IMPA - Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ, Brazil

Introduction to Computational Manifolds and Applications

Part 1 - Constructions

Prof. Marcelo Ferreira Siqueira

mfsiqueira@dimap.ufrn.br

Departmento de Informática e Matemática Aplicada
Universidade Federal do Rio Grande do Norte
Natal, RN, Brazil

Parametric Pseudo-Manifolds

Gluing Data

Let $K = \{(u, w) \in I \times I \mid \Omega_{uw} \neq \emptyset\}$.

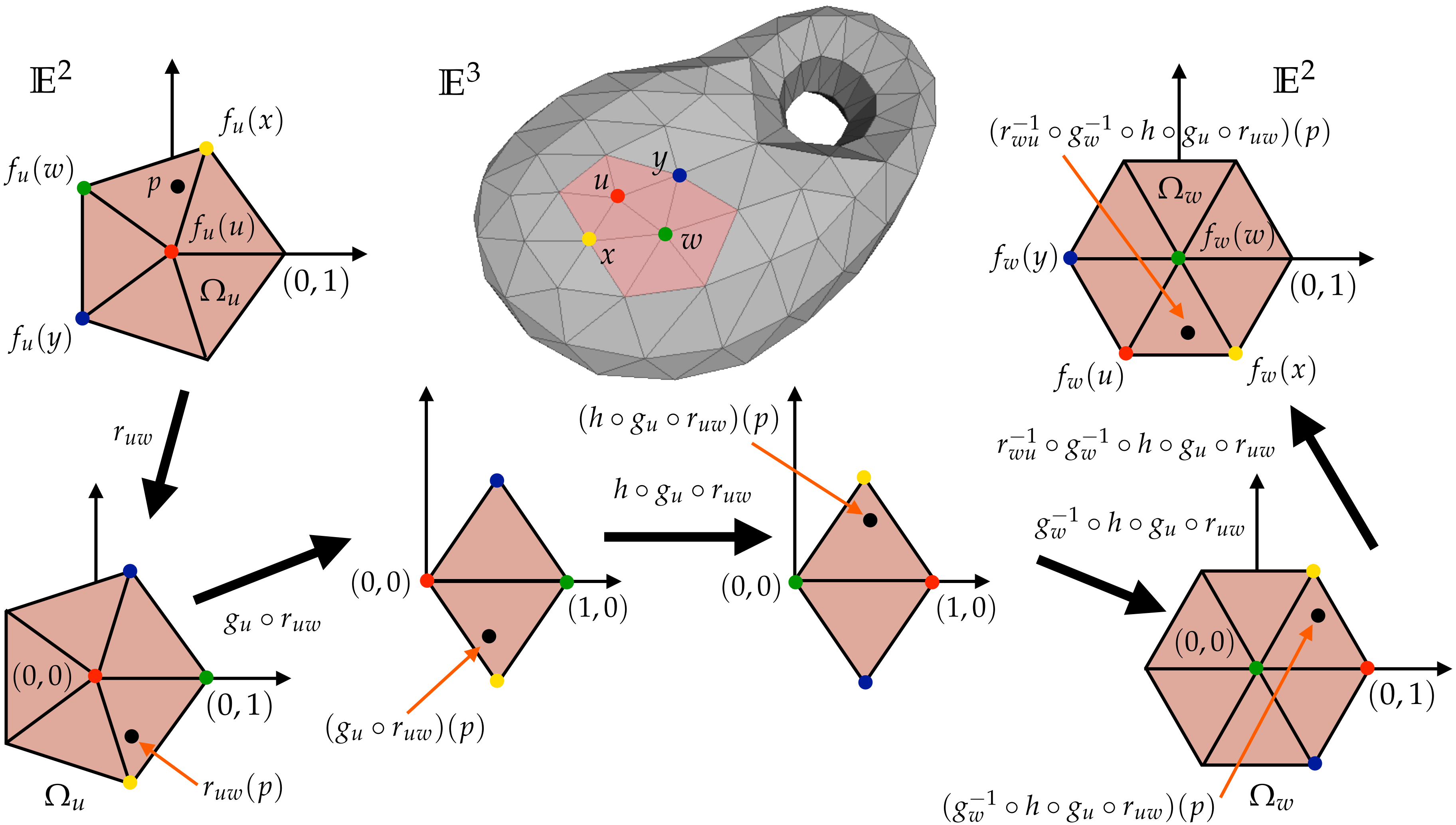
For every $(u, w) \in K$, we must define the transition map $\varphi_{wu} : \Omega_{uw} \to \Omega_{wu}$, which is a C^k-diffeomorphic function that takes Ω_{uw} onto Ω_{wu}, where k is a positive integer or $k = \infty$.

