IMPA - Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ, Brazil

Introduction to Computational Manifolds and Applications

Part 1 - Foundations

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Origins of Manifolds

Around 1860, Mobius, Jordan, and Dyck studied the topology of surfaces.

In a famous paper published in 1888, Dyck already uses the term manifold (in German).

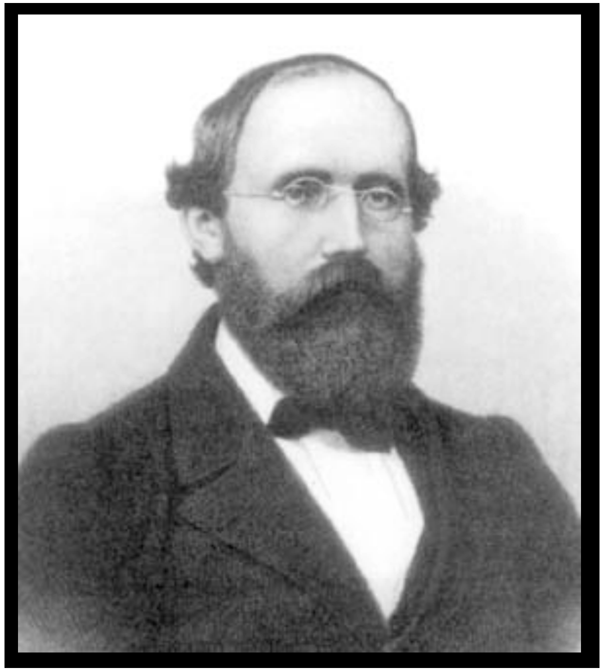
In the early 1900's, Dehn, Heegaard, Veblen, and Alexander routinely used the term manifold.

Hermann Weyl was among the first to give a rigorous definition (1913).

Manifolds

Origins of Manifolds

Georg Friedrich Bernhard Riemann
1828-1866



Manifolds

Origins of Manifolds

Hermann Klaus Hugo Weyl

1885-1955

Manifolds

Origins of Manifolds

Hermann Weyl (again)

Manifolds

Origins of Manifolds

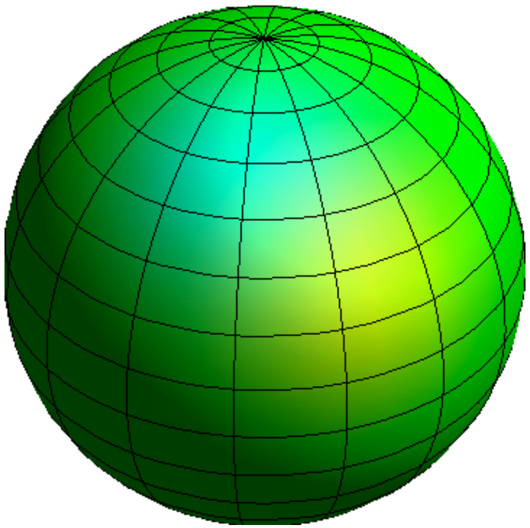
Hassler Whitney

1907-1989

Manifolds

Origins of Manifolds

We've seen that manifolds are generalizations of surfaces to arbitrary dimensions.



This is the image to have. However we should not think of a manifold as *always* sitting inside a fixed Euclidean space, but rather as a space of its own. Let us see how...

Manifolds

Formal Definition

Given a set E, a *topology on E (or a topological structure on E)*, is defined as a family $\mathcal{O}$ of subsets of E called *open sets*, and satisfying the following three properties:

(1) For every finite family $(U_i)_{1 \leq i \leq n}$ of sets $U_i \in \mathcal{O}$, we have $U_1 \cap \cdots \cap U_n \in \mathcal{O}$, i.e., $\mathcal{O}$ is closed under finite intersections.

(2) For every arbitrary family $(U)_{i \in I}$ of sets $U_i \in \mathcal{O}$, we have $\bigcup_{i \in I} U_i \in \mathcal{O}$, i.e., $\mathcal{O}$ is closed under arbitrary unions.

(3) $\emptyset \in \mathcal{O}$, and $E \in \mathcal{O}$, i.e., $\emptyset$ and E belong to $\mathcal{O}$.

A set E together with a topology $\mathcal{O}$ on E is called a *topological space*. Given a topological space $(E, \mathcal{O})$, a subset F of E is a *closed set* if $F = E - U$ for some open set $U \in \mathcal{O}$, i.e., F is the complement of some open set.

Manifolds

Formal Definition

Given $\mathbb{R}^n$, recall that the projection functions, $pr_i : \mathbb{R}^n \to \mathbb{R}$, are defined by

