IMPA - Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ, Brazil

Introduction to Computational Manifolds and Applications

Part 1 - Constructions

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Parametric Pseudo-Manifolds

Simplicial Surfaces

We will start investigating the construction of 2-dimensional PPM's in $\mathbb{E}^3$.

In the previous lecture, we considered a polygon as a sketch of the shape of the curve we wanted to build. Now, we need another object to play the same role the polygon did.

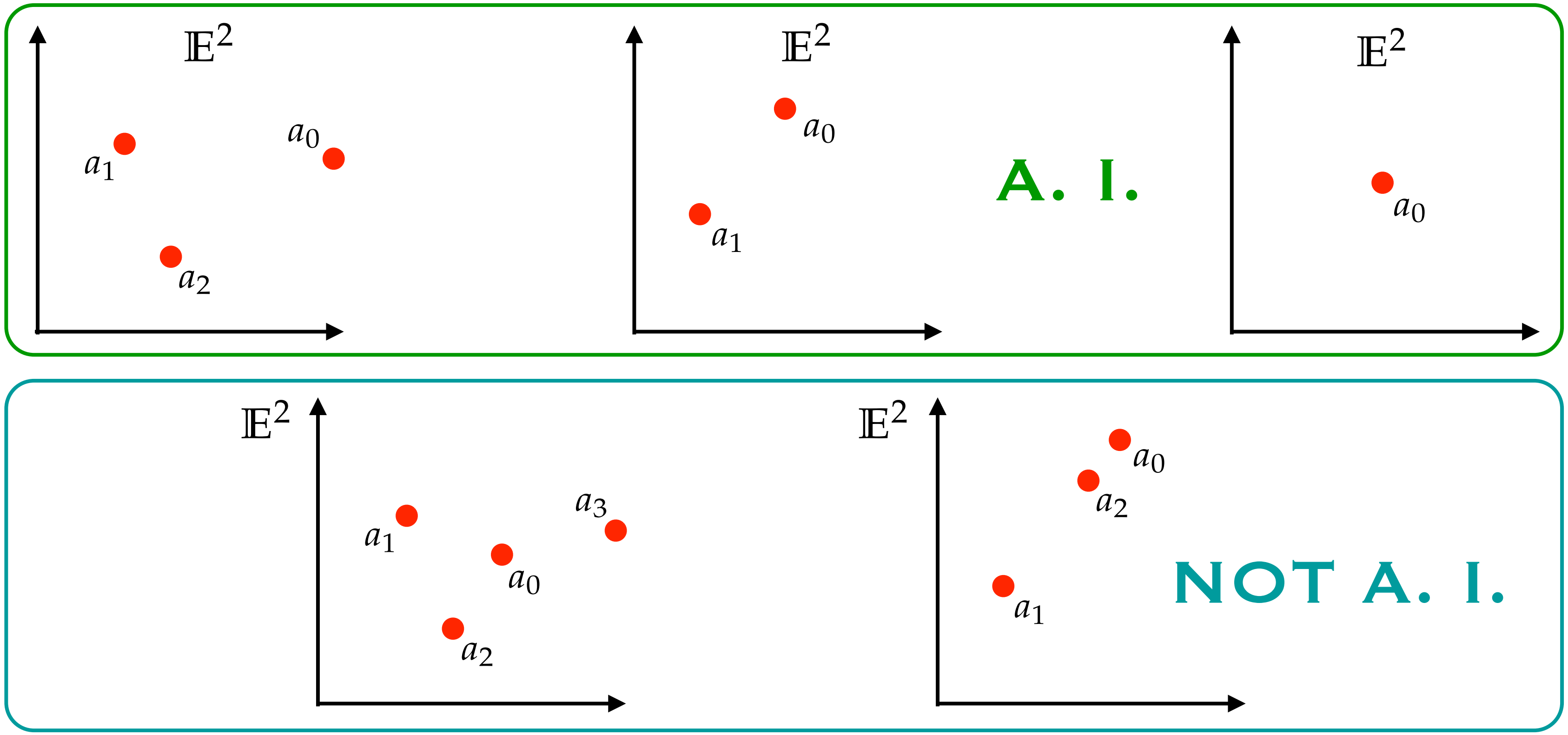
We can think of a few choices, but the easiest one is arguably a polygonal mesh.

So, let us start with a triangle mesh, which is a formally known as a simplicial surface.

Parametric Pseudo-Manifolds

Simplicial Surfaces

Definition 9.1. Given a finite family, $(a_i)_{i \in I}$, of points in $\mathbb{E}^n$, we say that $(a_i)_{i \in I}$ is *affinely independent* if the family of vectors, $(\boldsymbol{a_i a_j})_{j \in (I - \{i\})}$, is linearly independent for some $i \in I$.



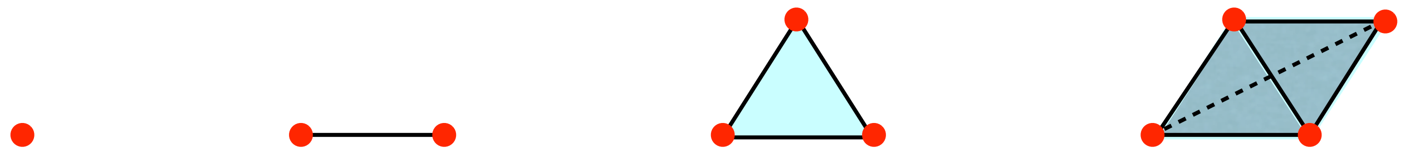
Parametric Pseudo-Manifolds

Simplicial Surfaces

Definition 9.2. Let $a_0, \ldots, a_d$ be any $d+1$ affinely independent points in $\mathbb{E}^n$, where d is a non-negative integer. The *simplex* σ spanned by the points $a_0, \ldots, a_d$ is the convex hull of these points, and is denoted by $[a_0, \ldots, a_d]$. The points $a_0, \ldots, a_d$ are the *vertices* of σ. The *dimension*, $\dim(\sigma)$, of the simplex σ is d, and σ is also called a *d-simplex*.

In $\mathbb{E}^n$, the largest number of affinely independent points is $n+1$.

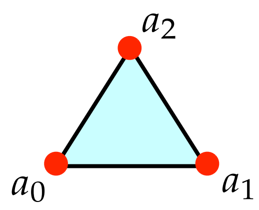
So, in $\mathbb{E}^n$, we have simplices of dimension $0, 1, \ldots, n$. A 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. Furthermore, the convex hull of any nonempty subset of vertices of a simplex is a simplex.

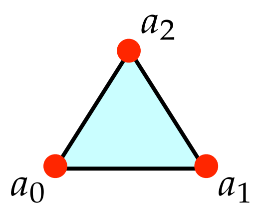


Parametric Pseudo-Manifolds

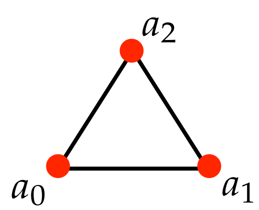
Simplicial Surfaces

Definition 9.3. Let $\sigma = [a_0, \ldots, a_d]$ be a d-simplex in $\mathbb{E}^n$. A *face* of σ is a simplex spanned by a nonempty subset of $\{a_0, \ldots, a_d\}$; if this subset is proper then the face is called a *proper face*. A face of σ whose dimension is k, i.e., a k-simplex, is called a *k-face*.