Here, we will assume that φ_{wu} is a composition of five distinct maps: two rotations, a double reflection, and two more general planar transformations. The latter transformations are the main obstacles for obtaining maps satisfying condition (3) of Definition 7.1.

The picture in the next slide illustrates the roles of each of the five maps.

Parametric Pseudo-Manifolds

Gluing Data



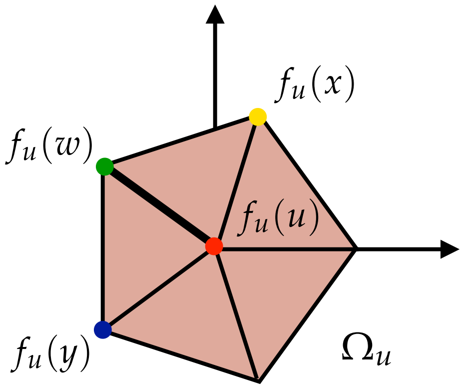
Parametric Pseudo-Manifolds

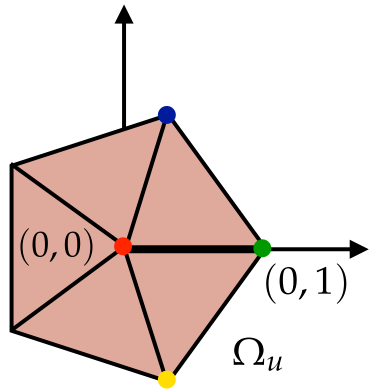
Gluing Data

Let $[u, w]$ be an edge of $\mathcal{K}$. Then,

$$r_{uw} : \mathbb{E}^2 \longrightarrow \mathbb{E}^2$$

is the rotation around the origin that identifies the edge $[f_u(u) = u', f_u(w)]$ of $\mathcal{K}_u$ with the edge $[u', u'_0]$, where u' and u'_0 have coordinates $(0,0)$ and $(1,0)$, respectively.



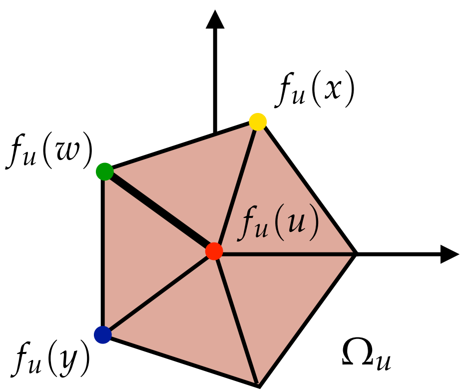


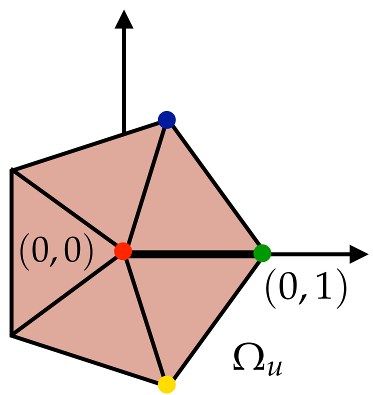
Parametric Pseudo-Manifolds

Gluing Data

If $f_u(w) = u'_i$, for some $i \in \{0, \ldots, n_u - 1\}$, then the rotation angle, θ_{uw}, of r_{uw} is

$$\theta_{uw} = -\, i \cdot \frac{2\pi}{n_u}\,.$$



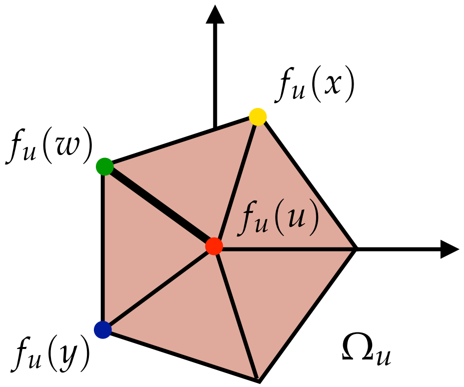


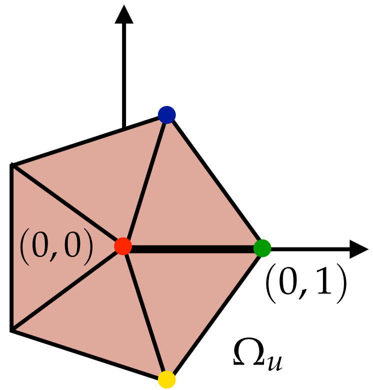
Parametric Pseudo-Manifolds

Gluing Data

Note that

$$
\begin{aligned}
f_u(u) &= (0,0) \\
f_u(x) &= \left(\cos\left((i-1)\cdot\frac{2\pi}{n_u}\right), \sin\left((i-1)\cdot\frac{2\pi}{n_u}\right)\right) \\
f_u(w) &= \left(\cos\left(i\cdot\frac{2\pi}{n_u}\right), \sin\left(i\cdot\frac{2\pi}{n_u}\right)\right) \\
f_u(y) &= \left(\cos\left((i+1)\cdot\frac{2\pi}{n_u}\right), \sin\left((i+1)\cdot\frac{2\pi}{n_u}\right)\right) .
\end{aligned}
$$



