

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Using these notations, $\dot{C}(t)$ is expressed as follows:

$$\dot{C}(t) = \boldsymbol{X}_u(t)\dot{u}(t) + \boldsymbol{X}_v(t)\dot{v}(t)\,,$$

or simply as

$$\dot{C} = \boldsymbol{X}_u\dot{u} + \boldsymbol{X}_v\dot{v}\,.$$

Now, if we want $\dot{C} \neq 0$ for all regular curves

$$t \mapsto (u(t), v(t))\,,$$

we must require that

$$\boldsymbol{X}_u \quad \text{and} \quad \boldsymbol{X}_v$$

be linearly independent. Equivalently, we must require that the cross product, $\boldsymbol{X}_u \times \boldsymbol{X}_v$ be non-null.

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Definition 3.1. A *surface patch* X is a map

$$X : \Omega \to \mathbb{E}^3,$$

where Ω is some open subset of the plane $\mathbb{R}^2$ and where X is at least C^3-continuous.

We say that the surface X is *regular at* $(u, v) \in \Omega$ iff $\boldsymbol{X}_u \times \boldsymbol{X}_v \neq \boldsymbol{0}$, and we also say that $p = X(u, v)$ is a *regular point of* X. If $\boldsymbol{X}_u \times \boldsymbol{X}_v = \boldsymbol{0}$, we say that $p = X(u, v)$ is a *singular point of* X. The surface X is *regular on* Ω iff $\boldsymbol{X}_u \times \boldsymbol{X}_v \neq \boldsymbol{0}$, for all points (u, v) in Ω.

The subset $X(\Omega)$ of $\mathbb{E}^3$ is called the *trace* of the surface X.

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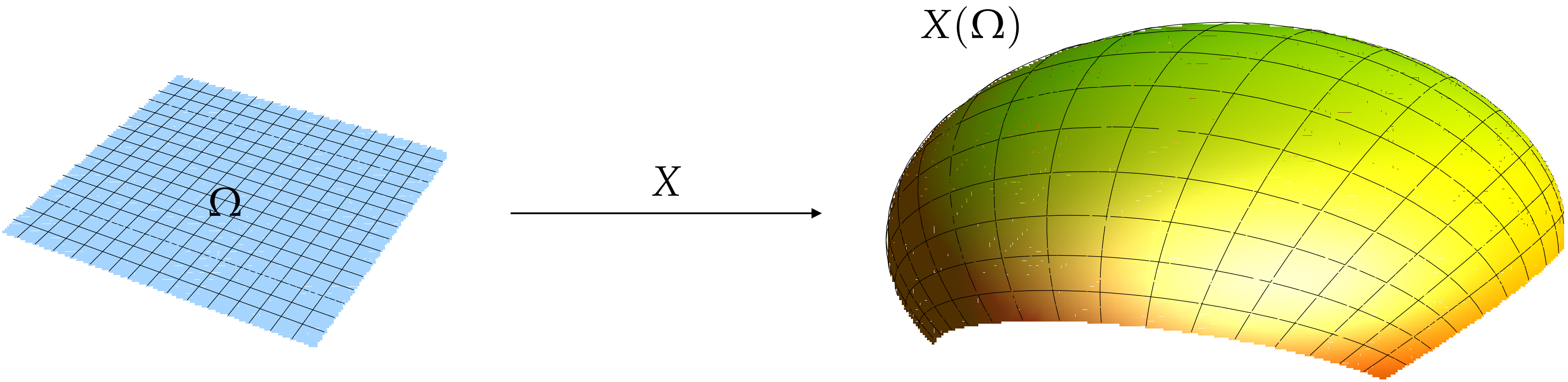
Let $\Omega =]-1,1[\times]-1,1[$, and let X be the surface patch defined by

$$x = \frac{2au}{u^2+v^2+1},\ y = \frac{2bv}{u^2+v^2+1},\ z = \frac{c(1-u^2-v^2)}{u^2+v^2+1},$$

where $a, b, c > 0$.

The surface X is a portion of an *ellipsoid*, and it is shown below, for $a = 5$, $b = 4$, $c = 3$.