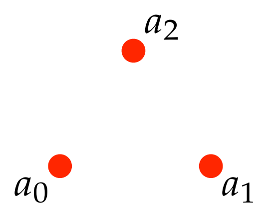




a 2-face: $[a_0, a_1, a_2]$



3 proper 1-faces: $[a_0, a_1]$, $[a_0, a_2]$, $[a_1, a_2]$



3 proper 0-faces: $[a_0]$, $[a_1]$, $[a_2]$

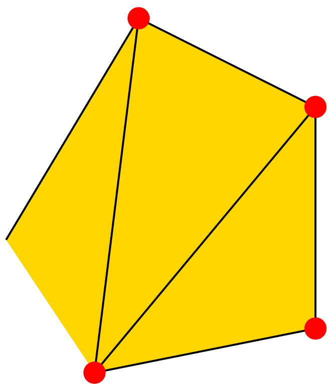
Parametric Pseudo-Manifolds

Simplicial Surfaces

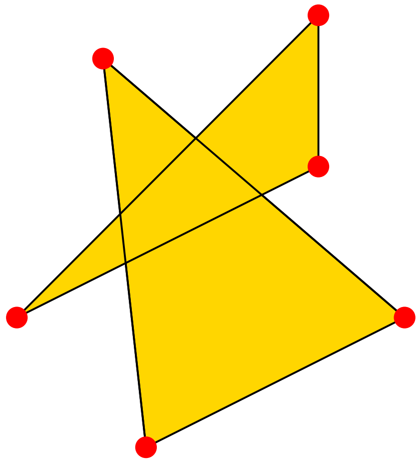
Definition 9.4. A *simplicial complex* $\mathcal{K}$ in $\mathbb{E}^n$ is a finite collection of simplices in $\mathbb{E}^n$ such that

(1) if a simplex is in $\mathcal{K}$, then all its faces are in $\mathcal{K}$;

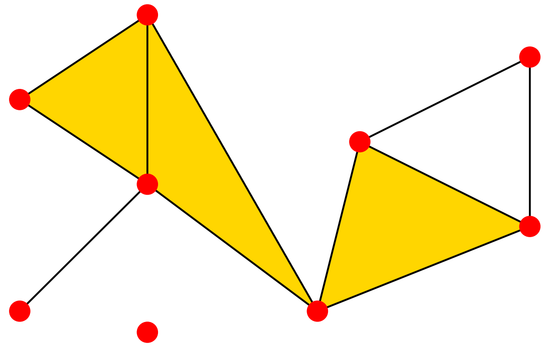
(2) if $\sigma, \tau \in \mathcal{K}$ are simplices such that $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau$ is a face of both σ and τ.



violates (1)



violates (2)

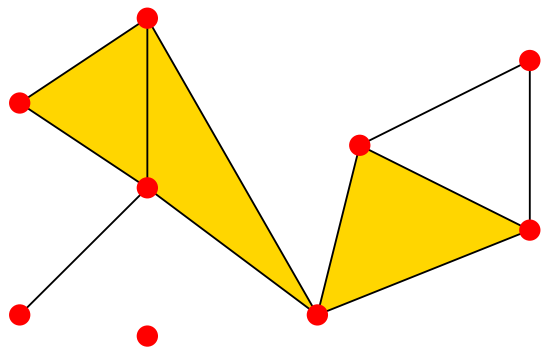


a simplicial complex

Parametric Pseudo-Manifolds

Simplicial Surfaces

Definition 9.5. The *dimension*, $\dim(\mathcal{K})$, of a simplicial complex, $\mathcal{K}$, is the largest dimension of a simplex in $\mathcal{K}$, i.e., $\dim(\mathcal{K}) = \max\{\dim(\sigma) \mid \sigma \in \mathcal{K}\}$. We refer to a d-dimensional simplicial complex as simply a *d-complex*. The set consisting of the union of all points in the simplices of $\mathcal{K}$ is called the *underlying space of $\mathcal{K}$*, and it is denoted by $|\mathcal{K}|$. The underlying space, $|\mathcal{K}|$, of $\mathcal{K}$ is also called the *geometric realization* of $\mathcal{K}$.



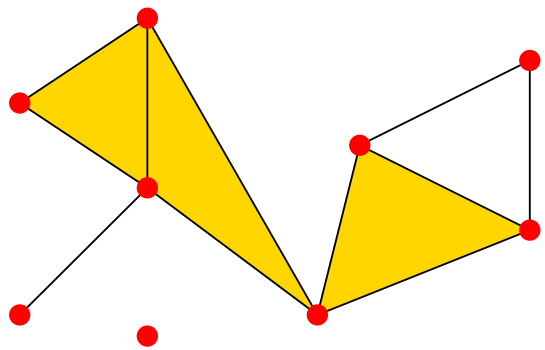
a 2-complex

Parametric Pseudo-Manifolds

Simplicial Surfaces

A simplicial complex is a combinatorial object (i.e., a finite collection of simplices).

The underlying space of a simplicial complex is a topological object, a subset of some $\mathbb{E}^n$.



a 2-complex

Parametric Pseudo-Manifolds

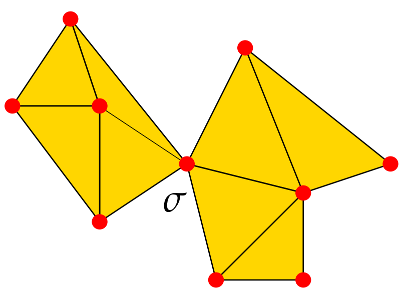
Simplicial Surfaces

Definition 9.6. Let $\mathcal{K}$ be a simplicial complex in $\mathbb{E}^n$. Then, for any simplex σ in $\mathcal{K}$, we define two other complexes, the *star*, $\mathrm{st}(\sigma, \mathcal{K})$, and the *link*, $\mathrm{lk}(\sigma, \mathcal{K})$, of σ in $\mathcal{K}$, as follows:

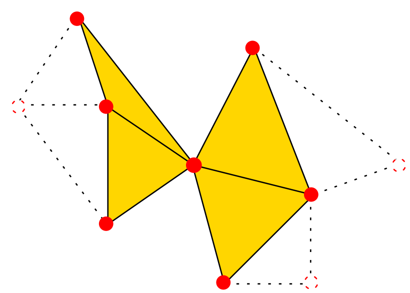
$$\mathrm{st}(\sigma, \mathcal{K}) = \{\tau \in \mathcal{K} \mid \exists \eta \text{ in } \mathcal{K} \text{ such that } \sigma \text{ is a face of } \eta \text{ and } \tau \text{ is a face of } \eta\}$$

and

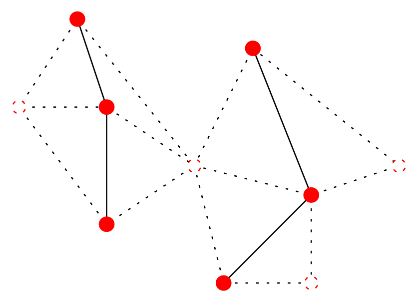
$$\mathrm{lk}(\sigma, \mathcal{K}) = \{\tau \in \mathcal{K} \mid \tau \text{ is in } st(\sigma, \mathcal{K}) \text{ and } \tau \text{ and } \sigma \text{ have no face in common}\}.$$



$\mathcal{K}$



$\mathrm{st}(\sigma, \mathcal{K})$

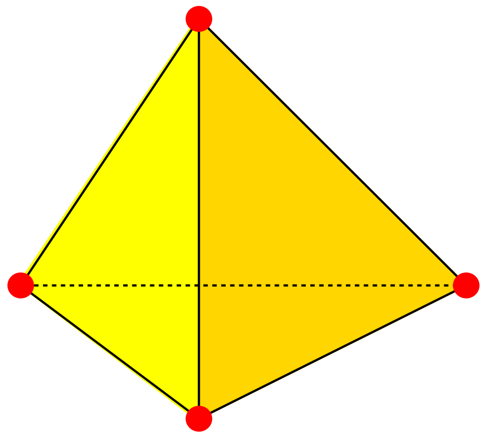


$\mathrm{lk}(\sigma, \mathcal{K})$

Parametric Pseudo-Manifolds

Simplicial Surfaces

Definition 9.7. A 2-complex $\mathcal{K}$ in $\mathbb{E}^n$ is called a *simplicial surface without boundary* if every 1-simplex of $\mathcal{K}$ is the face of precisely two simplices of $\mathcal{K}$, and the underlying space of the link of each 0-simplex of $\mathcal{K}$ is homeomorphic to the unit circle, $S^1 = \{x \in \mathbb{E}^2 \mid \|x\| = 1\}$.

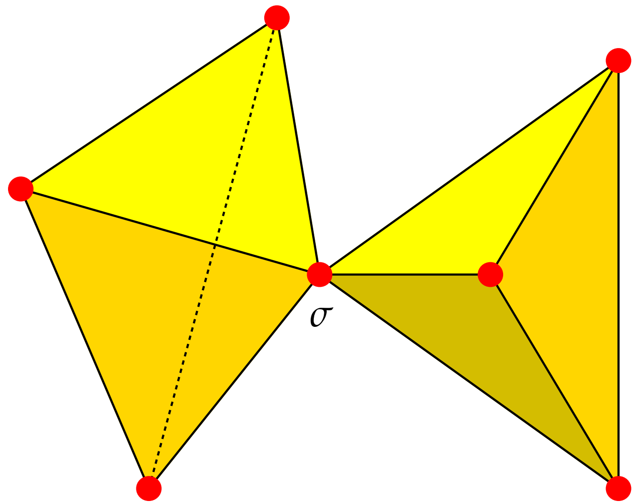


The set consisting of the 0-, 1-, and 2-faces of a 3-simplex is a simplicial surface without boundary.