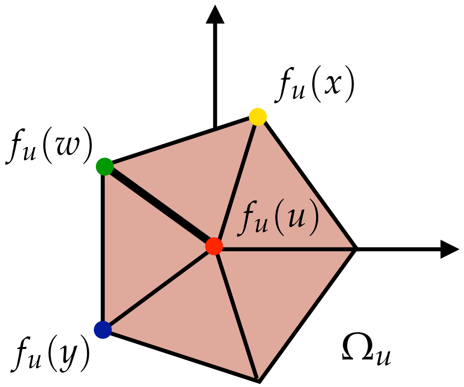


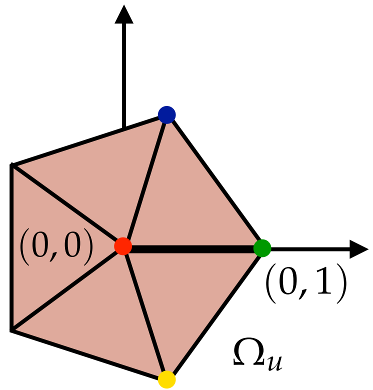
Parametric Pseudo-Manifolds

Gluing Data

So, r_{uw} takes the quadrilateral $Q_{uw} = [f_u(u), f_u(x), f_u(w), f_u(y)]$ onto the quadrilateral

$$r_{uw}(Q_{uw}) = \left[(0,0), \left(\cos\left(-\frac{2\pi}{n_u}\right), \sin\left(-\frac{2\pi}{n_u}\right)\right), (1,0), \left(\cos\left(\frac{2\pi}{n_u}\right), \sin\left(-\frac{2\pi}{n_u}\right)\right)\right].$$





Parametric Pseudo-Manifolds

Gluing Data

The next map, $g_u : \mathbb{E}^2 \to \mathbb{E}^2$, is *ideally* a C^k-diffeomorphism of the plane. This map must take the interior of the quadrilateral, $r_{uw}(Q_{uw})$, *onto* the so-called *canonical quadrilateral*,

$$Q = \left[(0,0), \left(\cos\left(-\frac{\pi}{3}\right), \sin\left(-\frac{\pi}{3}\right)\right), (1,0), \left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)\right)\right].$$

Parametric Pseudo-Manifolds

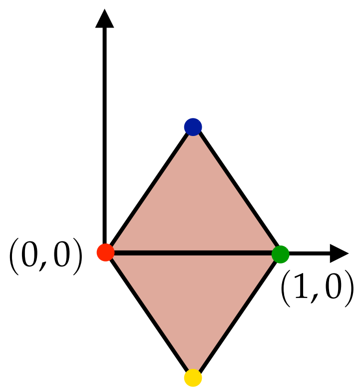
Gluing Data

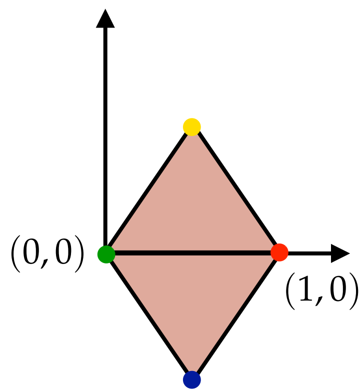
The reason for requiring that $\mathring{Q}$ be the codomain of g_u is somewhat arbitrary and unclear at this moment. But, it has to do with the cocycle condition. We will get back to this point in a few minutes. For the time being, let us assume that such a map g_u exists.

Parametric Pseudo-Manifolds

Gluing Data

Finally, we have the map $h : \mathbb{E}^2 \to \mathbb{E}^2$ given by $h(x, y) = (1 - x, -y)$. The map h performs a reflection with respect to the x axis and another with respect to the line $x = 0.5$.