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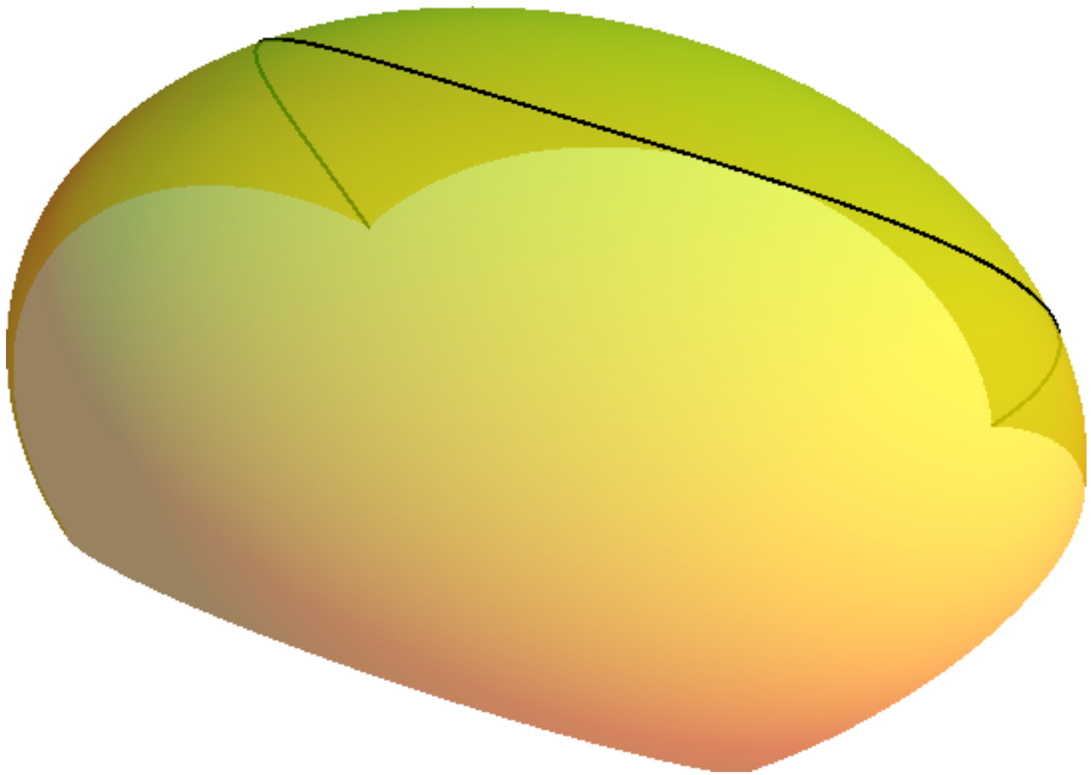
The curve $C(t) = X(t, t^2)$ on the surface X is also displayed, for $t \in]-1, 1[$.



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For a more exotic example, the function $X : \mathbb{E}^2 \to \mathbb{E}^3$ defined by

$$F_1(u,v) = u,$$
$$F_2(u,v) = v,$$
$$F_3(u,v) = u^3 - 3v^2u,$$

represents what is known as the *monkey saddle*.

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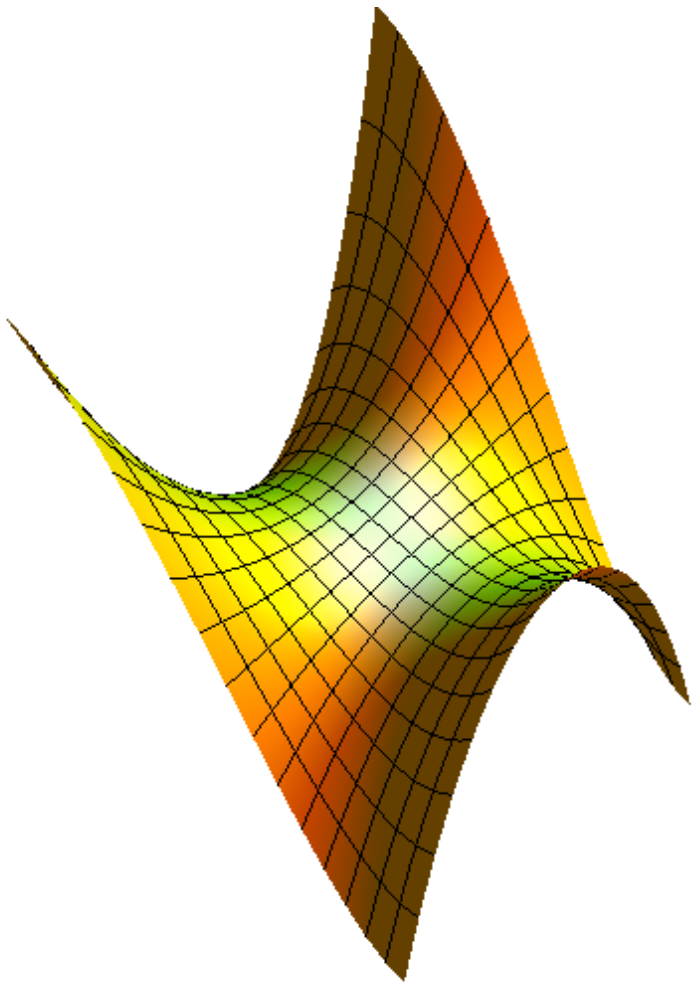

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It often desirable to define a (regular) surface patch $X : \Omega \to \mathbb{E}^3$, where Ω is a *closed* subset of $\mathbb{E}^2$.

If Ω is a closed set, we assume that there is some open subset U containing Ω and such that X can be extended to a (regular) surface over U (i.e., that X is at least C^3-continuous).