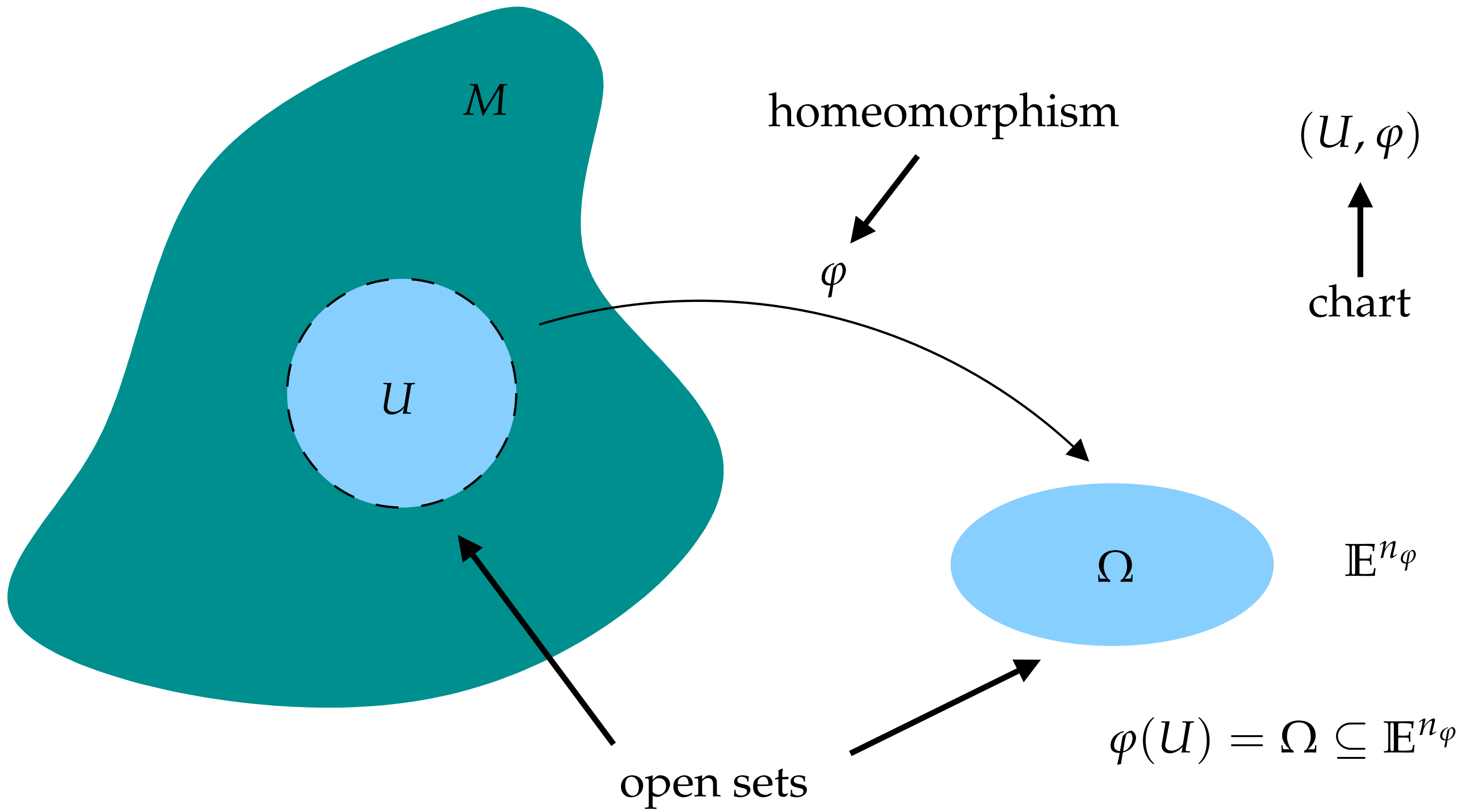
$$pr_i(x_1, \ldots, x_n) = x_i\,, \quad 1 \leq i \leq n\,.$$

Definition 5.1. Given a topological space, M, a *chart* (or *local coordinate map*) is a pair, (U, φ), where U is an open subset of M and $\varphi : U \to \Omega$ is a homeomorphism onto an open subset, $\Omega = \varphi(U)$, of $\mathbb{E}^{n_\varphi}$ (for some $n_\varphi \geq 1$). For any $p \in M$, a chart, (U, φ), is a *chart at p* iff $p \in U$. If (U, φ) is a chart, then the functions $x_i = pr_i \circ \varphi$ are called *local coordinates* and for every $p \in U$, the tuple $(x_1(p), \ldots, x_n(p))$ is the set of *coordinates of p* w.r.t. the chart.

The inverse, (Ω, φ^{-1}), of a chart is called a *local parametrization*.