Parametric Pseudo-Manifolds

Simplicial Surfaces

The simplicial complex consisting of the proper faces of two 3-simplices (i.e., two tetrahedra) sharing a common vertex **is not** a simplicial surface without boundary as the link of the common vertex of the two 3-simplices is not homeomorphic to the unit circle, S^1.

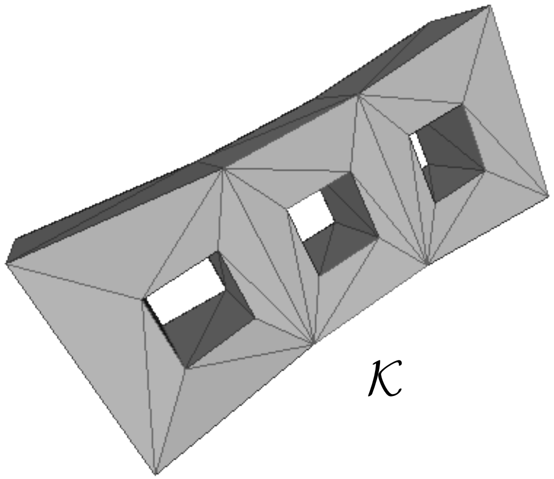


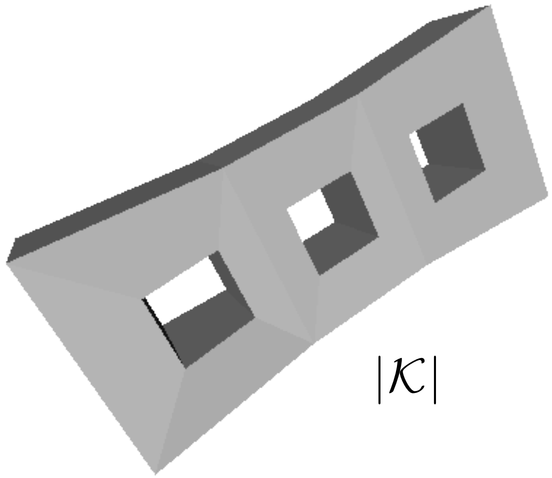
Parametric Pseudo-Manifolds

Simplicial Surfaces

From now on, we will refer to a simplicial surface without boundary as simply a simplicial surface. The underlying space of a simplicial surface is called its *underlying surface*.

The underlying surface of a simplicial surface is a topological 2-manifold in $\mathbb{E}^n$.





Parametric Pseudo-Manifolds

Simplicial Surfaces

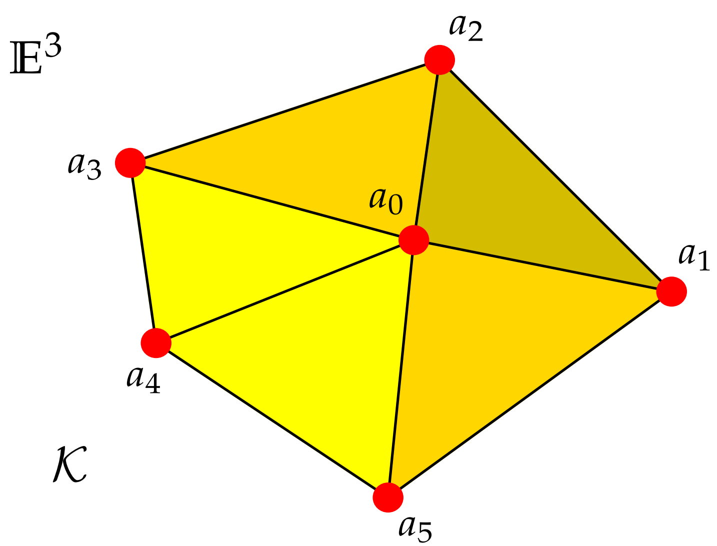
Definition 9.8. Let $\mathcal{K}$ be a simplicial complex in $\mathbb{E}^n$. For each integer i, with $0 \leq i \leq dim(\mathcal{K})$, we define $\mathcal{K}^{(i)}$ as the simplicial complex consisting of all j-simplices of $\mathcal{K}$, for every j such that $0 \leq j \leq i$. Moreover, if $\mathcal{L}$ is a simplicial complex in $\mathbb{E}^m$, then a map

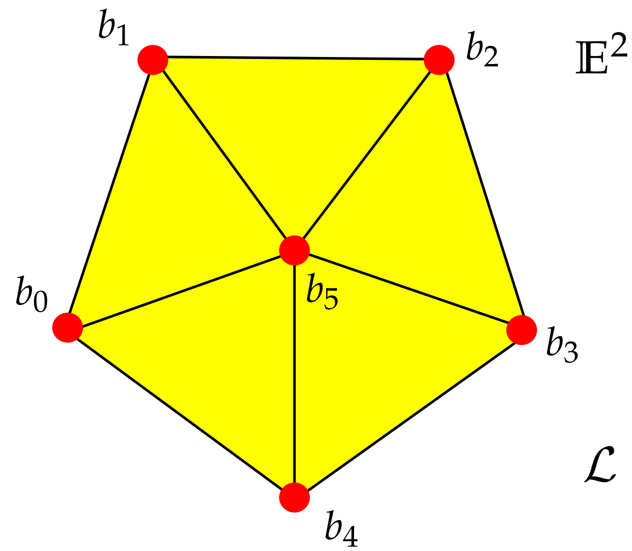
$$f : \mathcal{K}^{(0)} \to \mathcal{L}^{(0)}$$

is called a *simplicial map* if whenever $[a_0, \ldots, a_d]$ is a simplex in $\mathcal{K}$, then $[f(a_0), \ldots, f(a_d)]$ is a simplex in $\mathcal{L}$. A simplicial map is a *simplicial isomorphism* if it is a bijective map, and if its inverse is also a simplicial map. Finally, if there exists a simplicial isomorphism from $\mathcal{K}$ to $\mathcal{L}$, then we say that $\mathcal{K}$ and $\mathcal{L}$ are *simplicially isomorphic*.

Parametric Pseudo-Manifolds

Simplicial Surfaces

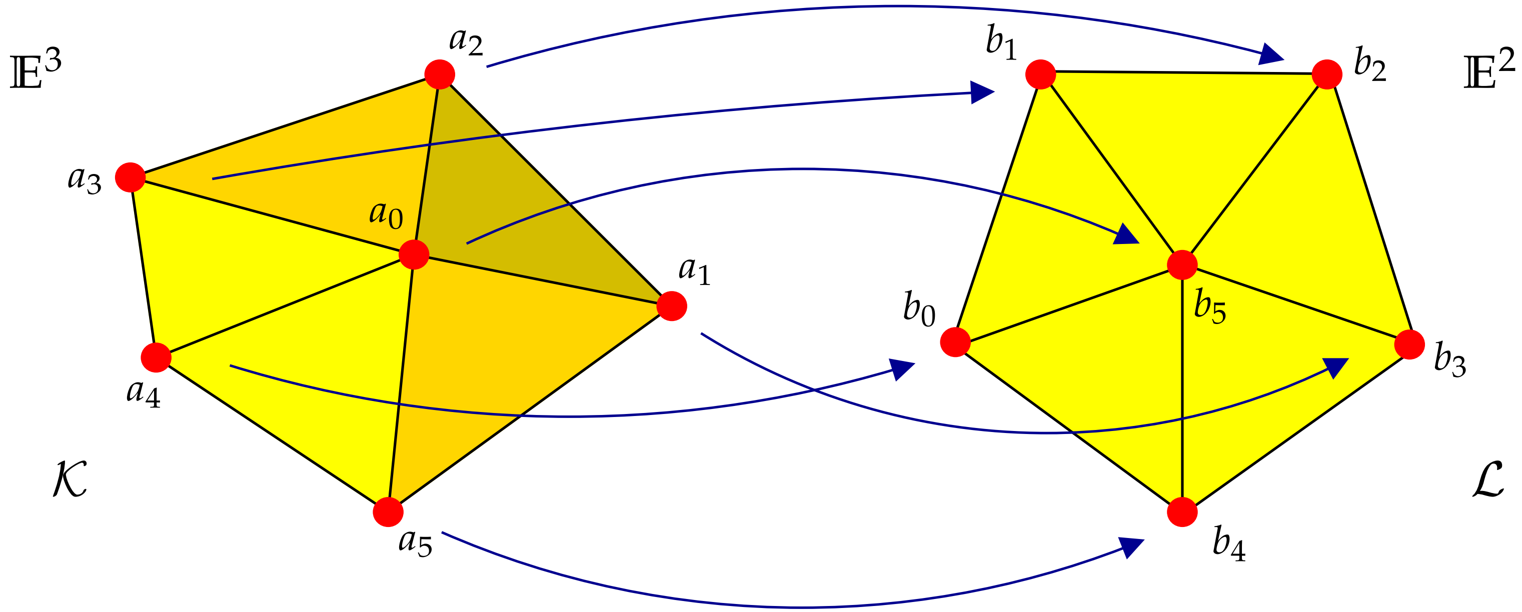




$\mathcal{K}$ and $\mathcal{L}$ are simplicially isomorphic.