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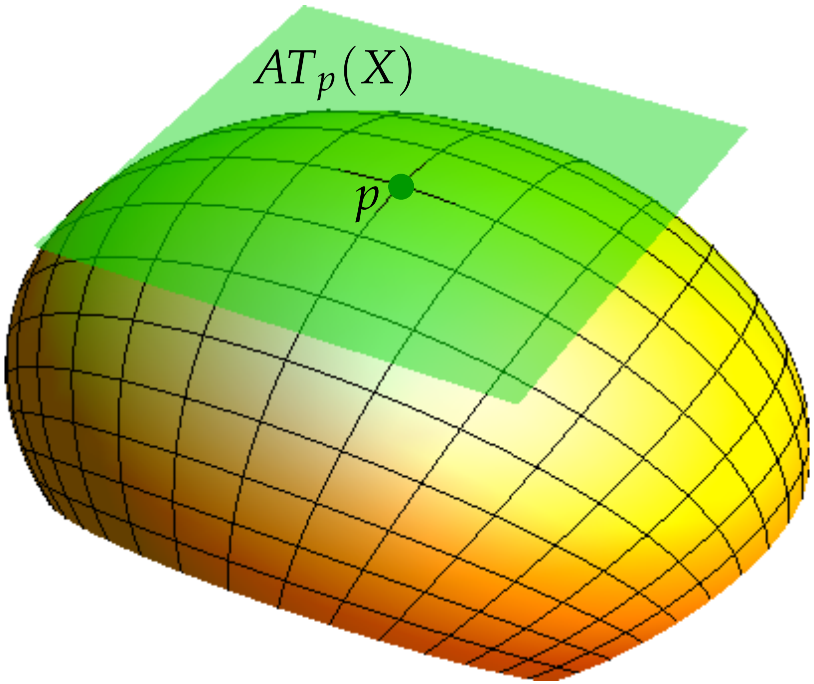
Given a regular point $p = X(u, v)$, since the tangent vectors to all the curves passing through a given point are of the form $\boldsymbol{X}_u \dot{u} + \boldsymbol{X}_v \dot{v}$, it is obvious that they form a vector space of dimension 2 isomorphic to $\mathbb{R}^2$, called the *tangent space at p*, and denoted as $T_p(X)$.

Note that $(\boldsymbol{X}_u, \boldsymbol{X}_v)$ is a basis of this vector space $T_p(X)$.

Surfaces

Parametric Surfaces

The set of tangent lines passing through p and having some tangent vector in $T_p(X)$ as direction is an affine plane called the *(affine) tangent plane at p*. Geometrically, this is an object different from $T_p(X)$ and it should be denoted differently (perhaps as $AT_p(X)$?).



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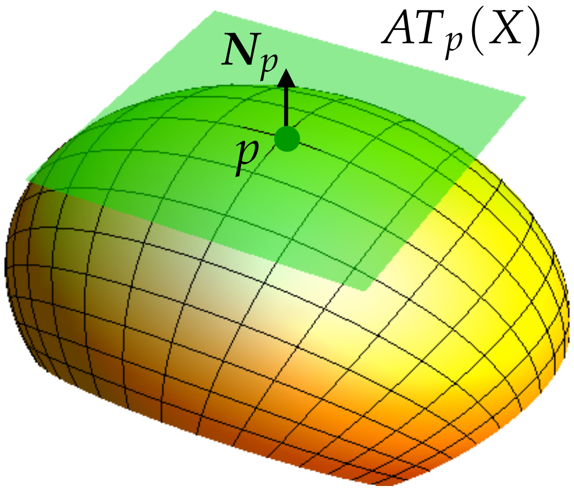
Parametric Surfaces

The unit vector

$$\boldsymbol{N}_p = \frac{\boldsymbol{X}_u \times \boldsymbol{X}_v}{\|\boldsymbol{X}_u \times \boldsymbol{X}_v\|}$$

is called the *unit normal vector at p*, and the line through p of direction $\boldsymbol{N}_p$ is the *normal line to X at p*.



This time, we can use the notation N_p for the line, to distinguish it from the vector $\boldsymbol{N}_p$.

Surfaces

Parametric Surfaces

The fact that we are not requiring the map X defining a surface $X : \Omega \to \mathbb{E}^3$ to be injective may cause problems. Indeed, if the map X is not injective, it may happen that

$$p = X(u_0, v_0) = X(u_1, v_1)$$

for some (u_0, v_0) and (u_1, v_1) such that

$$(u_0, v_0) \neq (u_1, v_1)\,.$$

Indeed, we really have two pairs of partial derivatives, i.e., $(\boldsymbol{X}_u(u_0, v_0), \boldsymbol{X}_v(u_0, v_0))$ and $(\boldsymbol{X}_u(u_1, v_1), \boldsymbol{X}_v(u_1, v_1))$, and the planes spanned by these pairs could be distinct.

Surfaces

Parametric Surfaces

In this case, there are really two tangent planes

$$T_{(u_0,v_0)}(X) \qquad \text{and} \qquad T_{(u_1,v_1)}(X)$$

at the point p where X has a self-intersection.

Similarly, the normal $\boldsymbol{N}_p$ is not well defined, and we really have two normals, $\boldsymbol{N}_{(u_0,v_0)}$ and $\boldsymbol{N}_{(u_1,v_1)}$, at p.

We can avoid the problem entirely by assuming that X is injective, although this will rule out some surfaces that come up in practice.

Surfaces

Parametric Surfaces

The tangent space, $T_p(X)$, may also be undefined when p is not a regular point. For example, consider the parametric surface $X = (x(u,v), y(u,v), z(u,v))$ defined such that

$$x = u(u^2 + v^2),$$
$$y = v(u^2 + v^2),$$
$$z = u^2 v - v^3/3\,.$$

Note that all the partial derivatives at $(u,v) = (0,0)$ are zero. So, the tangent space is undefined at the origin, and hence the origin is a singular point of the surface X. Indeed, one can check that the tangent lines to the surface at the origin do not lie in a plane.

Surfaces

Parametric Surfaces

Surfaces

Surfaces in a More General Sense

In most applications, we are interested in the trace

$$S = X(\Omega)$$

of a surface $X : \Omega \to \mathbb{E}^3$, rather than the actual parametrization of S by X.

Since S is a subset of $\mathbb{E}^3$, it inherits the subspace topology from $\mathbb{E}^3$; namely, a subset $U \subseteq S$ is open iff $U = S \cap B$, for some open subset $B \subseteq \mathbb{E}^3$.

