IMPA - Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ, Brazil

Introduction to Computational Manifolds and Applications

Part 1 - Foundations

Prof. Jean Gallier

jean@cis.upenn.edu

Department of Computer and Information Science
University of Pennsylvania
Philadelphia, PA, USA

Manifolds

Submanifolds embedded in $\mathbb{R}^N$

Definition 4.1. Given any integers N, m, with $N \geq m \geq 1$, an *m-dimensional smooth manifold in $\mathbb{E}^N$, for short a manifold*, is a nonempty subset M of $\mathbb{E}^N$ such that for every point $p \in M$, there are two open subsets, $\Omega \subseteq \mathbb{E}^m$ and $U \subseteq M$, with $p \in U$, and a smooth function,

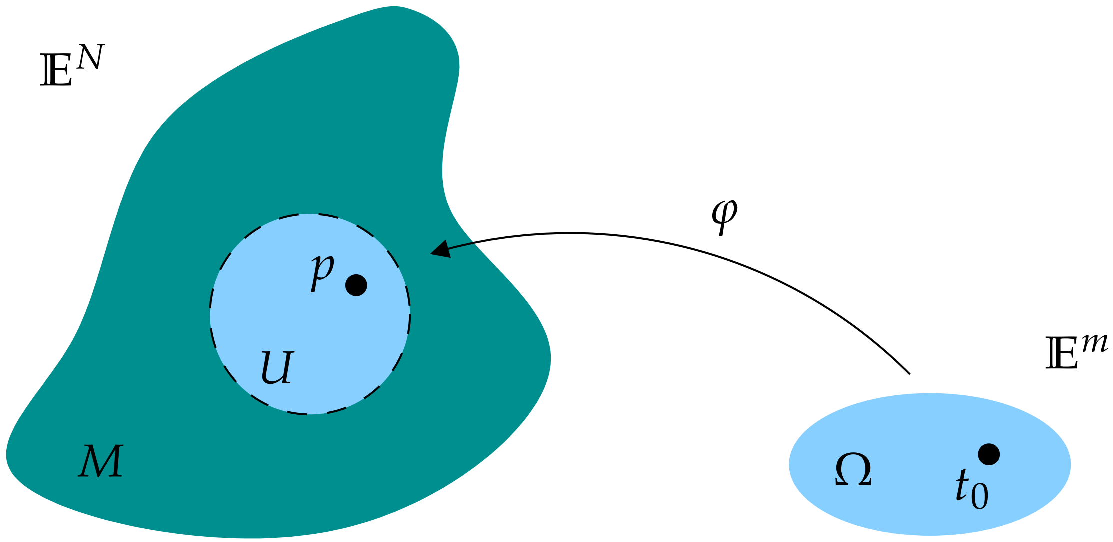
$$\varphi : \Omega \to \mathbb{E}^N,$$

such that φ is a homeomorphism between Ω and $U = \varphi(\Omega)$, and $(d\varphi)_{t_0}$ is injective, for

$$t_0 = \varphi^{-1}(p).$$

Manifolds

Submanifolds embedded in $\mathbf{R}^N$



The function

$$\varphi : \Omega \to U$$

is called a *(local) parametrization of M at p*. If $0_m \in \Omega$ and $\varphi(0_m) = p$, we say that φ is *centered at p*.

Manifolds

Submanifolds embedded in $\mathbf{R}^\mathbf{N}$

Recall that

$$M \subseteq \mathbb{E}^N$$

is a topological space under the subspace topology, and U is some open subset of M in the subspace topology, which means that $U = M \cap W$ for some open subset W of $\mathbb{E}^N$.