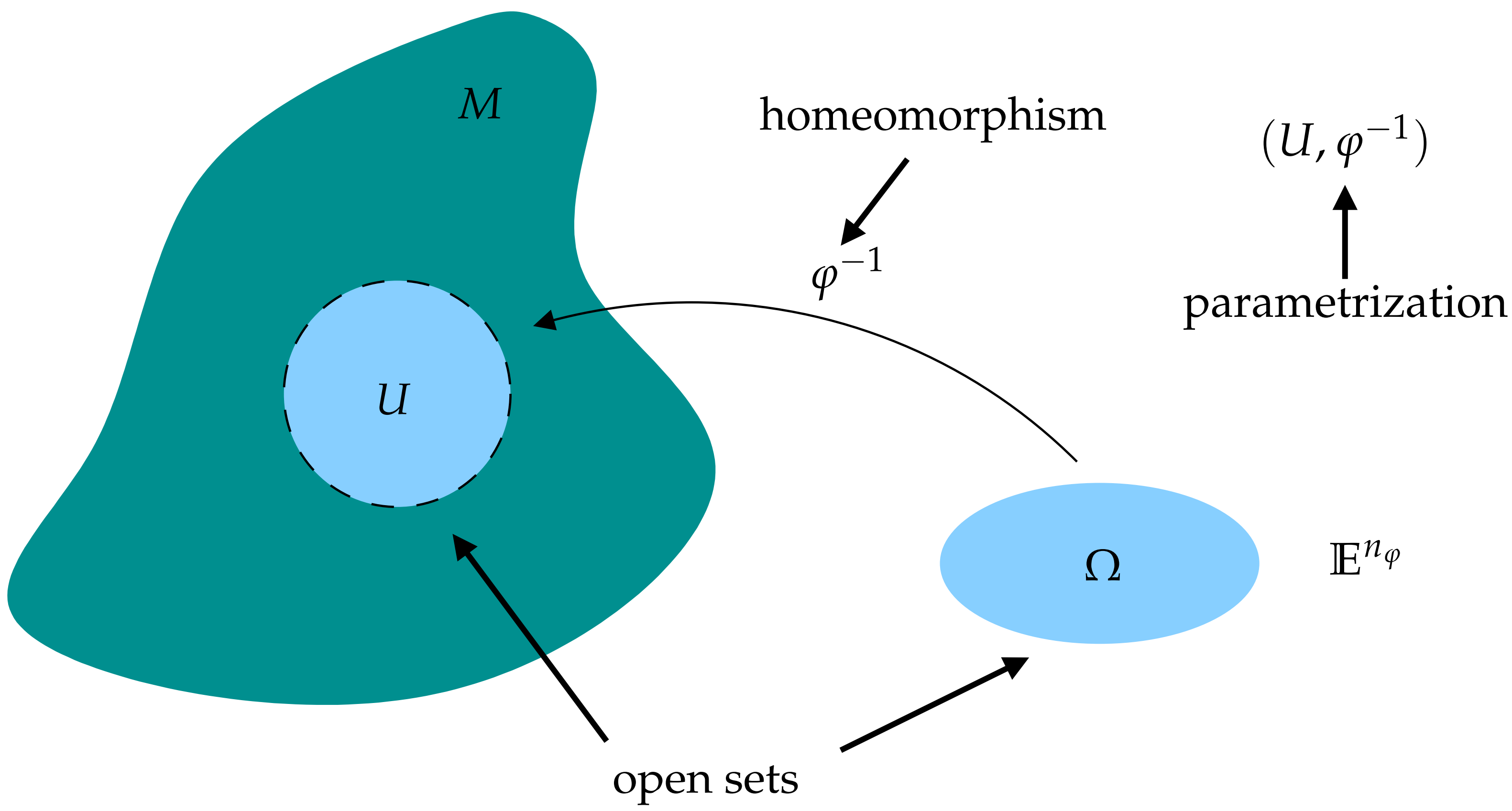
Manifolds

Formal Definition



Manifolds

Formal Definition



Manifolds

Formal Definition

Definition 5.2. Given any two charts, (U_i, φ_i) and (U_j, φ_j), if $U_i \cap U_j \neq \emptyset$, we define the *transition maps*,

$$\varphi_{ji} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j) \quad \text{and} \quad \varphi_{ij} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)\,,$$

as

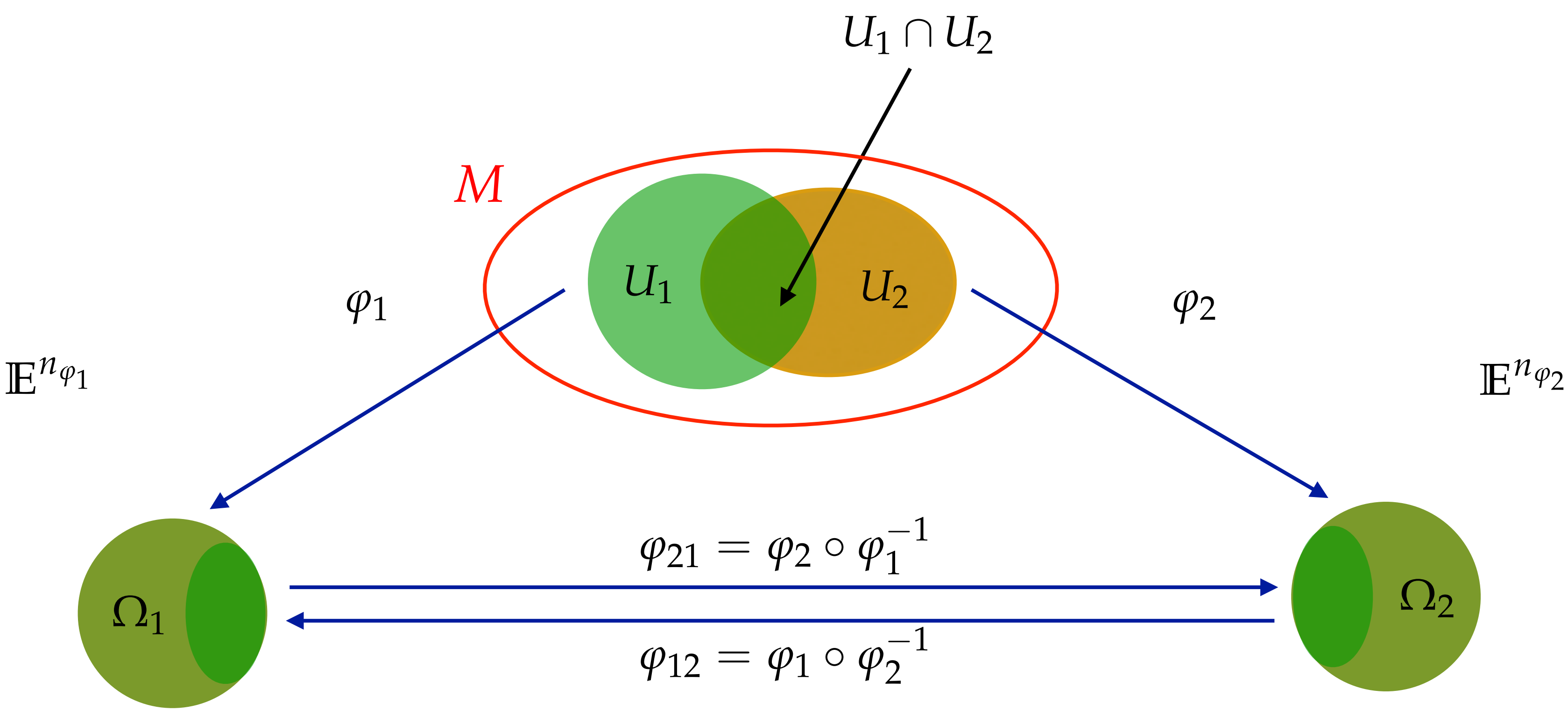
$$\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} \quad \text{and} \quad \varphi_{ij} = \varphi_i \circ \varphi_j^{-1}\,.$$

Clearly, $\varphi_{ji} = (\varphi_{ij})^{-1}$.