Parametric Pseudo-Manifolds

Simplicial Surfaces



Let

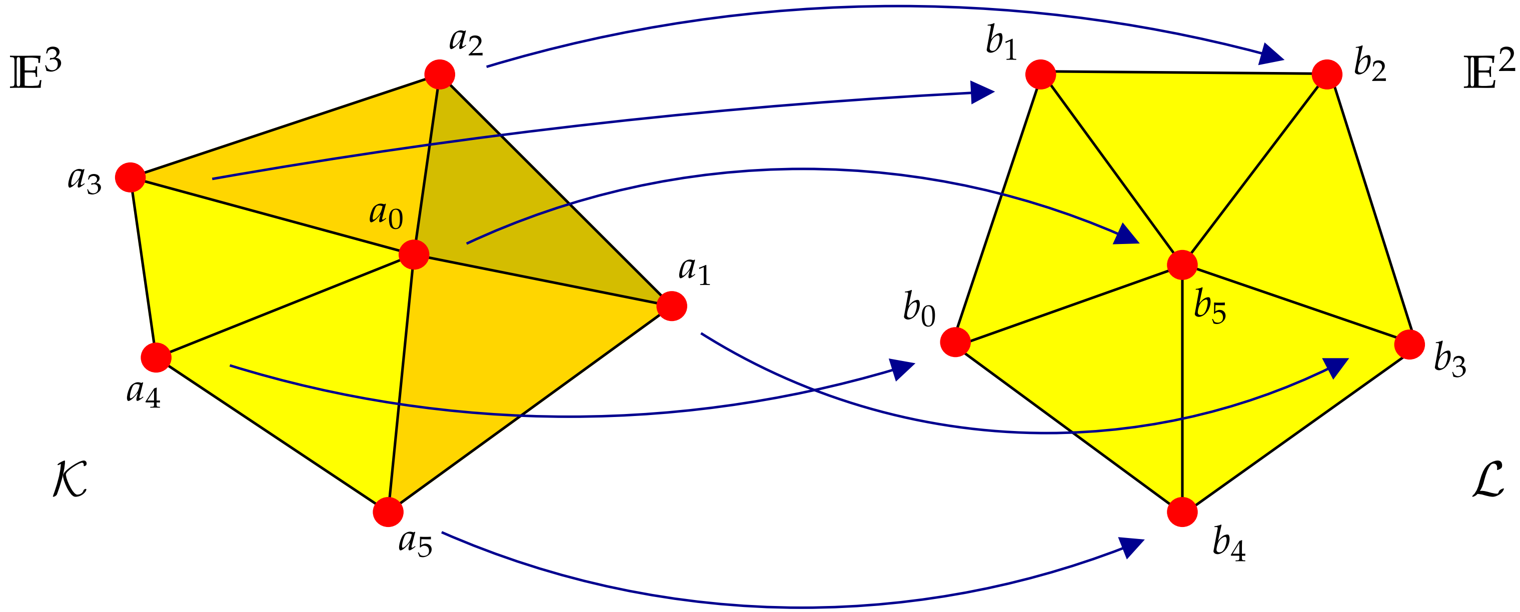
$$f : \mathcal{K}^{(0)} \to \mathcal{L}^{(0)}$$

be given by

$$f(a_0) = b_5, \quad f(a_1) = b_3, \quad f(a_2) = b_2, \quad f(a_3) = b_1, \quad f(a_4) = b_0, \quad f(a_5) = b_4 .$$

Parametric Pseudo-Manifolds

Simplicial Surfaces

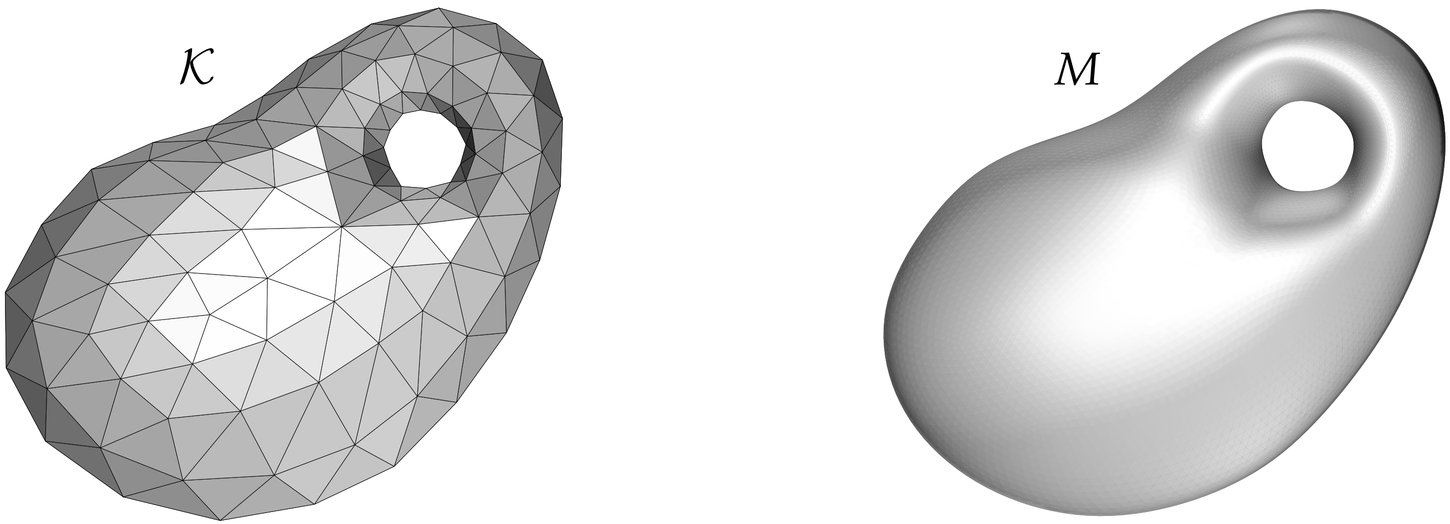


It is easily verified that f is a simplicial isomorphism.

Parametric Pseudo-Manifolds

Gluing Data

Given a simplicial surface, $\mathcal{K}$, in $\mathbb{E}^3$, we are interested in building a parametric pseudo-surface, $\mathcal{M}$, in $\mathbb{E}^3$ such that the image, M, of $\mathcal{M}$ is homeomorphic to the underlying surface, $|\mathcal{K}|$, of $\mathcal{K}$, and such that M also *approximates* the geometry of $|\mathcal{K}|$.



Parametric Pseudo-Manifolds

Gluing Data

As we did before, let us first focus on the definition of a set of gluing data.

Unfortunately, this task is not as easy as it was in the one-dimensional case.

The key is to notice that the simplicial surface, $\mathcal{K}$, which is a combinatorial object, explicitly defines a topological structure on $|\mathcal{K}|$ (via the adjacency relations of all simplices) .