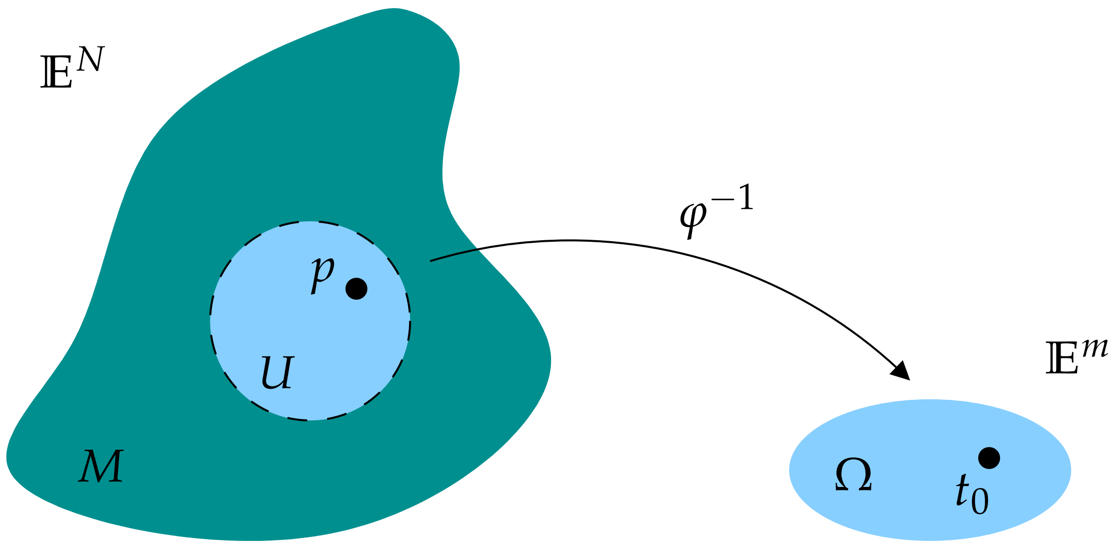
Manifolds

Submanifolds embedded in $\mathbb{R}^N$

Since $\varphi : \Omega \to U$ is a homeomorphism, it has an inverse,

$$\varphi^{-1} : U \to \Omega \,,$$

that is also a homeomorphism, called a *(local) chart*.



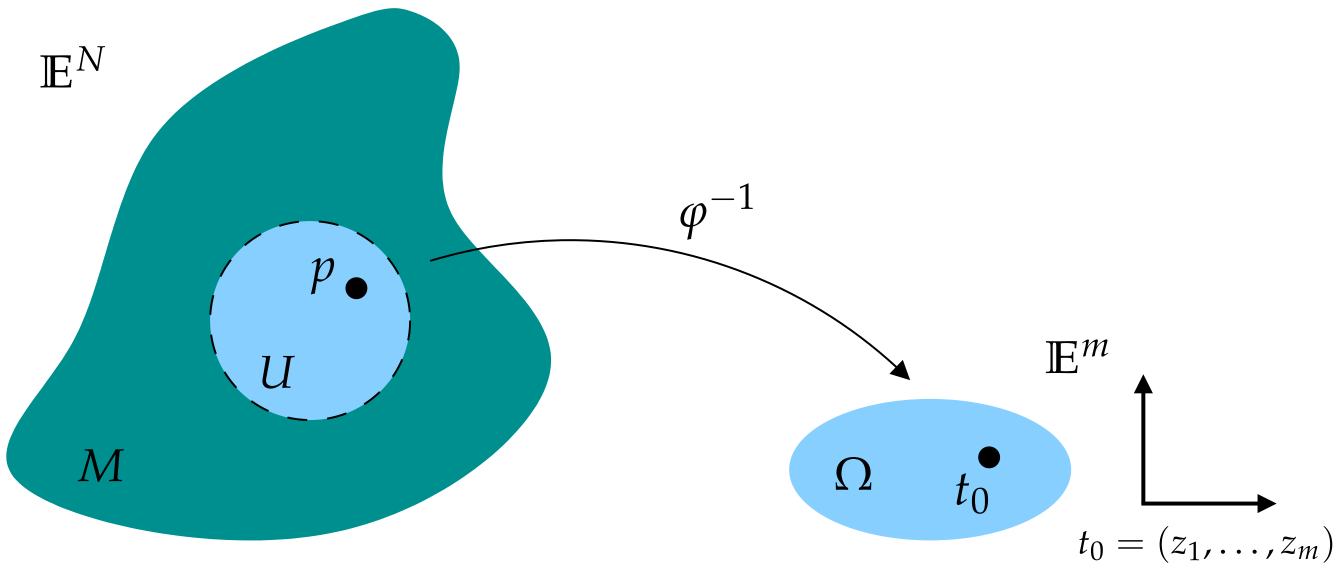
Manifolds

Submanifolds embedded in R^N

Since $\Omega \subseteq \mathbb{E}^m$, for every $p \in M$ and every parametrization $\varphi : \Omega \to U$ of M at p, we have

$$\varphi^{-1}(p) = (z_1, \ldots, z_m)\,,$$

for some $z_i \in \mathbb{E}$, and we call $z_1, \ldots, z_m$ the *local coordinates of p (with respect to φ^{-1})*.