Observe that the transition maps φ_{ji} (resp. φ_{ij}) are maps between open subsets of $\mathbb{E}^n$.

Manifolds

Formal Definition



Manifolds

Formal Definition

A topological space $(E, \mathcal{O})$ is said to satisfy the *Hausdorff separation axiom (or T_2-separation axiom)* if for any two distinct points $a \neq b$ in E, there exist two open sets U_a and U_b such that, $a \in U_a$, $b \in U_b$, and $U_a \cap U_b = \emptyset$. When the T_2-separation axiom is satisfied, we also say that $(E, \mathcal{O})$ is a *Hausdorff space*.

A topological space E is called *second-countable* if there is a countable basis for its topology, i.e., if there is a countable family, $(U_i)_{i \geq 0}$, of open sets such that every open set of E is a union of open sets U_i.

Manifolds

Formal Definition

Definition 5.3. Given a topological space, M, given some integer $n \geq 1$ and given some k such that k is either an integer, with $k \geq 1$, or $k = \infty$, a C^k *n-atlas* (or *n-atlas of class* C^k),

$$\mathcal{A} = \{(U_i, \varphi_i)\}_i \,,$$

is a family of charts such that

(1) $\varphi_i(U_i) \subseteq \mathbb{E}^n$ for all i ;

(2) The U_i cover M, i.e.,

$$M = \bigcup_i U_i \,;$$

(3) Whenever

$$U_i \cap U_j \neq \emptyset \,,$$

the transition map φ_{ji} (and φ_{ij}) is a C^k-diffeomorphism.

Manifolds

Formal Definition

Given a C^k n-atlas, $\mathcal{A}$, on M, for any other chart, (U, φ), we say that (U, φ) is *compatible* with the atlas $\mathcal{A}$ iff every map $\varphi_i \circ \varphi^{-1}$ and $\varphi \circ \varphi_i^{-1}$ is C^k (whenever $U \cap U_i \neq \emptyset$).

Two atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are *compatible* iff every chart of one is compatible with the other atlas. This is equivalent to saying that the union of the two atlases is still an atlas.

It is immediately verified that compatibility induces an equivalence relation on C^k n-atlases on M. In fact, given an atlas, $\mathcal{A}$, for M, the collection, $\widetilde{\mathcal{A}}$, of all charts compatible with $\mathcal{A}$ is a maximal atlas in the equivalence class of charts compatible with $\mathcal{A}$.

Finally, we have our generalized notion of a manifold.

Manifolds

Formal Definition

Definition 5.4. Given some integer $n \geq 1$ and given some k such that k is either an integer, with $k \geq 1$, or $k = \infty$, a *C^k-manifold of dimension n* consists of a topological space, M, together with an equivalence class, $\overline{\mathcal{A}}$, of C^k n-atlases, on M. Any atlas, $\mathcal{A}$, in the equivalence class $\overline{\mathcal{A}}$ is called a *differentiable structure of class C^k (and dimension n) on M*. We say that M is *modeled on $\mathbb{E}^n$*. When $k = \infty$, we say that M is a *smooth manifold*.

It might have been better to use the terminology *abstract manifold* rather than manifold, to emphasize the fact that the space M is not a priori a subspace of $\mathbb{E}^N$, for some suitable N.

To avoid pathological cases and to ensure the existence of partitions of unity, we further require that the topology of M be Hausdorff and second-countable.

Manifolds

Formal Definition

We can allow $k = 0$ in the above definitions. In this case, condition (3) in Definition 5.3 is void, since a C^0-diffeomorphism is just a homeomorphism, but φ_{ji} is always a homeomorphism.

In this case, M is called a *topological manifold of dimension n*.

We do not require a manifold to be connected but we require all the components to have the same dimension, n. Actually, on every connected component of M, it can be shown that the dimension, n_φ, of the range of every chart is the same (i.e., $n_\varphi = n$). This is quite easy to show if $k \geq 1$ but for $k = 0$, this requires a deep theorem due to Brouwer.

Manifolds

Formal Definition

Brouwer's *Invariance of Domain Theorem* states the following: if $U \subseteq \mathbb{E}^n$ is an open set and if $f : U \to \mathbb{E}^n$ is a continuous and injective map, then the set $f(U)$ is open in $\mathbb{E}^n$.

Using Brouwer's Theorem, we can show the following fact: If $U \subseteq \mathbb{E}^m$ and $V \subseteq \mathbb{E}^n$ are two open subsets and if $f : U \to V$ is a homeomorphism between U and V, then $m = n$.

If $m > n$, then consider the injection, $i : \mathbb{E}^n \to \mathbb{E}^m$, where $i(x) = (x, 0_{m-n})$. Clearly, i is injective and continuous, so $i \circ f : U \to i(V)$ is injective and continuous and Brouwer's Theorem implies that $i(V)$ is open in $\mathbb{E}^m$, which is a contradiction, as $i(V) = V \times \{0_{m-n}\}$ is not open in $\mathbb{E}^m$. If $m < n$, consider the homeomorphism $f^{-1} : V \to U$.

Manifolds

Formal Definition

What happens if $n = 0$?

In this case, every one-point subset of M is open, so every subset of M is open, i.e., M is any (countable if we assume M to be second-countable) set with the discrete topology!

Observe that since $\mathbb{E}^n$ is locally compact and locally connected, so is every manifold.