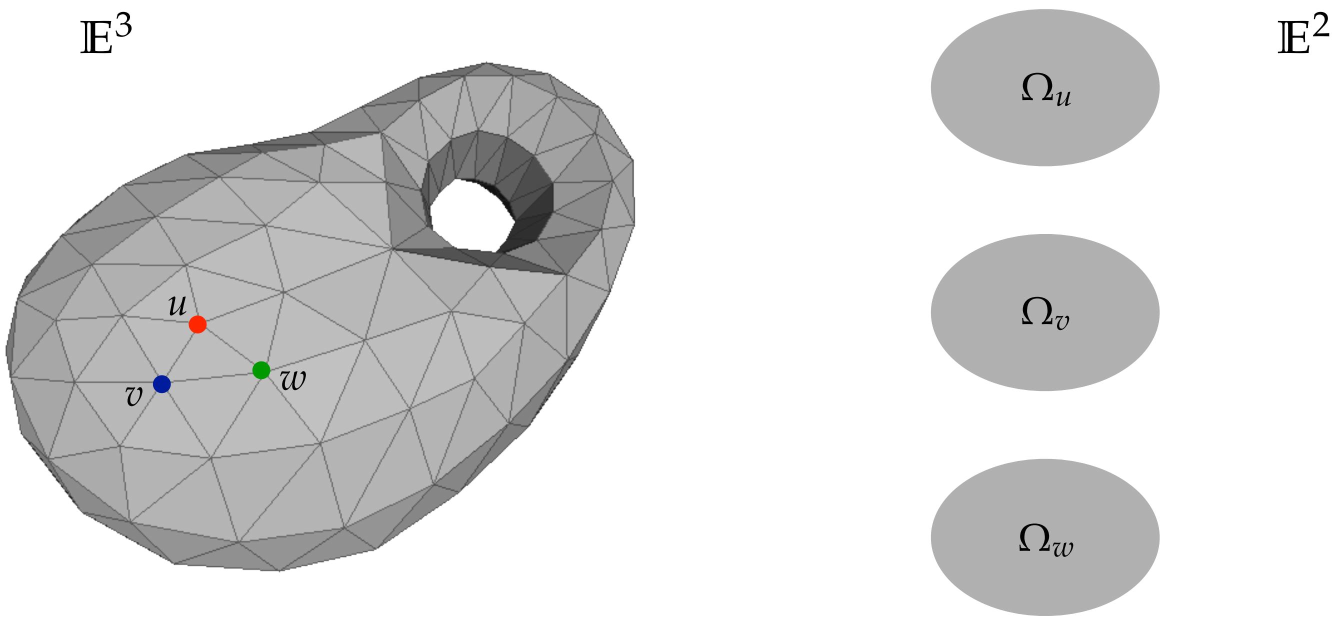
So, we should define p-domains, gluing domains, and transition functions based on $\mathcal{K}$.

Parametric Pseudo-Manifolds

Gluing Data

As we will see during the next lectures, there are many choices for p-domains. But, in general, p-domains are associated with simplices of $\mathcal{K}$. For instance, the vertices of $\mathcal{K}$.

We can define a one-to-one correspondence between p-domains and vertices of $\mathcal{K}$.

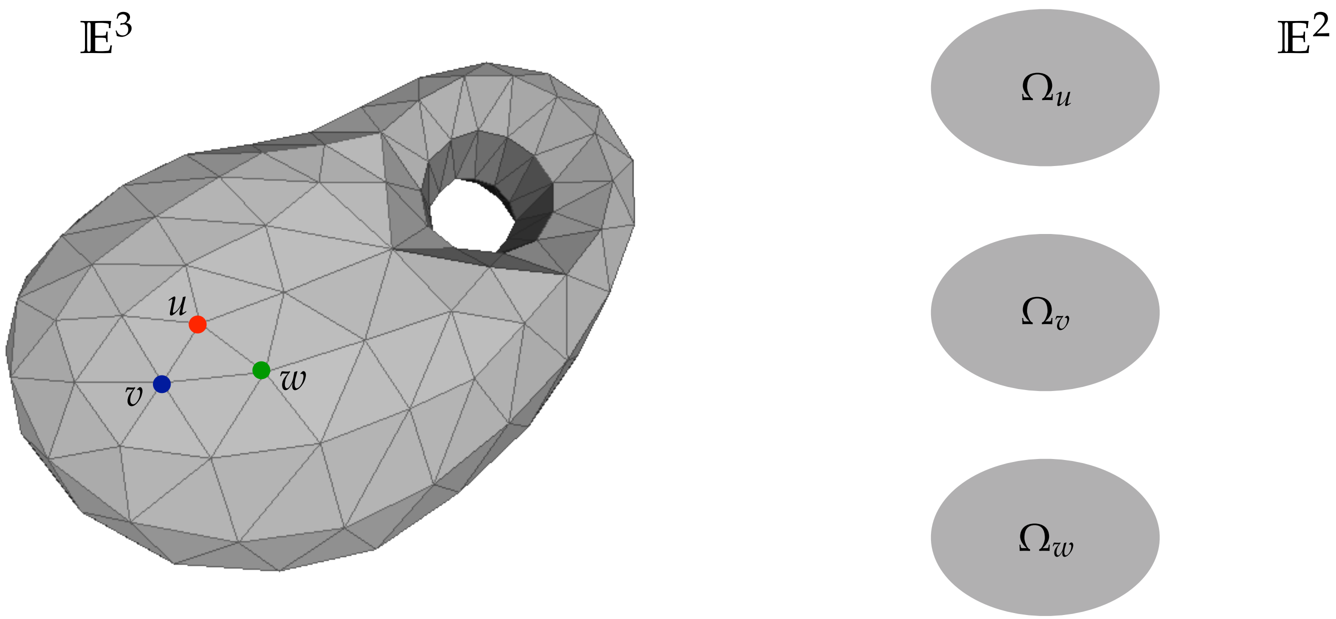


Parametric Pseudo-Manifolds

Gluing Data

The previous correspondence implies that the number of p-domains is equal to the number of vertices of $\mathcal{K}$. A distinct choice of correspondence may yield a different number.

The choice of a geometry for the p-domains is a key decision too.

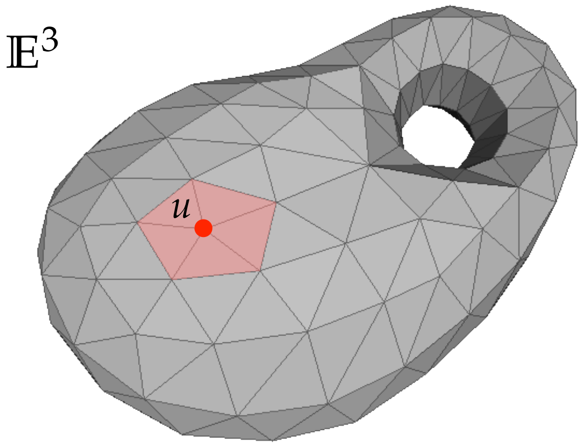


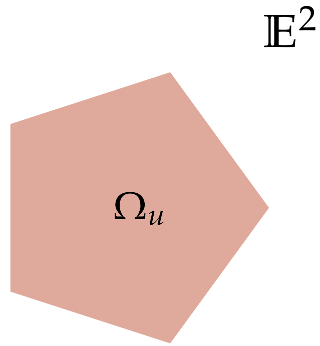
Parametric Pseudo-Manifolds

Gluing Data

Intuitively, each p-domain is an open "disk" that is consistently glued to other p-domains in order to define the topology of the image, M, of the parametric pseudo-surface.

Since a vertex u of $\mathcal{K}$ is connected only to the vertices of $\mathcal{K}$ that belong to the link, $\mathrm{lk}(u, \mathcal{K})$, of u in $\mathcal{K}$, it is natural to think of the p-domain, Ω_u, which is associated with vertex u, as the interior of a polygon in $\mathbb{E}^2$ with the same number of vertices as $\mathrm{lk}(u, \mathcal{K})$.

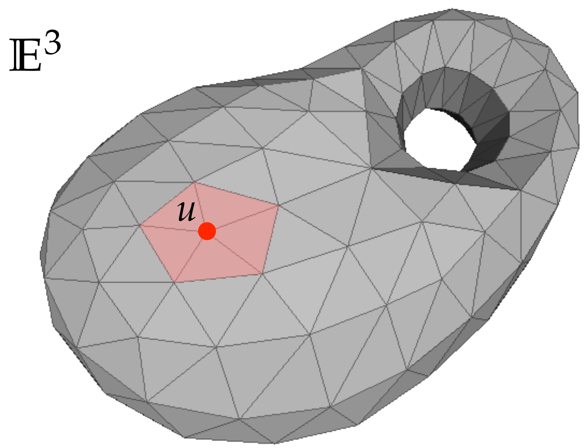


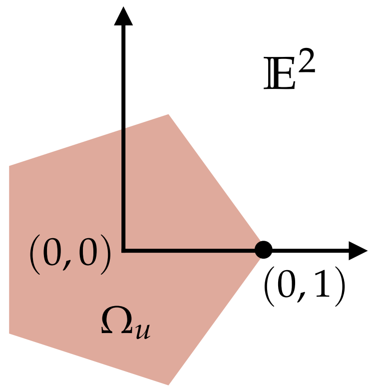


Parametric Pseudo-Manifolds

Gluing Data

To simplify calculations, we can assume that Ω_u is a regular polygon inscribed in a unit circle centered at the origin of a local coordinate system of $\mathbb{E}^2$. We can also assume that one vertex of Ω_u is located at the point $(0,1)$. Now, Ω_u is uniquely defined.