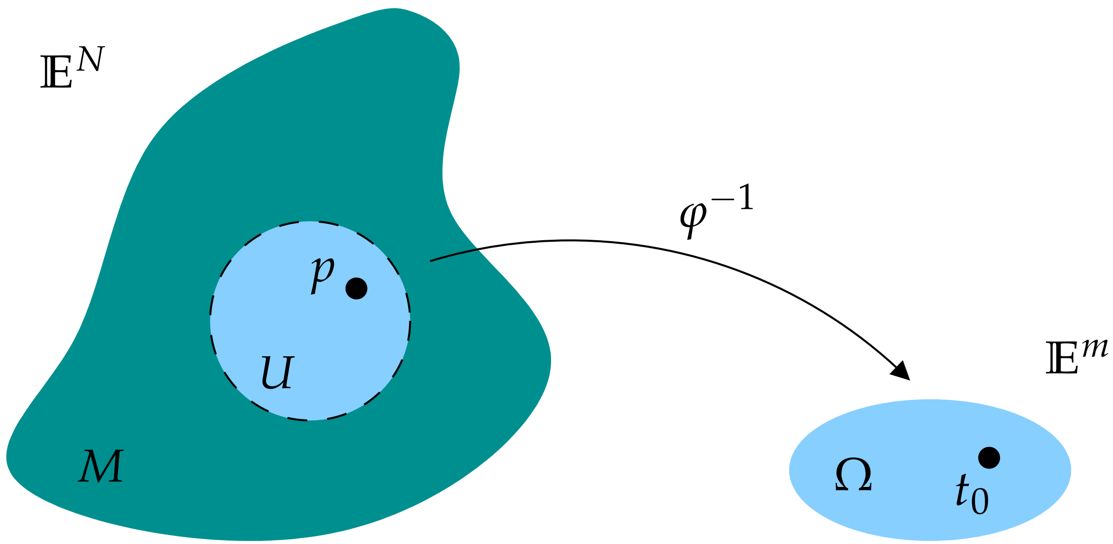


Manifolds

Submanifolds embedded in R^N

We often refer to a manifold M without explicitly specifying its dimension (the integer m).

Intuitively, a chart provides a "flattened" local map of a region on a manifold.

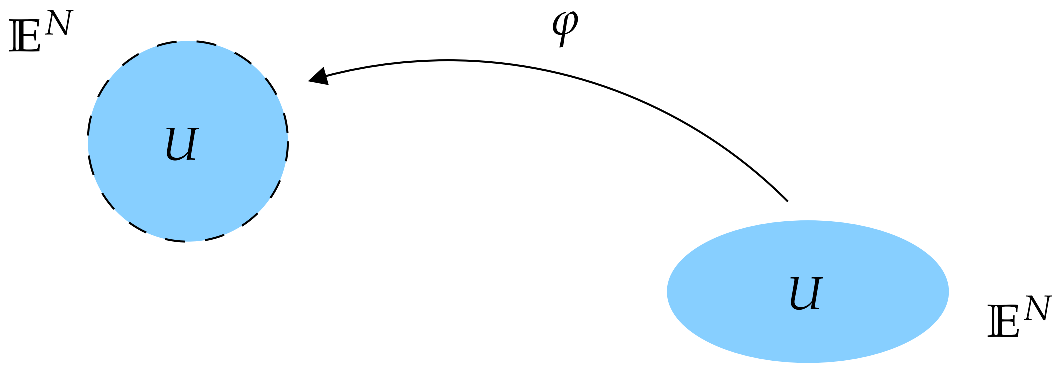


Manifolds

Submanifolds embedded in $\mathbf{R}^N$

Example 4.1.

Every open subset, U, of $\mathbb{E}^N$ is a manifold in a trivial way. Indeed, we can use the inclusion map, $\varphi : U \to \mathbb{E}^N$, where $\varphi(p) = p$ for every $p \in U$, as a parametrization. Note that, in this case, there is a single parametrization, namely φ, for every point p in U.



Manifolds

Submanifolds embedded in $\mathbb{R}^N$

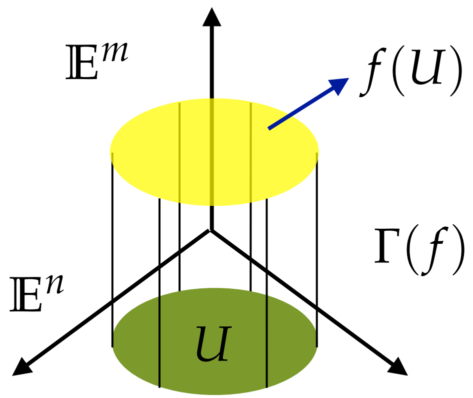
Example 4.2.

For U an open subset of $\mathbb{E}^n$ and

$$f : U \to \mathbb{E}^m$$

the graph of f, $\Gamma(f)$, is defined to be the subspace

$$\Gamma(f) = \{(x, f(x)) \in U \times \mathbb{E}^m\}.$$



Manifolds

Submanifolds embedded in $\mathbb{R}^N$

If f is smooth, then $\Gamma(f)$ is a manifold of dimension n in $\mathbb{E}^{n+m}$.

Indeed, if we let

$$\varphi : U \to \Gamma(f) \qquad \text{and} \qquad \psi : \Gamma(f) \to U$$

such that

$$\varphi(x) = (x, f(x)) \qquad \text{and} \qquad \psi((x, f(x)) = x\,,$$

then φ and ψ are smooth and inverse to each other, and hence they are homeomorphisms.

Manifolds

Submanifolds embedded in $\mathbb{R}^N$

The derivative, $(d\varphi)_x$, of φ at x, which is equal to $(\mathrm{id}_n, (df)_x)$, is clearly injective. So,

$$\Gamma(f)$$

is a manifold in $\mathbb{E}^{n+m}$. That's why many of the familiar surfaces from calculus, for instance, an elliptic or a hyperbolic paraboloid, which are graphs of functions, are manifolds.