Manifolds

Formal Definition

To get a better grasp of the notion of manifold it is useful to consider examples of non-manifolds. First, consider the curve in $\mathbb{E}^2$ given by the zero locus of the equation

$$y^2 = x^2 - x^3,$$

namely, the set of points

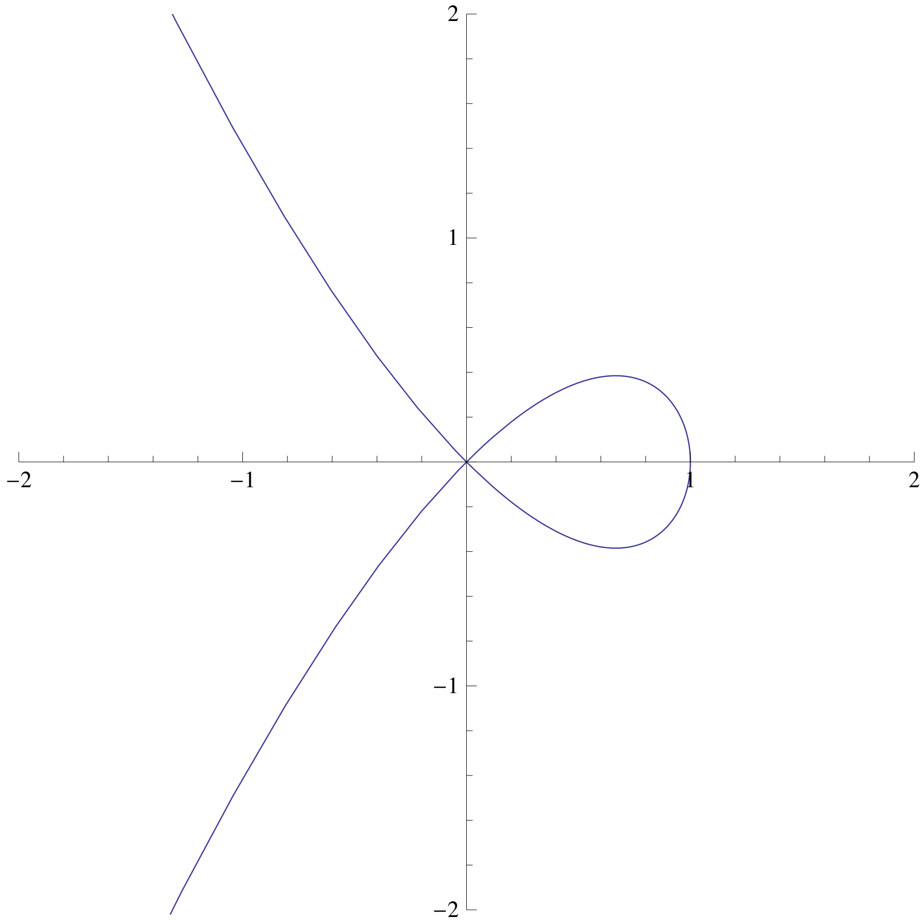
$$M_1 = \{(x, y) \in \mathbb{E}^2 \mid y^2 = x^2 - x^3\}.$$

This curve is called a *nodal cubic* and we saw its parametric equation in our first lecture:

$$\begin{aligned} x &= 1 - t^2 \\ y &= t(1 - t^2). \end{aligned}$$

Manifolds

Formal Definition



Manifolds

Formal Definition

We claim that M_1 is not even a topological manifold.

The problem is that the nodal cubic has a self-intersection at the origin.

If M_1 was a topological manifold, then there would be a connected open subset, $U \subseteq M_1$, containing the origin, $O = (0, 0)$, namely the intersection of a small enough open disc centered at O with M_1, and a local chart, $\varphi : U \to \Omega$, where Ω is some connected open subset of $\mathbb{E}$ (that is, an open interval), since φ is a homeomorphism.

However, $U - \{O\}$ consists of four disconnected components and $\Omega - \varphi(O)$ consists of two disconnected components, contradicting the fact that φ is a homeomorphism.

Manifolds

Formal Definition

Let us now consider the curve in $\mathbb{E}^2$ given by the zero locus of the equation

$$y^2 = x^3,$$

namely, the set of points

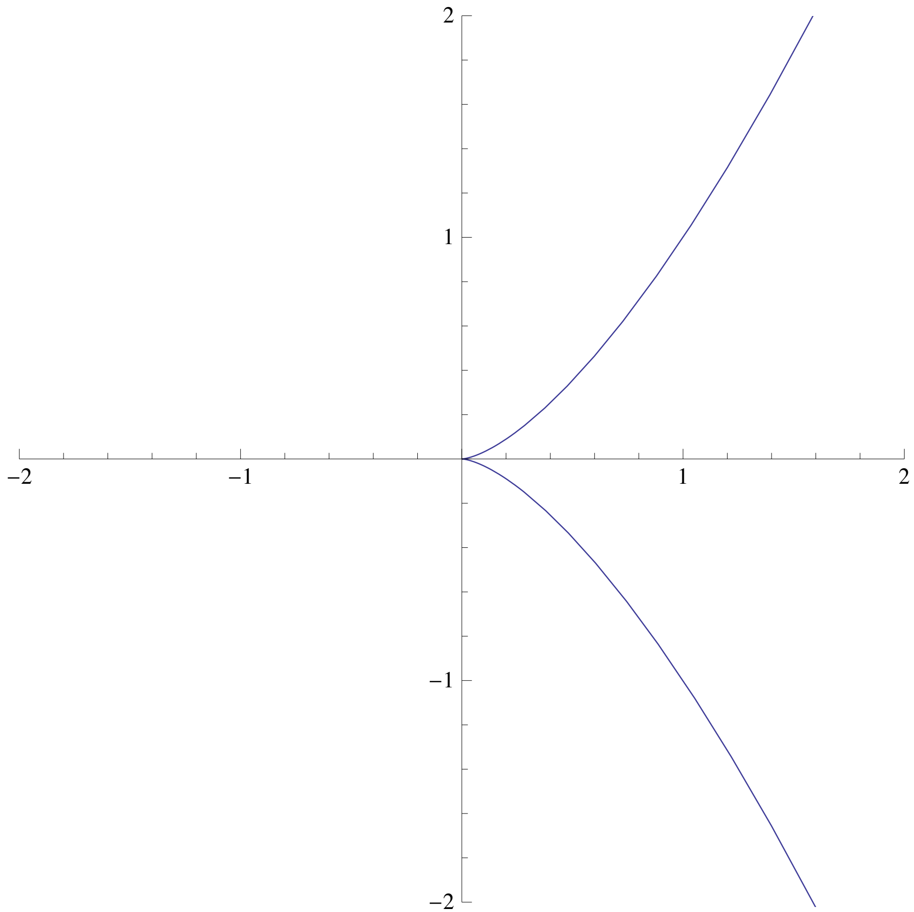
$$M_2 = \left\{(x, y) \in \mathbb{E}^2 \mid y^2 = x^3\right\}.$$