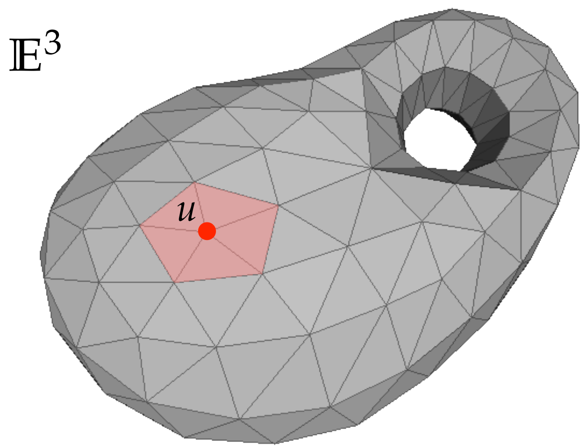
Parametric Pseudo-Manifolds

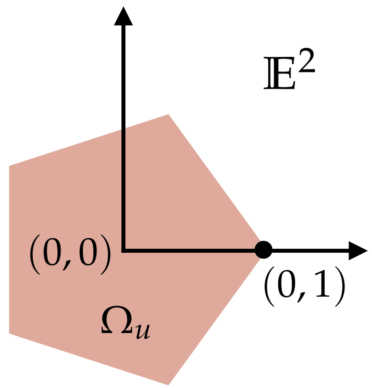
Gluing Data

Formally, let $I = \{u \mid u \text{ is a vertex in } \mathcal{K}\}$, n_u be the number of vertices of the link, $\mathrm{lk}(u, \mathcal{K})$, of u in $\mathcal{K}$, and P_u be the regular, n_u-polygon whose vertices are located at the points

$$\left(\cos\left(i \cdot \frac{2\pi}{n_u}\right), \sin\left(i \cdot \frac{2\pi}{n_u}\right)\right),$$

for all $i = 0, 1, \ldots, n_u - 1$. Then, we can define $\Omega_u = \mathring{P}_u$, where $\mathring{P}_u$ is the interior of P_u.





Parametric Pseudo-Manifolds

Gluing Data

Checking...

(1) For every $i \in I$, the set Ω_i is a nonempty open subset of $\mathbb{E}^n$ called parametrization domain, for short, p-domain, and any two distinct p-domains are pairwise disjoint, i.e.,

$$\Omega_i \cap \Omega_j = \emptyset,$$

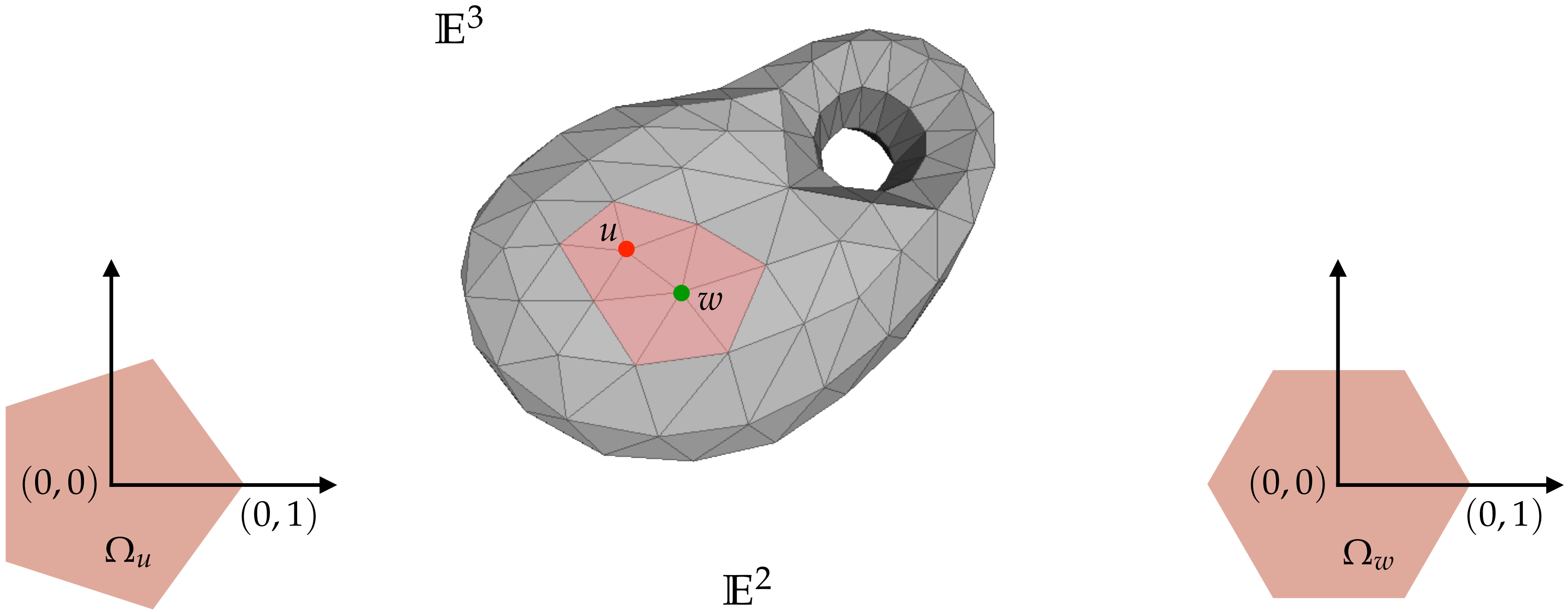
for all $i \neq j$.

Our p-domains are (connected) open subsets of $\mathbb{E}^2$. If we assume that they live in distinct copies of $\mathbb{E}^2$, then they will not overlap, and hence condition (1) of Definition 7.1 holds.

Parametric Pseudo-Manifolds

Gluing Data

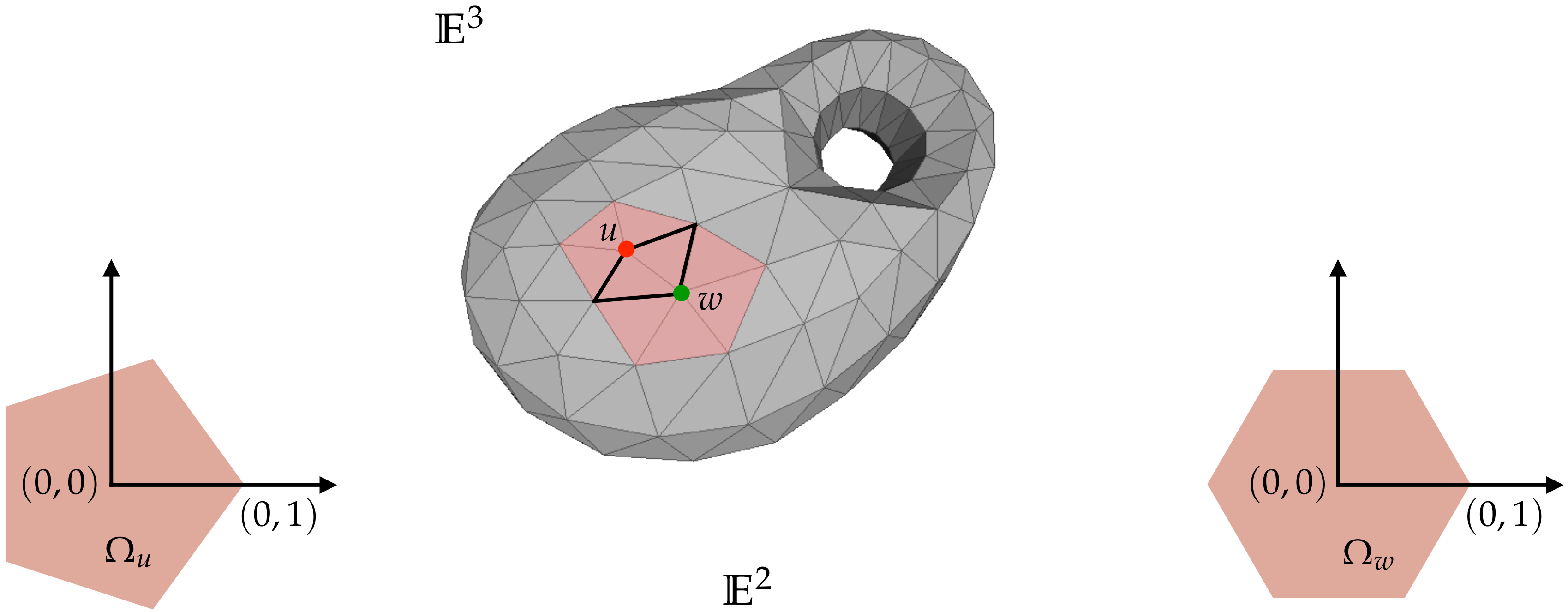
What about gluing domains? The following picture should help us find a good choice:



Parametric Pseudo-Manifolds

Gluing Data

As we can see, the intersection of the stars, $\mathrm{st}(u, \mathcal{K})$ and $\mathrm{st}(w, \mathcal{K})$, of u and w consists of exactly two triangles. These triangles share an edge in both $\mathrm{st}(u, \mathcal{K})$ and $\mathrm{st}(w, \mathcal{K})$. So, we can think of defining the gluing domains as *diamond-shaped*, open subsets of the p-domains.