Manifolds

Submanifolds embedded in $\mathbb{R}^N$

Example 4.3.

For any two positive integers, m and n, let $M_{m,n}(\mathbb{R})$ be the vector space of all $m \times n$ matrices.

Since $M_{m,n}(\mathbb{R})$ is isomorphic to $\mathbb{R}^{mn}$, we give it the topology of $\mathbb{R}^{mn}$.

The *general linear group*, $\mathbf{GL}(n, \mathbb{R})$, is by definition the set of matrices

$$\mathbf{GL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\} = \det^{-1}(\mathbb{R} - \{0\}).$$

$\mathbf{GL}(n, \mathbb{R})$ is a manifold in $\mathbb{R}^{n^2}$.

Manifolds

Submanifolds embedded in R^N

Indeed, since the function

$$\det : M_n(\mathbb{R}) \to \mathbb{R}$$

is continuous and $\mathbb{R} - \{0\}$ is open in $\mathbb{R}$, and since $\mathbf{GL}(n, \mathbb{R})$ is the inverse image of (the open set) $\mathbb{R} - \{0\}$ under the function det, $\mathbf{GL}(n, \mathbb{R})$ is an open set of $M_n(\mathbb{R}) \approx \mathbb{R}^{n^2}$.

From Example 4.1, we conclude that $\mathbf{GL}(n, \mathbb{R})$ is a manifold in $\mathbb{R}^{n^2}$.

The following two lemmas provide the link with the definition of an abstract manifold:

Manifolds

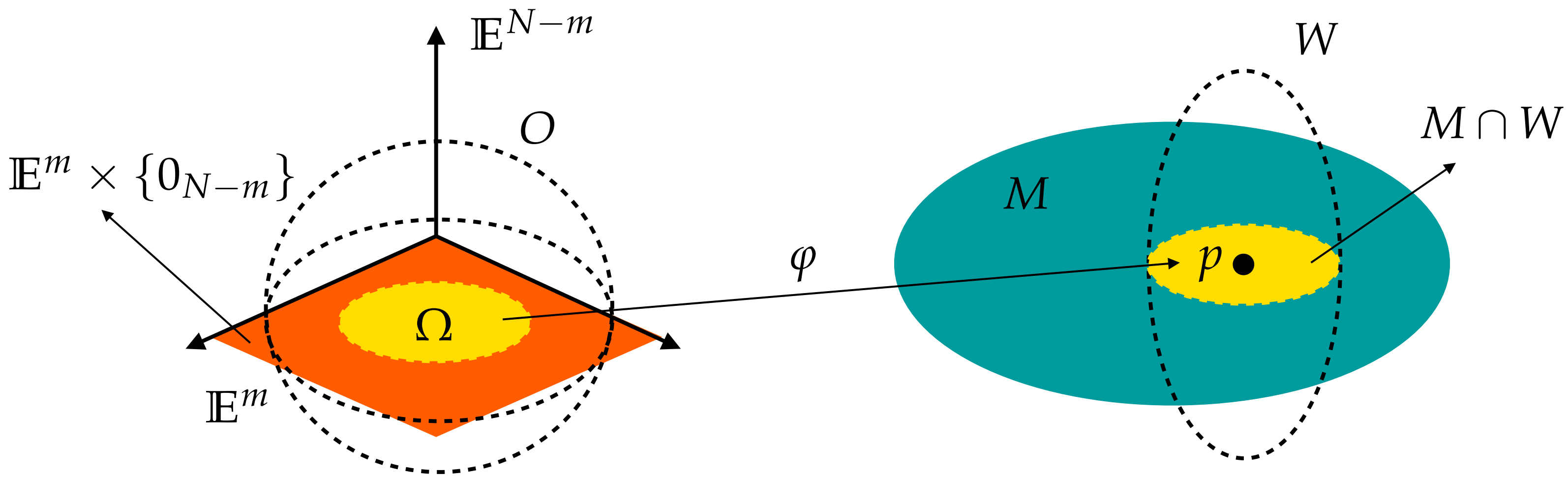
Submanifolds embedded in R^N

Lemma 4.1. Given an m-dimensional manifold M in $\mathbb{E}^N$, for every $p \in M$ there are two open sets $O, W \subseteq \mathbb{E}^N$, with $0_N \in O$ and $p \in (M \cap W)$, and a smooth diffeomorphism

$$\varphi : O \to W$$

such that

$$\varphi(0_N) = p \qquad \text{and} \qquad \varphi(O \cap (\mathbb{E}^m \times \{0_{N-m}\})) = M \cap W\,.$$



Manifolds

Submanifolds embedded in $\mathbf{R}^N$

The next lemma is easily shown from Lemma 4.1. It is a key technical result used to show that interesting properties of maps between manifolds do not depend on parametrizations.