Surfaces

Surfaces in a More General Sense

It is then natural to require not only that

$$X : \Omega \rightarrow \mathbb{E}^3$$

be injective and continuous, but that its inverse,

$$X^{-1} : S \rightarrow \Omega\,,$$

be continuous. This means that

$$X : \Omega \rightarrow \mathbb{E}^3$$

is a *homeomorphism* between $\Omega \subseteq \mathbb{E}^2$ and $S \subseteq \mathbb{E}^3$, considered as a topological space. One of the benefits of requiring that X is a homeomorphism is that S can't have self-intersections.

Surfaces

Surfaces in a More General Sense

We have the following provisional definition of a surface:

Definition 3.2. A *surface* is a subset $S \subseteq \mathbb{E}^3$, such that there is some open subset $\Omega \subseteq \mathbb{E}^2$, and some smooth map, $\varphi\colon \Omega \to \mathbb{E}^3$, such that φ is a homeomorphism from Ω to S, and $(d\varphi)_t$ is injective for every $t \in \Omega$; equivalently, the matrix $J(\varphi)(t)$ has rank 2.

The map φ is called a *parametrization* of the surface S.

Surfaces in a More General Sense

The reason for requiring $(d\varphi)_t$ to be injective for every $t \in \Omega$ is to ensure that the tangent plane at $p = \varphi(t)$ be defined for all $p \in S$.

Definition 3.2 is good, in the sense that it allows us to "do calculus" on the surface S, by making use of the continuous maps φ and φ^{-1}. However, Definition 3.2 imposes a major restriction on the surfaces defined in this fashion: *they cannot be compact spaces*.

Intuitively speaking, we can't define closed surfaces. This is because if S was compact, then Ω would be compact too, because φ^{-1} is continuous; but this is absurd since Ω is open.

Consequently, a simple sphere, $S^2 \subseteq \mathbb{E}^3$, is not a surface according to Definition 3.2.

Surfaces

Surfaces in a More General Sense

The problem is that Definition 3.2 is *too global*. We need a *local definition*.

Instead of requiring a single parametrization for S, for every point p on S, we require that some open subset $U \subseteq S$ containing p have a parametrization, $\varphi_U : \Omega_U \to \mathbb{E}^3$, where φ_U is a homeomorphism from Ω_U to U. This leads us to the following definition:

Definition 3.3. A *surface* is a subset $S \subseteq \mathbb{E}^3$, such that for every point $p \in S$, there is some open subset $\Omega \subseteq \mathbb{E}^2$, some open subset $B \subseteq \mathbb{E}^3$ with $p \in B$, and a smooth map

$$\varphi : \Omega \to \mathbb{E}^3,$$

such that φ is a homeomorphism from Ω to $\varphi(\Omega) = U = S \cap B$, and $(d\varphi)_q$ is injective, where $q = \varphi^{-1}(p)$; equivalently, the Jacobian matrix $J(\varphi)(q)$ of $d\varphi$ at $q \in \Omega$ has rank 2.

Surfaces

Surfaces in a More General Sense

Each map φ is called a *parametrization* of the surface S.

Clearly, we can define a surface in $\mathbb{E}^N$ (where $N > 3$) by using smooth maps,

$$\varphi : \Omega \to \mathbb{E}^N$$

which are homeomorphisms from an open subset, Ω, of $\mathbb{E}^2$ to an open subset, $U = \varphi(\Omega)$, of $\mathbb{E}^N$.

Surfaces in a More General Sense

The unit sphere S^2 in $\mathbb{E}^3$ defined such that

$$S^2 = \left\{(x, y, z) \in \mathbb{E}^3 \mid x^2 + y^2 + z^2 = 1\right\}$$

is a smooth surface, because it can be parametrized using the following two maps:

$$\varphi_1 : \mathbb{E}^2 \rightarrow S^2 - \{(0,0,1)\} \quad \text{and} \quad \varphi_2 : \mathbb{E}^2 \rightarrow S^2 - \{(0,0,-1)\}$$

where

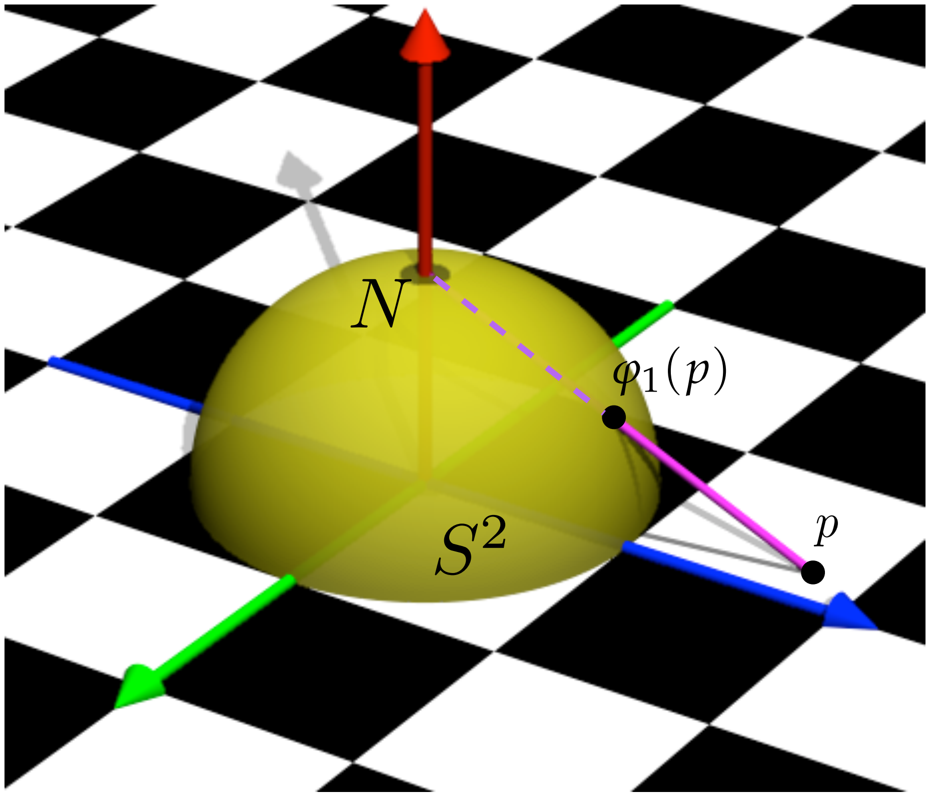
$$\varphi_1 \ : (u, v) \mapsto \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right)$$