This curve is called a *cuspidal cubic* and we also saw its parametric form in the first lecture:

$$\begin{aligned} x &= t^2 \\ y &= t^3. \end{aligned}$$

Manifolds

Formal Definition



Manifolds

Formal Definition

Consider the map, $\varphi : M_2 \to \mathbb{E}$, given by

$$\varphi(x, y) = y^{1/3}.$$

Since $x = y^{2/3}$ on M_2, we see that φ^{-1} is given by

$$\varphi^{-1}(t) = (t^2, t^3)$$

and clearly, φ is a homeomorphism, so M_2 is a topological manifold. However, in the atlas consisting of the single chart, $\varphi : M_2 \to \mathbb{E}$, the space M_2 is also a smooth manifold! Indeed, as there is a single chart, condition (3) of Definition 5.3 holds vacuously.

Manifolds

Formal Definition

This fact is somewhat unexpected because the cuspidal cubic is usually not considered smooth at the origin, since the tangent vector of the parametric curve,

$$c : t \mapsto (t^2, t^3) \,,$$

at the origin is the zero vector (the velocity vector at t, is $\dfrac{dc}{dt}(t) = (2t, 3t^2)$).

However, this apparent paradox has to do with the fact that, as a parametric curve, M_2 is not immersed in $\mathbb{E}^2$ since c' is not injective (see "immersion" in slide 39 of Lecture 2), whereas as an abstract manifold, with this single chart, M_2 is diffeomorphic to $\mathbb{E}$.

Manifolds

Formal Definition

Now, we also have the chart, $\psi : M_2 \to \mathbb{E}$, given by

$$\psi(x,y) = y\,,$$

with ψ^{-1} given by

$$\psi^{-1}(u) = (u^{2/3}, u)\,.$$

Then, observe that

$$\varphi \circ \psi^{-1}(u) = u^{1/3}\,,$$

a map that is **not** differentiable at $u = 0$. Therefore, the atlas consisting of the charts

$$(M_2, \varphi : M_2 \to \mathbb{E}) \quad \text{and} \quad (M_2, \psi : M_2 \to \mathbb{E})$$

is not C^1 and thus, with respect to that atlas, M_2 is not a C^1-manifold.

Manifolds

Formal Definition

The example of the cuspidal cubic shows a peculiarity of the definition of a C^k (or C^∞) manifold:

If a space, M, happens to be a topological manifold because it has an atlas consisting of a single chart, then it is automatically a smooth manifold!

Manifolds

Formal Definition

In particular, if

$$f : U \to \mathbb{E}^m$$

is any *continuous* function from some open subset, U, of $\mathbb{E}^n$, to $\mathbb{E}^m$, then the graph,

$$\Gamma(f) = \{(x, f(x)) \in \mathbb{E}^{n+m} \mid x \in U\},$$

of f is a smooth manifold with respect to the atlas consisting of the single chart, namely,

$$\{(\Gamma(f), \varphi : \Gamma(f) \to U)\},$$

where

$$\varphi(x, f(x)) = x \qquad \text{and} \qquad \varphi^{-1}(x) = (x, f(x)).$$

Manifolds

Formal Definition

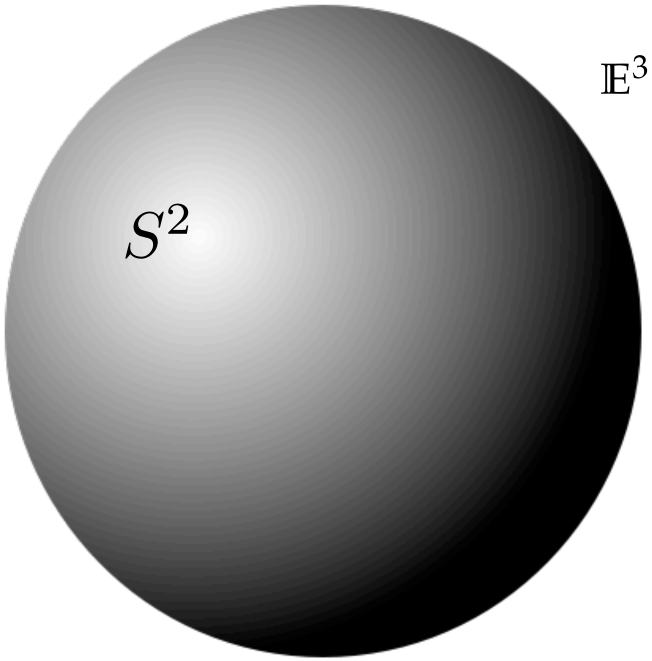
The example of the cuspidal cubic also shows clearly that whether a topological space is considered a C^k-manifold or a smooth manifold depends on the choice of atlas.

Manifolds

Examples

Example 5.1.