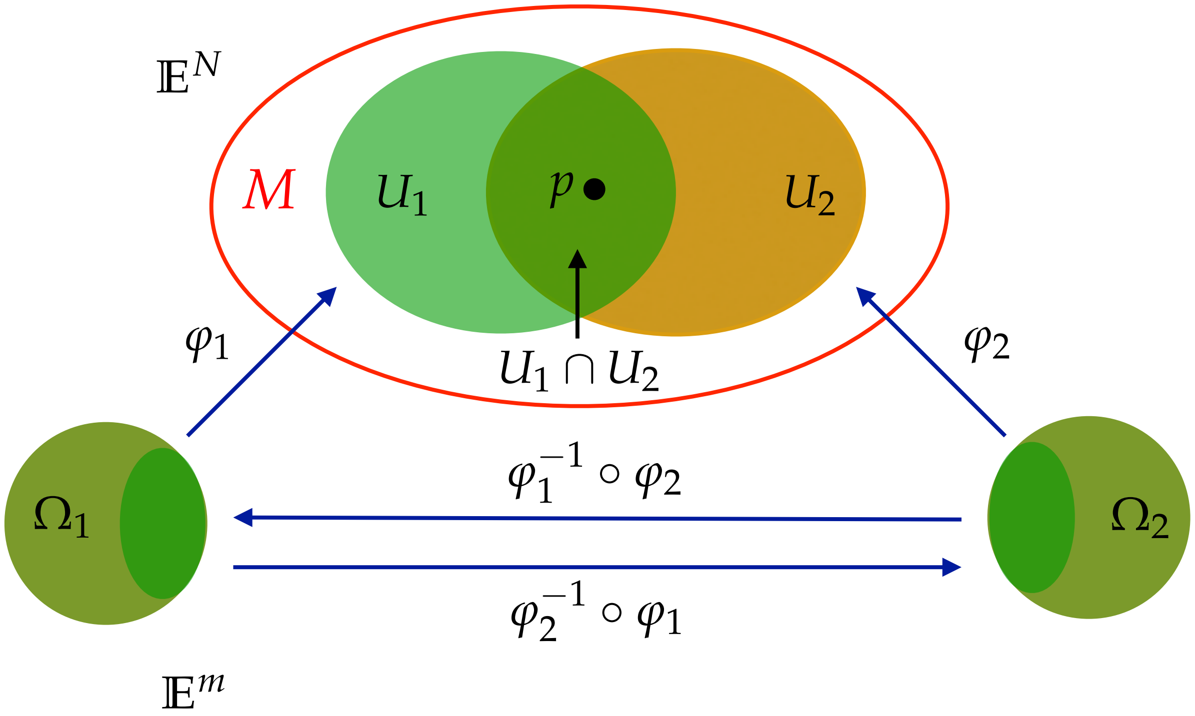
Lemma 4.2. Given an m-dimensional manifold M in $\mathbb{E}^N$, for every $p \in M$ and any two parametrizations, $\varphi_1 : \Omega_1 \to U_1$ and $\varphi_2 : \Omega_2 \to U_2$ of M at p, if $U_1 \cap U_2 \neq \emptyset$, the map

$$\varphi_2^{-1} \circ \varphi_1 : \varphi_1^{-1}(U_1 \cap U_2) \to \varphi_2^{-1}(U_1 \cap U_2)$$

is a smooth diffeomorphism.

Manifolds

Submanifolds embedded in R^N



The maps $\varphi_2^{-1} \circ \varphi_1$ and $\varphi_1^{-1} \circ \varphi_2$ are called *transition maps*.

Manifolds

Submanifolds embedded in $\mathbb{R}^N$

Using Definition 4.1, it may be quite hard to prove that a space is a manifold. Therefore, it is handy to have alternate characterizations such as those given in the next Proposition.

Proposition 4.3. A subset, $M \subseteq \mathbb{E}^{m+k}$, is an m-dimensional manifold if and only if either

(1) For every $p \in M$, there is some open subset, $W \subseteq \mathbb{E}^{m+k}$, with $p \in W$ and a (smooth) submersion, $f : W \to \mathbb{E}^k$, so that $W \cap M = f^{-1}(0)$, or

(2) For every $p \in M$, there is some open subset, $W \subseteq \mathbb{E}^{m+k}$, with $p \in W$ and a (smooth) map, $f : W \to \mathbb{E}^k$, so that $(df)_p$ is surjective and $W \cap M = f^{-1}(0)$.

Manifolds

Submanifolds embedded in $\mathbb{R}^N$

Condition (2), although apparently weaker than condition (1), is in fact equivalent to it.

This is because to say that $(df)_p$ is surjective means that the Jacobian matrix of $(df)_p$ has rank k, which means that some determinant is nonzero, and because the determinant function is continuous this must hold in some open subset $W_1 \subseteq W$ containing p.