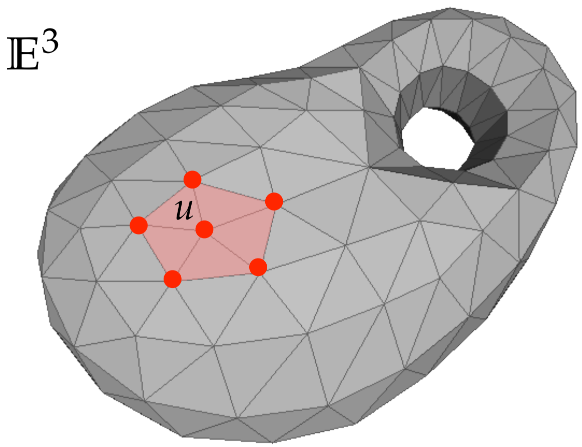


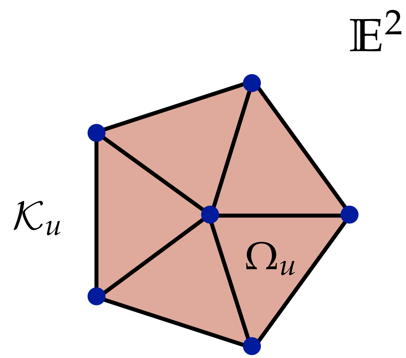
Parametric Pseudo-Manifolds

Gluing Data

To precisely define gluing domains, we associate a 2-dimensional simplicial complex, $\mathcal{K}_u$, with each p-domain Ω_u. The complex $\mathcal{K}_u$ satisfies the following two conditions: (1) $|\mathcal{K}_u|$, is the closure, $\overline{\Omega_u}$, of Ω_u and (2) $\mathcal{K}_u$ is isomorphic to the star, $\mathrm{st}(u, \mathcal{K})$, of u in K.

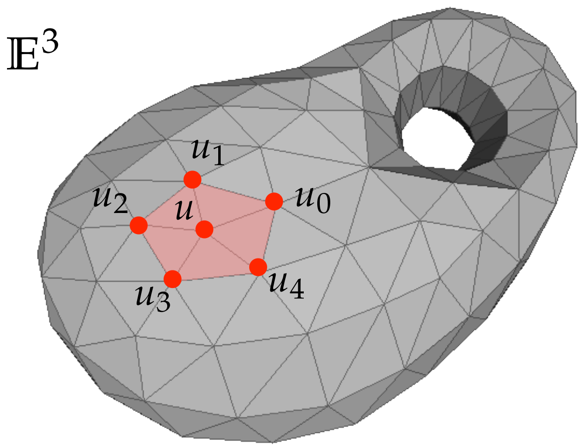
An obvious choice for $\mathcal{K}_u$ is the canonical triangulation of $\overline{\Omega_u}$:

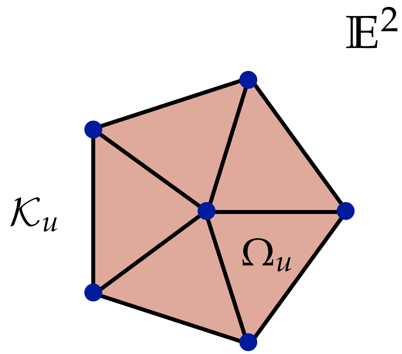




Gluing Data

Fix *any* counterclockwise enumeration, $u_0, u_1, \ldots, u_m$, of the vertices in $\mathrm{lk}(u, \mathcal{K})$.





Parametric Pseudo-Manifolds

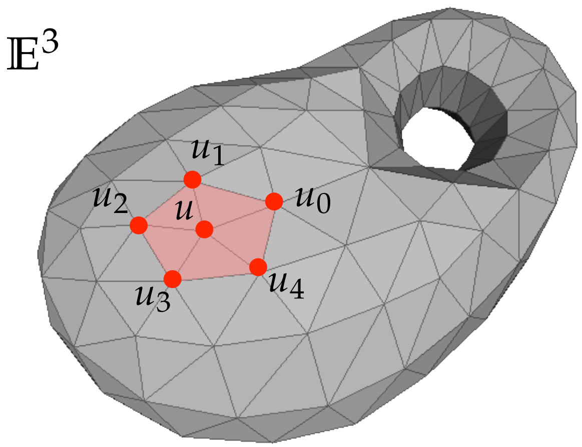
Gluing Data

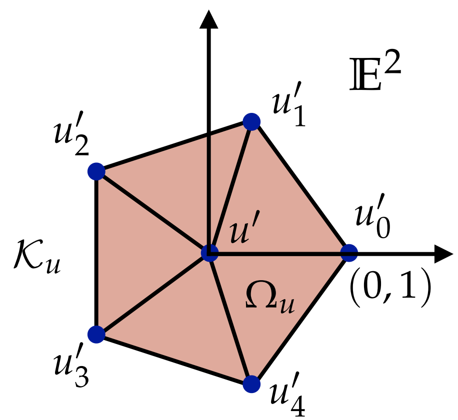
Let u'_0 be the vertex of $\mathcal{K}_u$ located at the point $(0, 1)$.

Let

$$u'_0, u'_1, \ldots, u'_m$$

be the counterclockwise enumeration of the vertices of $\mathrm{lk}(u', \mathcal{K}_u)$ starting with u'_0.





Parametric Pseudo-Manifolds

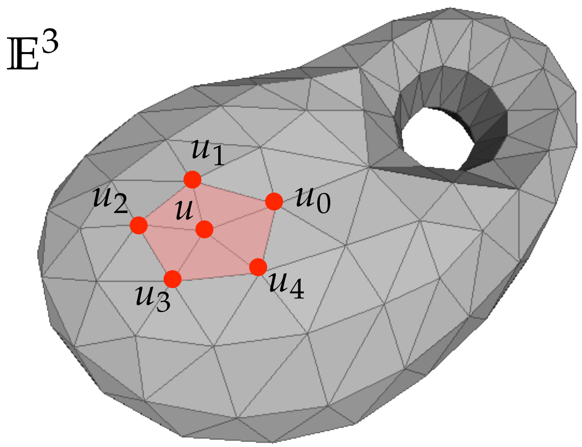
Gluing Data

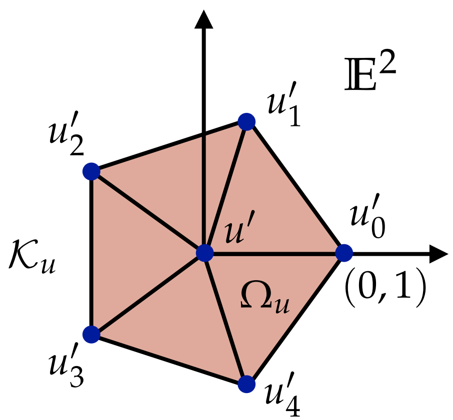
Let

$$f_u : \mathrm{st}(u, \mathcal{K})^{(0)} \to \mathcal{K}_u^{(0)}$$

be the simplicial map given by $f_u(u) = u'$ and $f_u(u_i) = u'_i$, for $i = 0, \ldots, m$.

It is easily verified that f_u is a simplicial isomorphism.



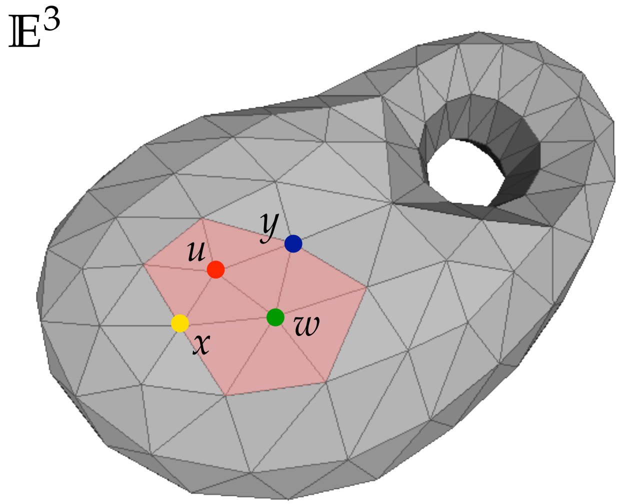


Parametric Pseudo-Manifolds

Gluing Data

Let u and w be any two vertices of $\mathcal{K}$ such that $[u, w]$ is an edge in $\mathcal{K}$.