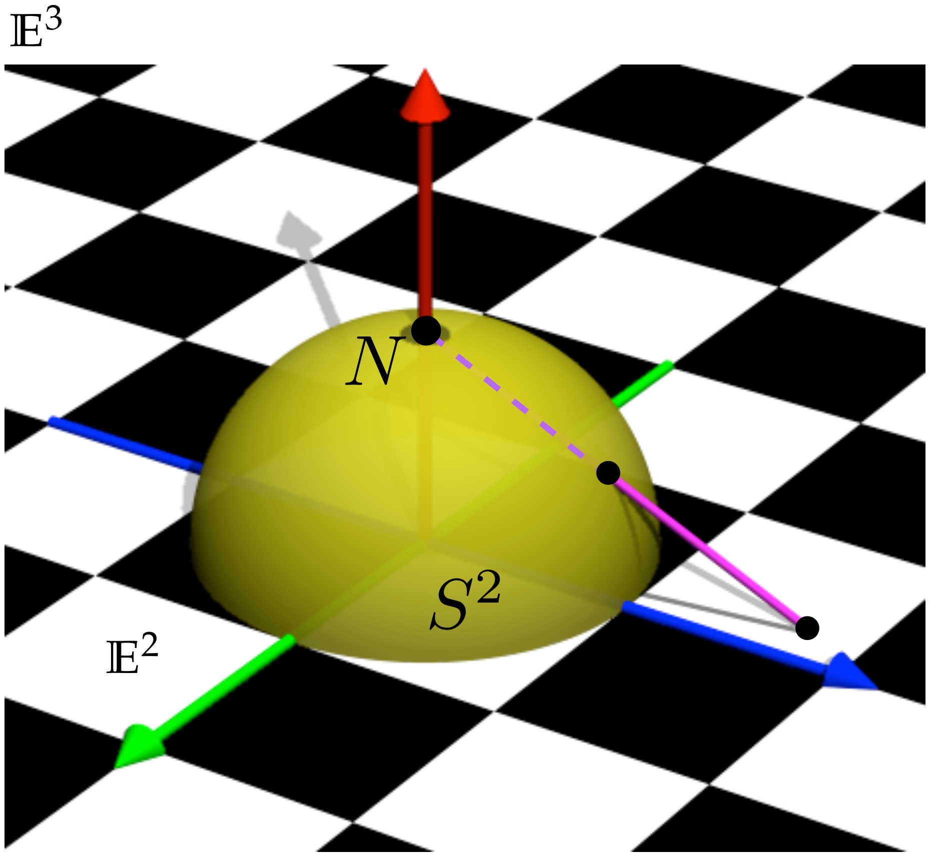
$$S^n = \{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{E}^{n+1} \mid x_1^2 + \cdots + x_n^2 + x_{n+1}^2 = 1\}$$



Manifolds

Examples

We use stereographic projection from the north pole...



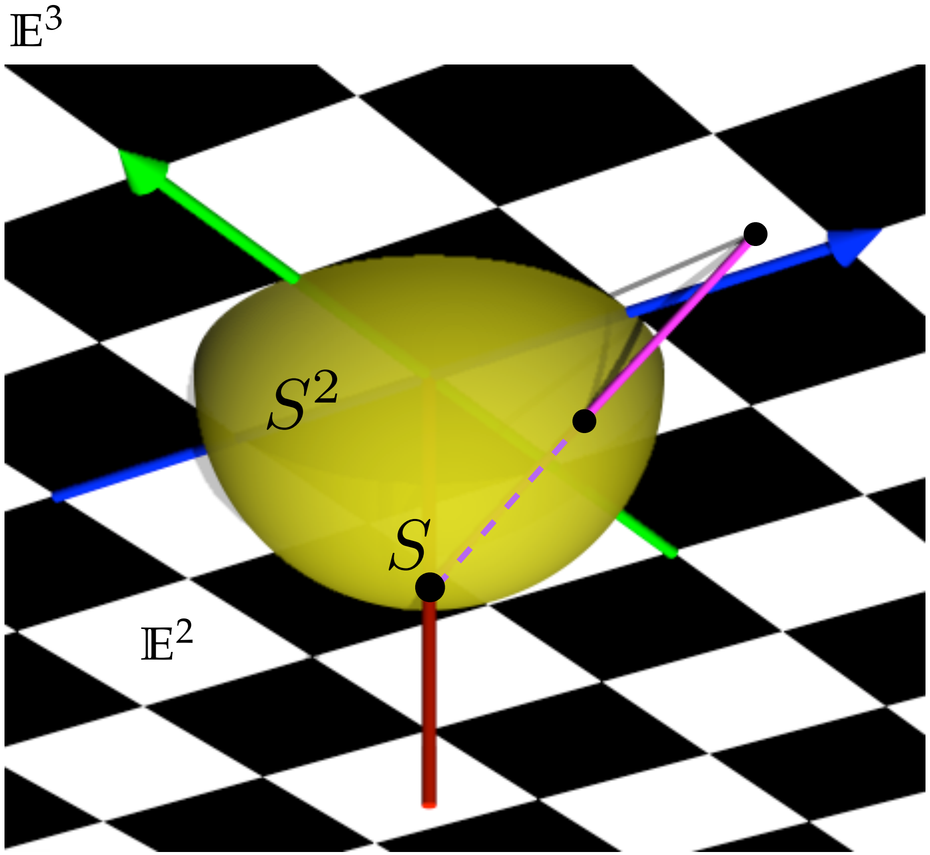
$$\sigma_N : S^n - \{N\} \to \mathbb{E}^n$$

$$\sigma_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n)$$

Manifolds

Examples

and stereographic projection from the south pole.