Consequently, the restriction, f_1, of f to W_1 is indeed a submersion and

$$f_1^{-1}(0) = W_1 \cap f^{-1}(0) = W_1 \cap (W \cap M) = W_1 \cap M\,.$$

The proof is based on two technical lemmas that are proved using the inverse function theorem.

Manifolds

Submanifolds embedded in $\mathbb{R}^N$

Lemma 4.4. Let $U \subseteq \mathbb{E}^m$ be an open subset of $\mathbb{E}^m$ and pick some $a \in U$. If

$$f : U \to \mathbb{E}^n$$

is a smooth immersion at a, i.e., if df_a is injective (and hence, $m \leq n$), then there is an open set, $V \subseteq \mathbb{E}^n$, with $f(a) \in V$, an open subset, $U' \subseteq U$, with $a \in U'$ and $f(U') \subseteq V$, an open subset $O \subseteq \mathbb{E}^{n-m}$, and a diffeomorphism, $\theta : V \to (U' \times O)$, so that

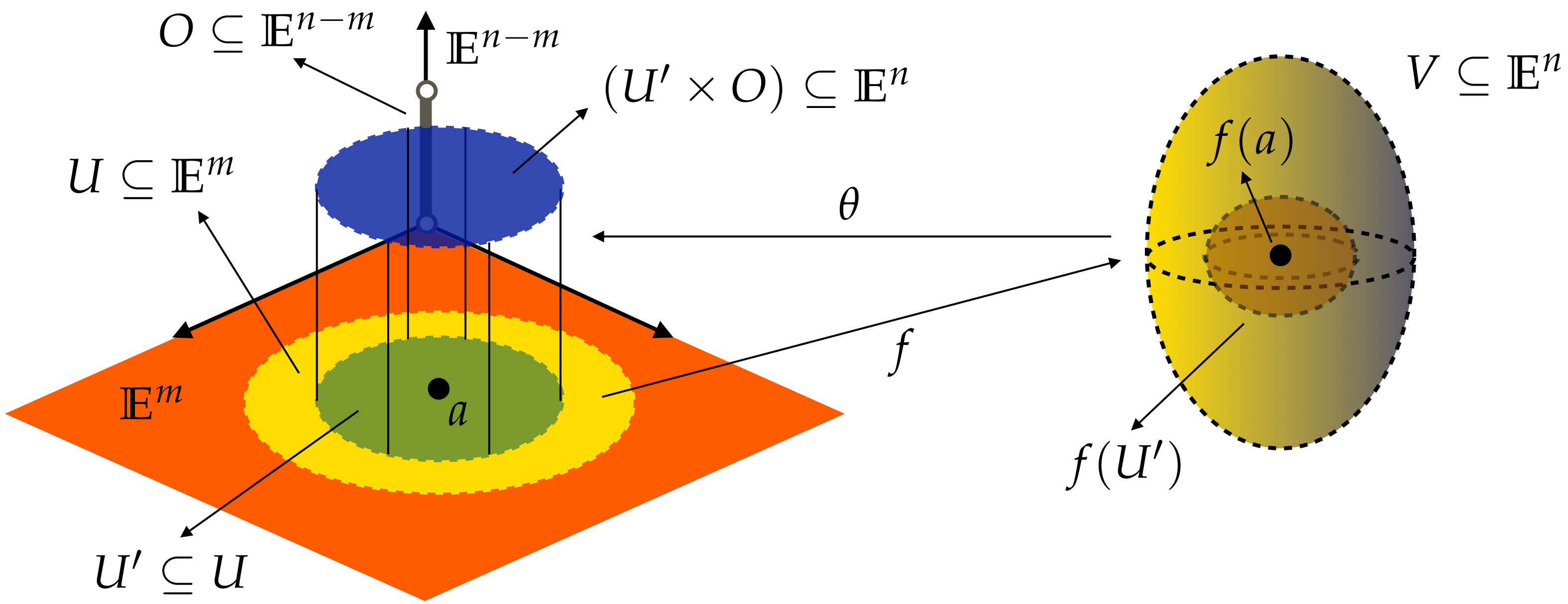
$$\theta(f(x_1, \ldots, x_m)) = (x_1, \ldots, x_m, 0, \ldots, 0),$$

for all

$$(x_1, \ldots, x_m) \in U'.$$

Manifolds

Submanifolds embedded in R^N



Manifolds

Submanifolds embedded in $\mathbb{R}^N$

Lemma 4.6. Let $W \subseteq \mathbb{E}^m$ be an open subset of $\mathbb{E}^m$ and pick some $a \in W$. If

$$f : W \to \mathbb{E}^n$$

is a smooth submersion at a, i.e., if df_a is surjective (and hence, $m \geq n$), then there is an open set, $V \subseteq W \subseteq \mathbb{E}^m$, with $a \in V$, and a diffeomorphism, ψ, with domain $O \subseteq \mathbb{E}^m$, so that

$$\psi(O) = V \qquad \text{and} \qquad f(\psi(x_1, \ldots, x_m)) = (x_1, \ldots, x_n)\,,$$