and

$$\varphi_2 : (u, v) \mapsto \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{1 - u^2 - v^2}{u^2 + v^2 + 1}\right) .$$

Surfaces

Surfaces in a More General Sense



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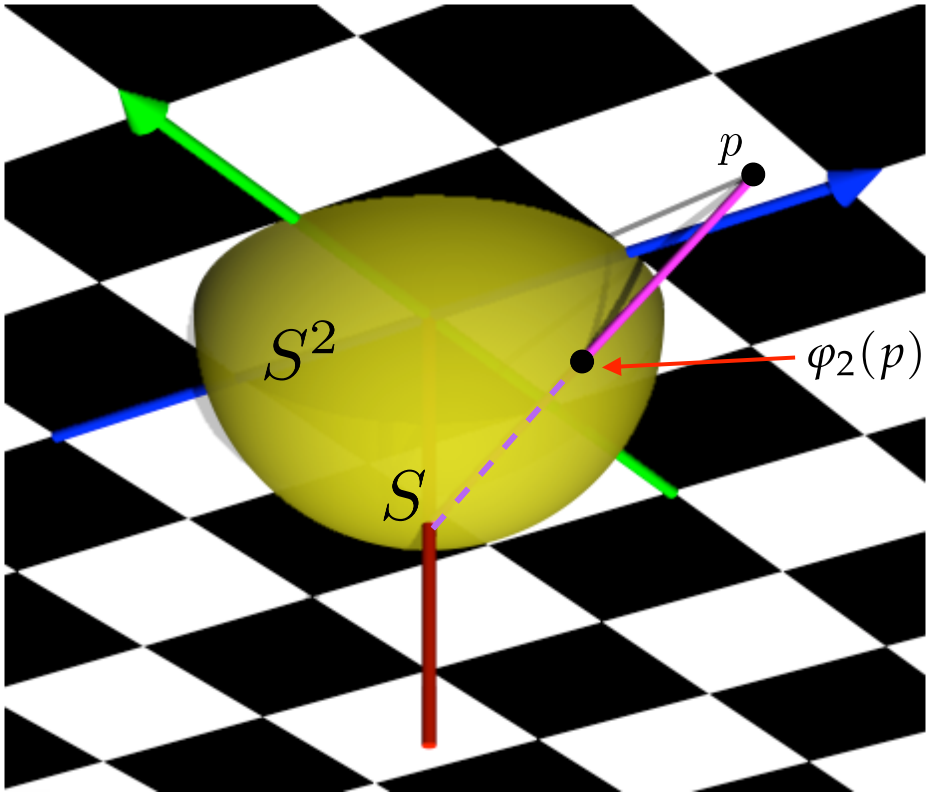
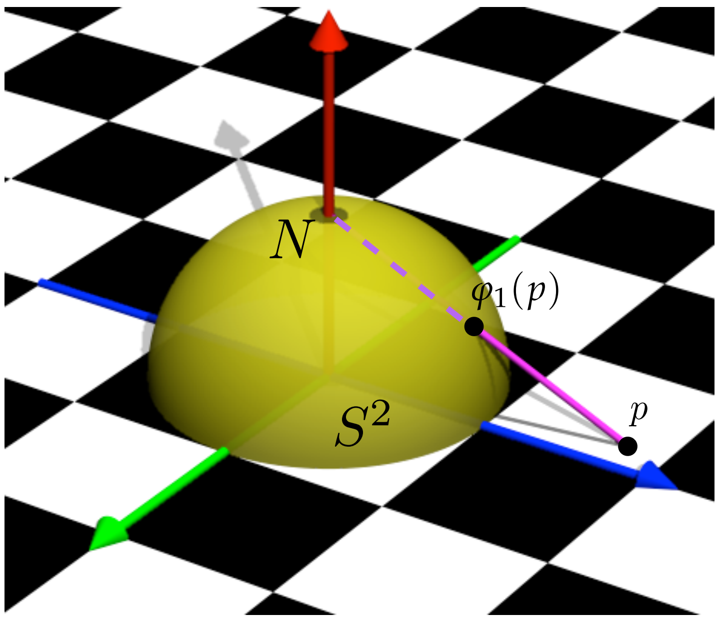
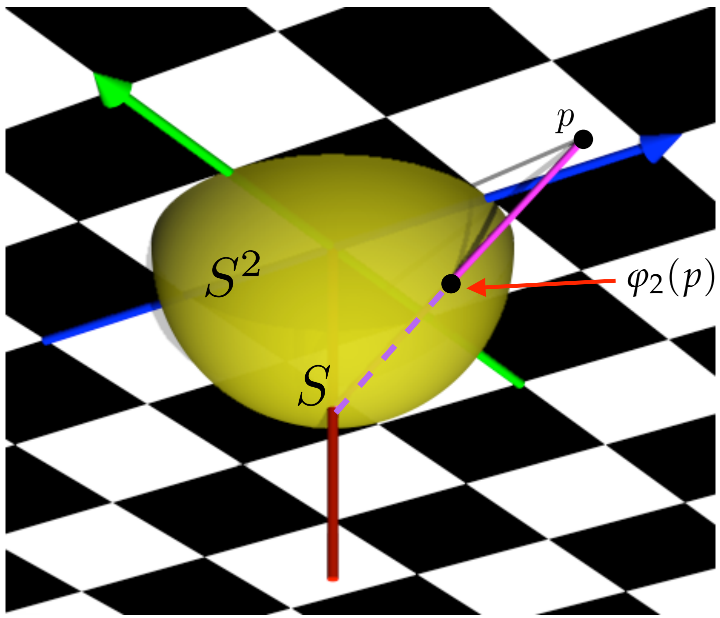
Surfaces in a More General Sense