$$\sigma_S : S^n - \{S\} \to \mathbb{E}^n$$

$$\sigma_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n)$$

Manifolds

Examples

The inverse stereographic projections are given by

$$\sigma_N^{-1}(x_1, \ldots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1} \left(2x_1, \ldots, 2x_n, \left(\sum_{i=1}^n x_i^2\right) - 1\right)$$

and

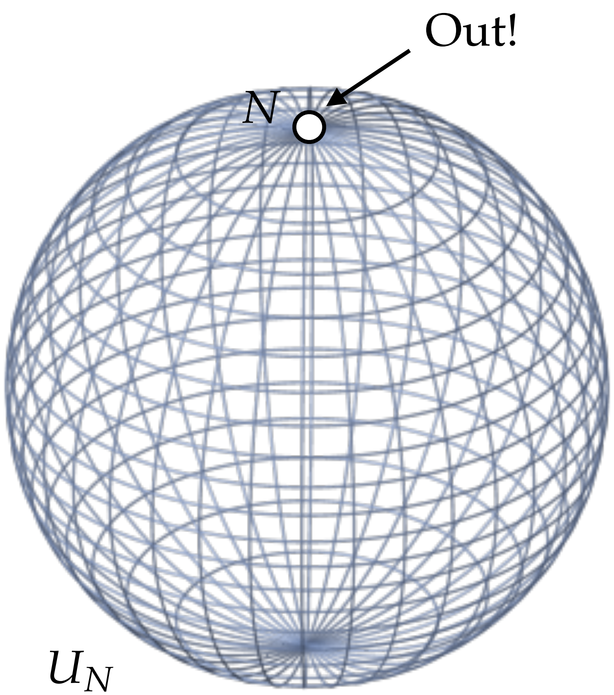
$$\sigma_S^{-1}(x_1, \ldots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1} \left(2x_1, \ldots, 2x_n, -\left(\sum_{i=1}^n x_i^2\right) + 1\right).$$

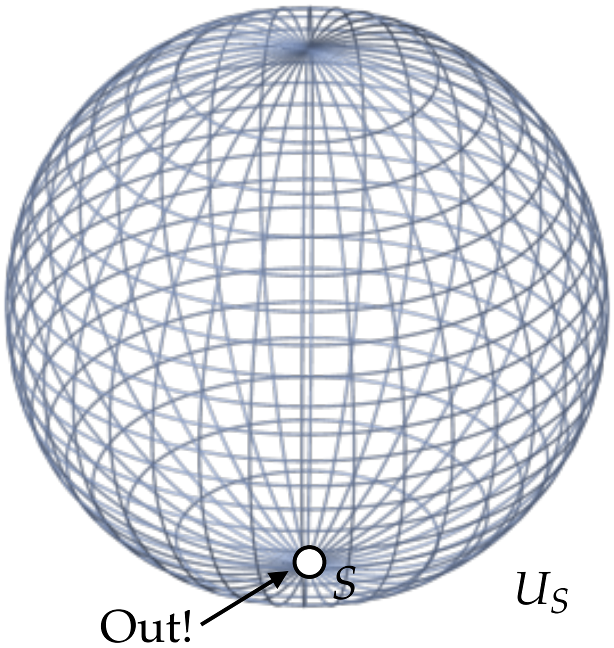
Manifolds

Examples

Consider the open cover consisting of

$$U_N = S^n - \{N\} \qquad \text{and} \qquad U_S = S^n - \{S\}\,.$$



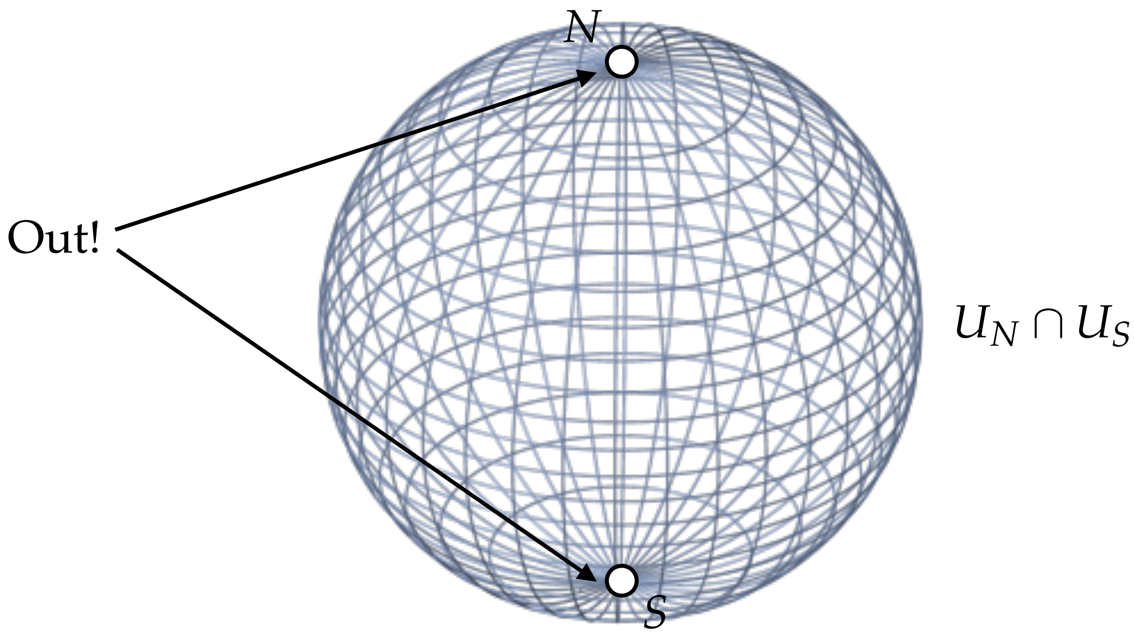


Manifolds

Examples

On the overlap

$$U_N \cap U_S = S^n - \{N, S\}\,.$$



Manifolds

Examples

It is easily checked that on the overlap, $U_N \cap U_S = S^n - \{N, S\}$, the transition maps

$$\sigma_S \circ \sigma_N^{-1} = \sigma_N \circ \sigma_S^{-1}$$

are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2}(x_1, \ldots, x_n),$$

that is, the inversion of center $O = (0, \ldots, 0)$ and power 1. Clearly, this map is smooth on $\mathbb{E}^n - \{O\}$, so we conclude that (U_N, σ_N) and (U_S, σ_S) form a smooth atlas for S^n.