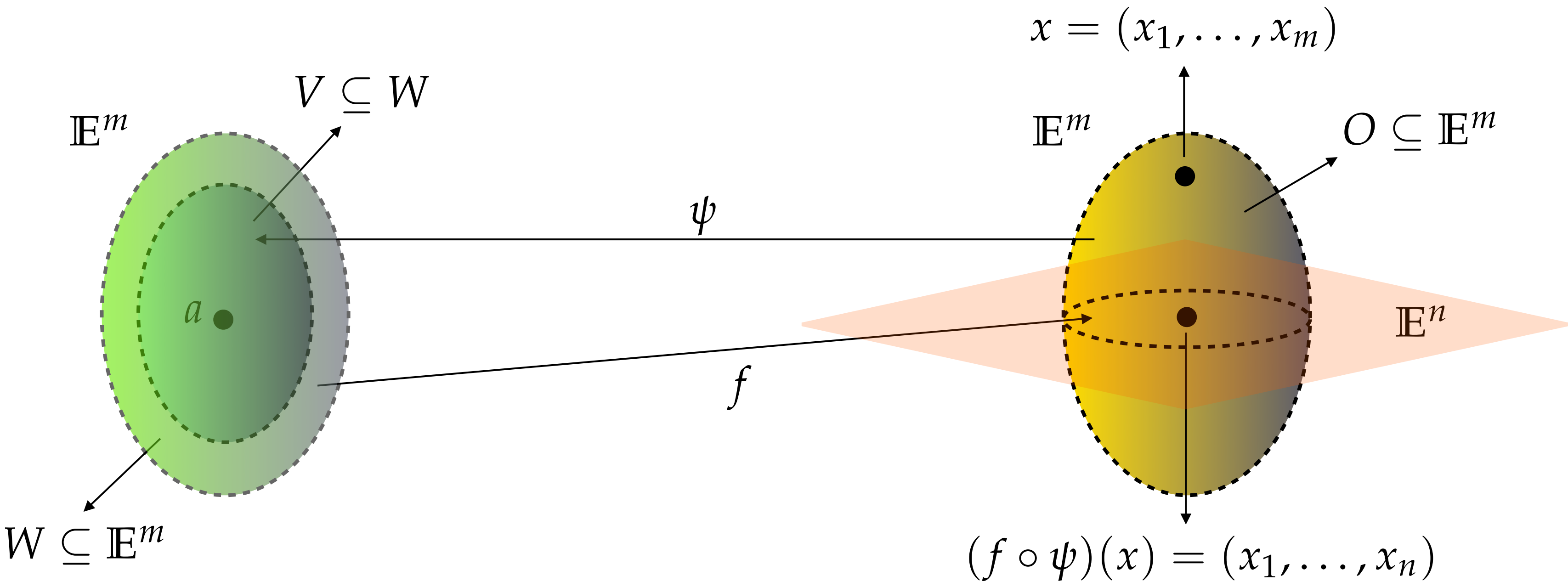
for all

$$(x_1, \ldots, x_m) \in O\,.$$

.

Manifolds

Submanifolds embedded in R^N



Using Lemmas 4.5 and 4.6, we can prove the following theorem:

Manifolds

Submanifolds embedded in $\mathbf{R}^N$

Theorem 4.7. A nonempty subset, $M \subseteq \mathbb{E}^N$, is an m-manifold (with $1 \leq m \leq N$) iff any of the following conditions hold:

(1) For every $p \in M$, there are two open subsets $\Omega \subseteq \mathbb{E}^m$ and $U \subseteq M$, with $p \in U$, and a smooth function

$$\varphi : \Omega \to \mathbb{E}^N$$

such that φ is a homeomorphism between Ω and $U = \varphi(\Omega)$, and $(d\varphi)_0$ is injective, where

$$p = \varphi(0)\,.$$

(2) For every $p \in M$, there are two open sets $O, W \subseteq \mathbb{E}^N$, with $0_N \in O$ and $p \in (M \cap W)$, and a smooth diffeomorphism $\varphi : O \to W$, such that $\varphi(0_N) = p$ and

$$\varphi(O \cap (\mathbb{E}^m \times \{0_{N-m}\})) = M \cap W.$$

Manifolds

Submanifolds embedded in $\mathbb{R}^N$

(3) For every $p \in M$, there is some open subset, $W \subseteq \mathbb{E}^N$, with $p \in W$ and a smooth submersion, $f : W \to \mathbb{E}^{N-m}$, so that $W \cap M = f^{-1}(0)$.