Surfaces in a More General Sense

The map φ_1 corresponds to the inverse of the *stereographic projection from the north pole*, $N = (0, 0, 1) \in \mathbb{E}^3$, onto the plane $z = 0$, and the map φ_2 corresponds to the inverse of the *stereographic projection from the south pole*, $S = (0, 0, -1) \in \mathbb{E}^3$, onto the plane $z = 0$.





Surfaces

Surfaces in a More General Sense

Using φ_1, the open lower hemisphere is parametrized by the open disk of center O and radius 1 contained in the plane $z = 0$. Similarly, using φ_2, the open upper hemisphere is parametrized by the open disk of center O and radius 1 contained in the plane $z = 0$.

The map φ_1^{-1} assigns local coordinates to the points in the open lower hemisphere of S^2, while the map φ_2^{-1} assigns local coordinates to the points in the open upper hemisphere.

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Surfaces in a More General Sense

From Definition 3.3, we see that a surface is the union of the images of a collection of parametrizations. So, if a point p belongs to the ranges of two different parametrizations, we will dispose of two different coordinate systems near p. More specifically, let

$$X_i : U_i \to X_i(U) \subseteq S\,, \quad \text{for } i = 1, 2 \quad ,$$

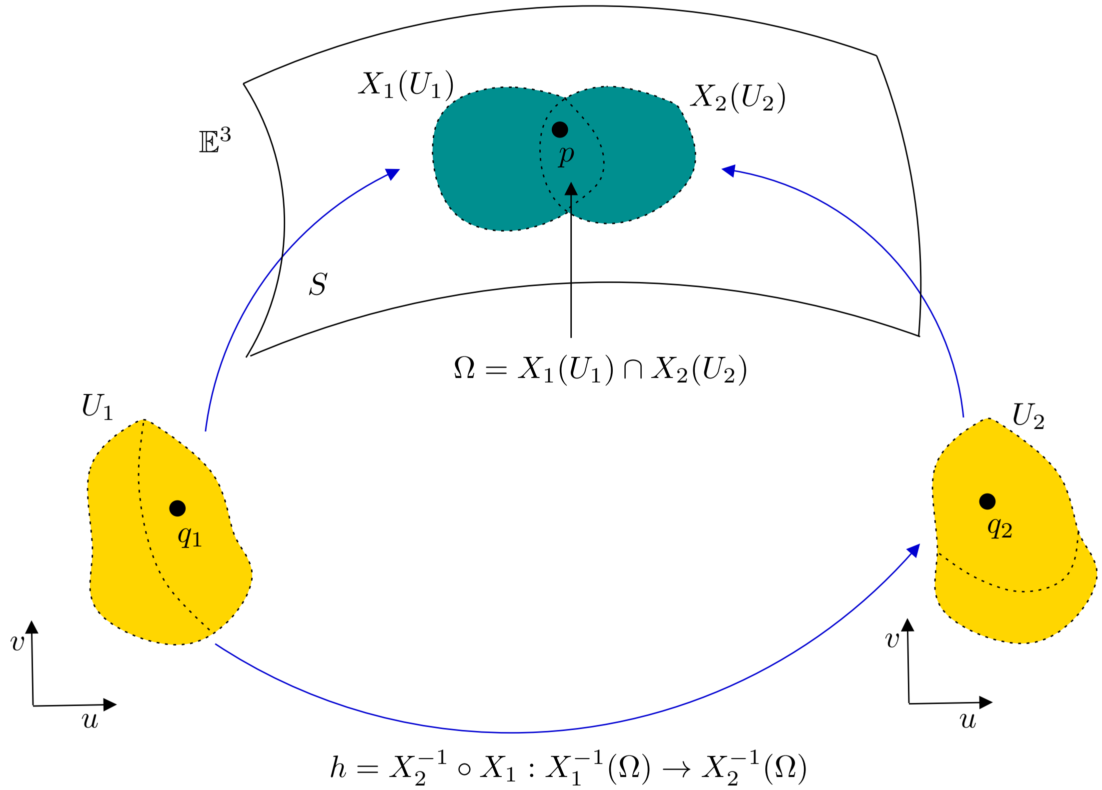
be two parametrizations of S such that $\Omega = X_1(U_1) \cap X_2(U_2)$ is nonempty. Then the map

$$h = X_2^{-1} \circ X_1 : X_1^{-1}(\Omega) \to X_2^{-1}(\Omega)$$

is a homeomorphism taking coordinates in U_1 with respect to X_1 into coordinates in U_2 with respect to X_2. The map h is said to be a *change of parameters* or a *change of coordinates*.