Let x and y be the other two vertices of $\mathcal{K}$ that also belong to both $\text{st}(u, \mathcal{K})$ and $\text{st}(w, \mathcal{K})$.



Assume that x precedes w in a counterclockwise traversal of the vertices of $\text{lk}(u, \mathcal{K})$ starting at y.

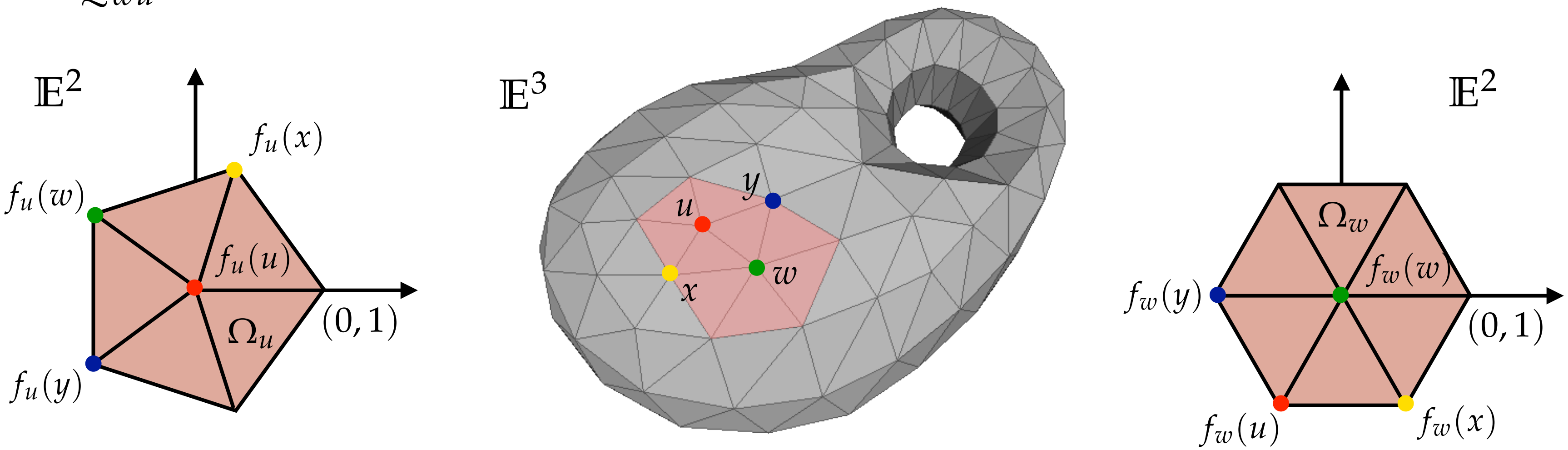
Parametric Pseudo-Manifolds

Gluing Data

We can now define the gluing domains, Ω_{uw} and Ω_{wu}, as $\Omega_{uw} = \mathring{Q}_{uw}$ and $\Omega_{wu} = \mathring{Q}_{wu}$, where

$$Q_{uw} = [f_u(u), f_u(x), f_u(w), f_u(y)] \quad \text{and} \quad Q_{wu} = [f_w(w), f_w(y), f_w(u), f_w(x)]$$

are the quadrilaterals given by the vertices $f_u(u)$, $f_u(x)$, $f_u(w)$, $f_u(y)$ of $\mathcal{K}_u$ and the vertices $f_w(w)$, $f_w(y)$, $f_w(u)$, $f_w(x)$ of $\mathcal{K}_w$, and $\mathring{Q}_{uw}$ and $\mathring{Q}_{wu}$ are the interiors of Q_{uw} and Q_{wu}.

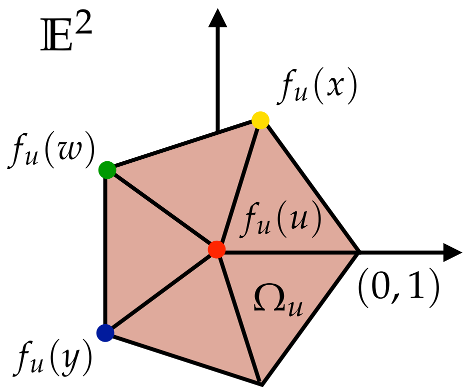


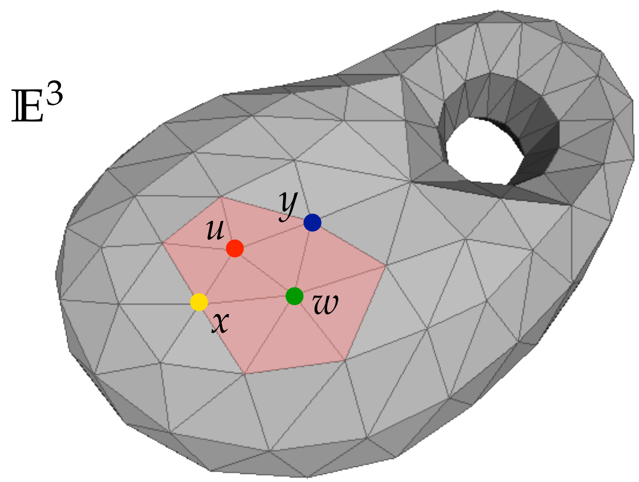
Parametric Pseudo-Manifolds

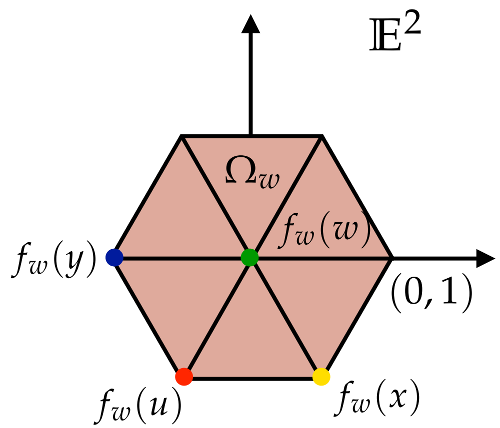
Gluing Data

Formally, for every $(u, w) \in I \times I$, we let

$$\Omega_{uw} = \begin{cases} \Omega_u & \text{if } u = w, \\ \emptyset & \text{if } u \neq w \text{ and } [u, w] \text{ is not an edge of } \mathcal{K}, \\ \mathring{Q}_{uw} & \text{if } u \neq w \text{ and } [u, w] \text{ is an edge of } \mathcal{K}. \end{cases}$$







Parametric Pseudo-Manifolds

Gluing Data

Checking...

(2) For every pair $(i, j) \in I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ji} \neq \emptyset$ if and only if $\Omega_{ij} \neq \emptyset$. Each nonempty subset Ω_{ij} (with $i \neq j$) is called a gluing domain.

By definition, the sets Ω_u, $\emptyset$, and $\mathring{Q}_{wu}$ are open in $\mathbb{E}^2$. Furthermore, the sets $\mathring{Q}_{uw}$ and $\mathring{Q}_{wu}$ are well-defined and nonempty, for every $u, w \in I$ such that $[u, w]$ is an edge of $\mathcal{K}$.

So, for every $u, w \in I$, we have that $\Omega_{uw} \neq \emptyset$ iff $[u, w]$ is an edge of $\mathcal{K}$. Thus, for every $u, w \in I$, $\Omega_{uw} \neq \emptyset$ iff $\Omega_{wu} \neq \emptyset$, and hence condition (2) of Definition 7.1 also holds.

Parametric Pseudo-Manifolds

Gluing Data

Our definitions of p-domain and gluing domain naturally lead us to a gluing process induced by the gluing of the stars of the vertices of $\mathcal{K}$ along their common edges and triangles.

The gluing strategy we adopted here does not depend on the geometry of the p-domains and gluing domains, but on the adjacency relations of vertices and edges of $\mathcal{K}$.

However, the geometry of the p-domains and gluing domains have a strong influence in the level of difficulty of the transition maps and parametrizations we choose to use.

Despite of our commitment to a particular geometry, we will present next an axiomatic way of defining the transition maps. Our axiomatic definition should be as much independent of the geometry of the p-domains and gluing domains as possible.