(4) For every $p \in M$, there is some open subset, $W \subseteq \mathbb{E}^N$, with $p \in W$, and $N - m$ smooth functions, $f_i : W \to \mathbb{E}$, so that the linear forms $(df_1)_p, \ldots, (df_{N-m})_p$ are linearly independent and

$$W \cap M = f_1^{-1}(0) \cap \cdots \cap f_{N-m}^{-1}(0).$$

Condition (4) says that locally (that is, in a small open set of M containing $p \in M$), M is "cut out" by $N - m$ smooth functions, $f_i : W \to \mathbb{E}$, in the sense that the portion of the manifold $M \cap W$ is the intersection of the $N - m$ hypersurfaces, $f_i^{-1}(0)$, (the zero-level sets of the f_i) and that this intersection is "clean", which means that the linear forms $(df_1)_p, \ldots, (df_{N-m})_p$ are linearly independent.

Manifolds

Submanifolds embedded in $\mathbb{R}^N$

Example 4.4.

The sphere

$$S^n = \{x \in \mathbb{E}^{n+1} \mid \|x\|_2^2 - 1 = 0\}$$

is an n-dimensional manifold in $\mathbb{E}^{n+1}$. Indeed, the map $f : \mathbb{E}^{n+1} - \{0\} \to \mathbb{E}$ given by

$$f(x) = \|x\|_2^2 - 1$$

is a submersion, since

$$(df)_x(y) = \begin{pmatrix} 2x_1 & 2x_2 & \cdots & 2x_{n+1} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{pmatrix}, \quad \text{for all } x, y \in (\mathbb{E}^{n+1} - \{0\}).$$

Manifolds

Submanifolds embedded in $\mathbf{R}^{\mathrm{N}}$

Using condition (3) of Theorem 4.7, namely,

(3) *For every $p \in M$, there is some open subset, $W \subseteq \mathbb{E}^N$, with $p \in W$ and a smooth submersion, $f : W \to \mathbb{E}^{N-m}$, so that $W \cap M = f^{-1}(0)$,*

with

$$M = S^n \quad \text{and} \quad W = \mathbb{E}^{n+1} - \{0\}\,,$$

we get $W \cap M = S^n = f^{-1}(0)$. So, by Theorem 4.7., S^n is indeed an n-manifold in $\mathbb{E}^{n+1}$.

Manifolds

Submanifolds embedded in R^N

Example 4.5.

Recall that the orthogonal group, $\boldsymbol{O}(n)$, is the group of all real $n \times n$ matrices, R, such that

$$RR^{\mathrm{T}} = R^{\mathrm{T}}R = I$$

and

$$\det(R) = \pm 1\,,$$

and that the rotation group, $\boldsymbol{SO}(n)$ is the subgroup of $\boldsymbol{O}(n)$ consisting of all matrices in $\boldsymbol{O}(n)$ such that

$$\det(R) = +1\,.$$

The group $\boldsymbol{SO}(n)$ is an $\left(\dfrac{n \cdot (n-1)}{2}\right)$-dimensional manifold in $\mathbb{R}^{n^2}$.

Manifolds

Submanifolds embedded in R^N

To see why, recall that

$$\boldsymbol{GL}^+(n) = \{A \in \boldsymbol{GL}(n) \mid \det(A) > 0\}$$

is an open set of $\mathbb{R}^{n^2}$. Now, note that $A^T A - I$ is a symmetric matrix, for all $A \in \boldsymbol{GL}^+(n)$.

So, let

$$f : \boldsymbol{GL}^+(n) \to \boldsymbol{S}(n)$$

be given by

$$f(A) = A^T A - I\,,$$

where $\boldsymbol{S}(n) \approx \mathbb{R}^{\frac{n(n+1)}{2}}$ is the vector space consisting of all $n \times n$ real symmetric matrices.

Manifolds

Submanifolds embedded in $\mathbf{R}^{\mathbf{N}}$

It is easy to show (using directional derivatives) that

$$(df)_A(H) = A^{\mathrm{T}}H + H^{\mathrm{T}}A \in \boldsymbol{S}(n)\,.$$

But then, $(df)_A$ is surjective for all $A \in \boldsymbol{SO}(n)$, because if S is any symmetric matrix, we see that

$$df(A)\left(\frac{AS}{2}\right) = A^{\mathrm{T}}\left(\frac{AS}{2}\right) + \left(\frac{AS}{2}\right)^{\mathrm{T}} A = \frac{1}{2}(A^{\mathrm{T}}AS + S^{\mathrm{T}}A^{\mathrm{T}}A) = \frac{1}{2}(S+S) = S\,.$$

As $\mathbf{SO}(n) = f^{-1}(0)$, we can use condition (3) of Theorem 4.7 again, with $W = \boldsymbol{GL}^{+}(n)$ and $M = \boldsymbol{SO}(n)$, to conclude that $\mathbf{SO}(n)$ is indeed a $\frac{n(n-1)}{2}$-manifold in $\mathbb{R}^{n^2}$.