Surfaces

Surfaces in a More General Sense



It turns out that the map h is a diffeomorphism.

Surfaces

Surfaces in a More General Sense

To prove this claim, we will use the following lemma:

Lemma 3.1. Let S be a surface and

$$X : U \to X(U) \subseteq S$$

a parametrization whose image contains the point p. Let $q \in U$ be such that $q = X^{-1}(p)$. Then, there exists an open subset, V, of U containing the point q and an orthogonal projection,

$$\pi : \mathbb{E}^3 \to \mathbb{E}^2 \,,$$

onto some of the three coordinates planes of $\mathbb{E}^3$ such that $W = (\pi \circ X)(V)$ is open in $\mathbb{E}^2$ and

$$\pi \circ X : V \to W$$

is a diffeomorphism.

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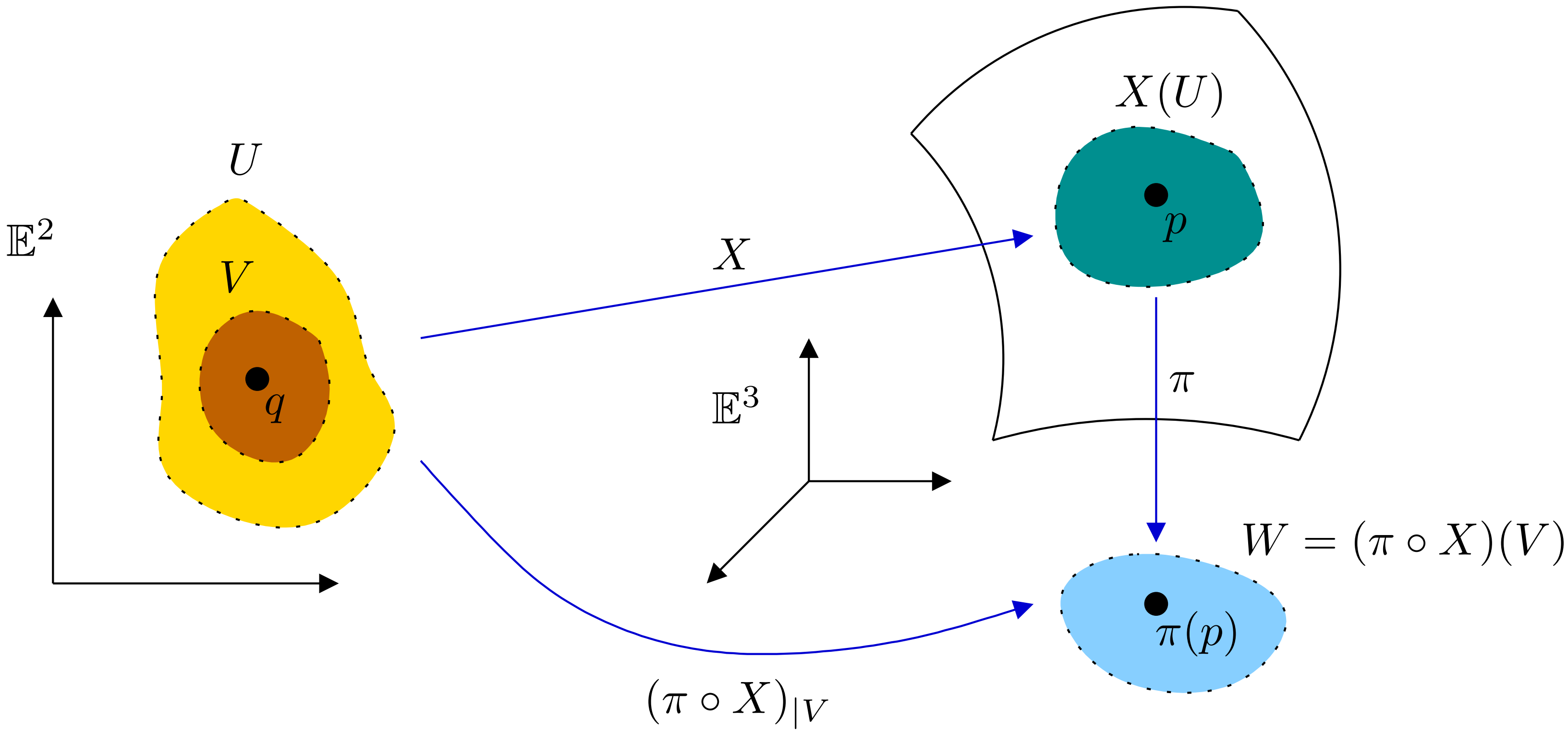
Surfaces in a More General Sense



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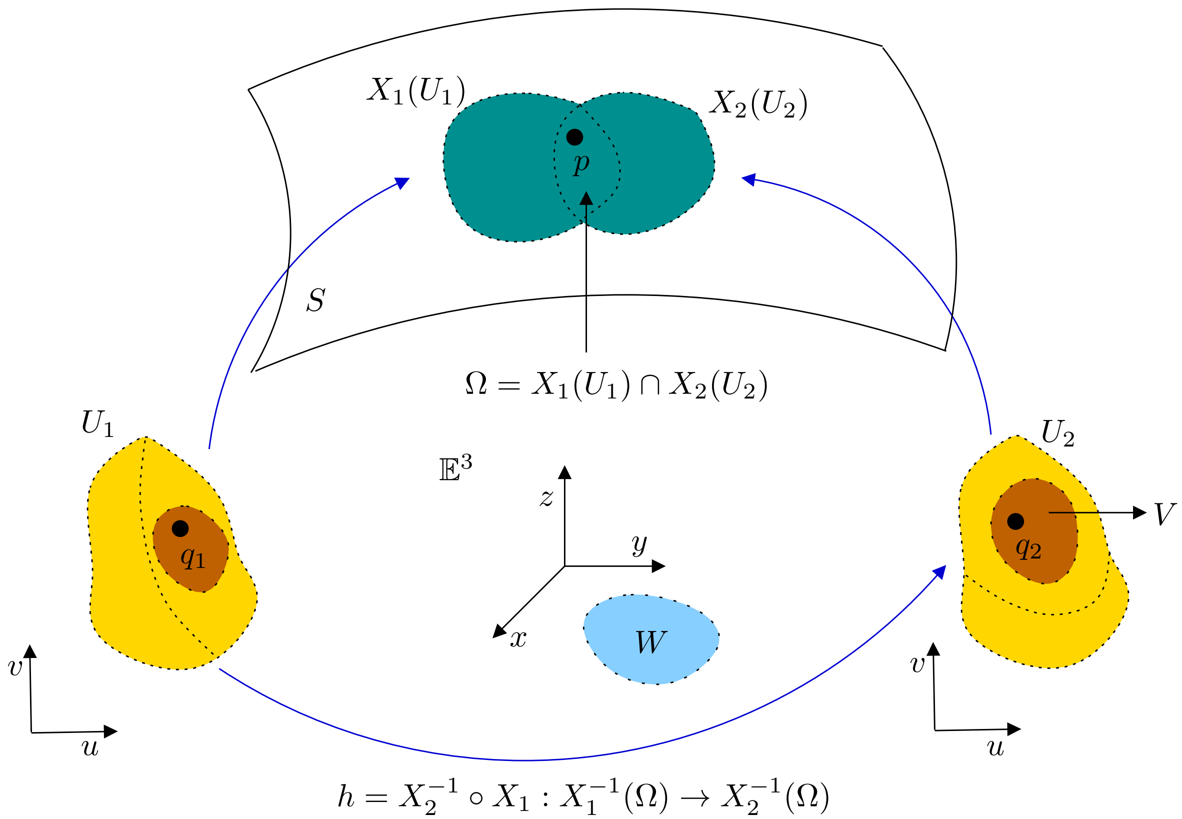
Theorem 3.2. Every change of coordinates is a diffeomorphism.



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Surfaces in a More General Sense

It is interesting to see how the unit normal vector $\mathbf{N}_p$ changes under a change of parameters.

Assume that $u = u(r, s)$ and $v = v(r, s)$, where $(r, s) \mapsto (u, v)$ is a diffeomorphism.

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Surfaces in a More General Sense

By the chain rule,

$$\begin{aligned}
\boldsymbol{X}_r \times \boldsymbol{X}_s &= \left(\boldsymbol{X}_u \frac{\partial u}{\partial r} + \boldsymbol{X}_v \frac{\partial v}{\partial r}\right) \times \left(\boldsymbol{X}_u \frac{\partial u}{\partial s} + \boldsymbol{X}_v \frac{\partial v}{\partial s}\right) \\
&= \left(\frac{\partial u}{\partial r}\frac{\partial v}{\partial s} - \frac{\partial u}{\partial s}\frac{\partial v}{\partial r}\right)(\boldsymbol{X}_u \times \boldsymbol{X}_v) \\
&= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} (\boldsymbol{X}_u \times \boldsymbol{X}_v) \\
&= \frac{\partial(u,v)}{\partial(r,s)} (\boldsymbol{X}_u \times \boldsymbol{X}_v),
\end{aligned}$$

denoting the Jacobian determinant of the map $(r,s) \mapsto (u,v)$ as $\dfrac{\partial(u,v)}{\partial(r,s)}$.

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Then, the relationship between the unit vectors $\boldsymbol{N}_{(u,v)}$ and $\boldsymbol{N}_{(r,s)}$ is

$$\boldsymbol{N}_{(r,s)} = \boldsymbol{N}_{(u,v)} \operatorname{sign}\left(\frac{\partial(u,v)}{\partial(r,s)}\right).$$

We will therefore restrict our attention to changes of variables such that the determinant

$$\frac{\partial(u,v)}{\partial(r,s)}$$

is positive.

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One should also note that the condition

$$\boldsymbol{X}_u \times \boldsymbol{X}_v \neq 0$$

is equivalent to the fact that the Jacobian matrix, $J(X)(u,v)$, of the derivative of the map $X : \Omega \rightarrow \mathbb{E}^3$ has rank 2, i.e., that the derivative $(dX)_{(u,v)}$ of X at (u,v) is injective.

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Indeed, the Jacobian matrix of the derivative of the map

$$(u,v) \mapsto X(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$$

is

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

and $\boldsymbol{X}_u \times \boldsymbol{X}_v \neq \mathbf{0}$ is equivalent to saying that one of the minors of order 2 is invertible.

Thus, a regular surface patch is an immersion of an open subset of $\mathbb{E}^2$ into $\mathbb{E}^3$.