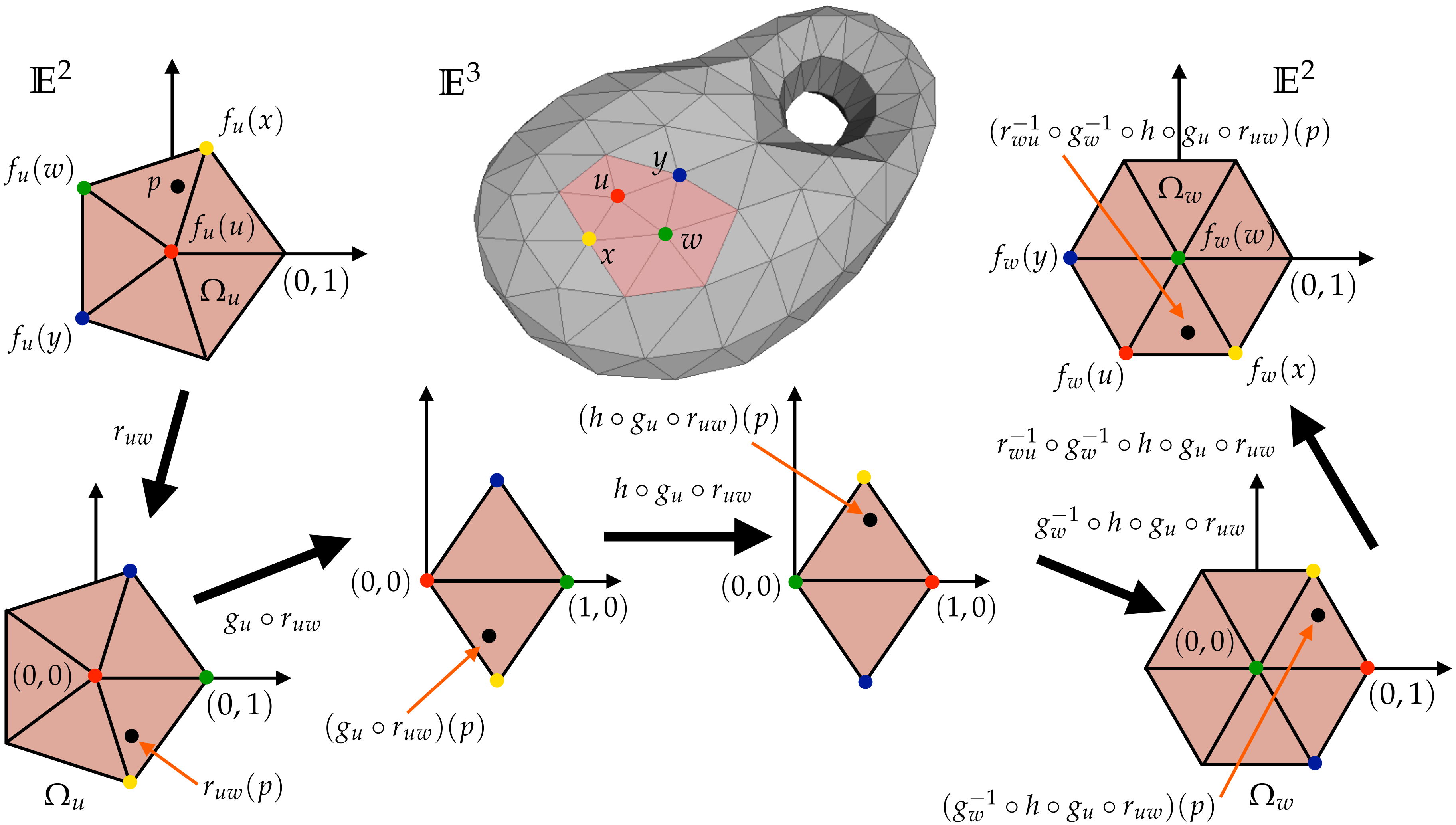




Note that h is a rotation of 180^o around the point $(0.5, 0)$. So, we have that $h = h^{-1}$.

Parametric Pseudo-Manifolds

Gluing Data



Parametric Pseudo-Manifolds

Gluing Data

By definition, r_{uw} and h are C^∞-diffeomorphisms of the plane. So, if the map g_u is a C^k-diffeomorphism of the plane satisfying $g_u(r_{uw}(\overset{\circ}{Q}_{uw})) = \overset{\circ}{Q}$, we could *tentatively* define

$$\varphi_{wu} : \Omega_{uw} \to \Omega_{wu}$$

such that

$$\varphi_{wu}(x) = \begin{cases} \mathrm{id}_{\Omega_{uw}} & \text{if } u = w \quad , \\ (r_{wu}^{-1} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw})(x) & \text{if } u \neq w \quad , \end{cases}$$

for every $x \in \Omega_{uw}$.

As we shall prove, the map φ_{wu} satisfies conditions 3(a) and 3(b) of Definition 7.1. But, without knowing the map g_u, we cannot say anything about condition 3(c) yet.

Parametric Pseudo-Manifolds

Gluing Data

Checking...

(3) If we let

$$K = \{(i,j) \in I \times I \mid \Omega_{ij} \neq \emptyset\},$$

then

$$\varphi_{ji} \colon \Omega_{ij} \to \Omega_{ji}$$

is a C^k bijection for every $(i,j) \in K$ called a transition (or gluing) map.

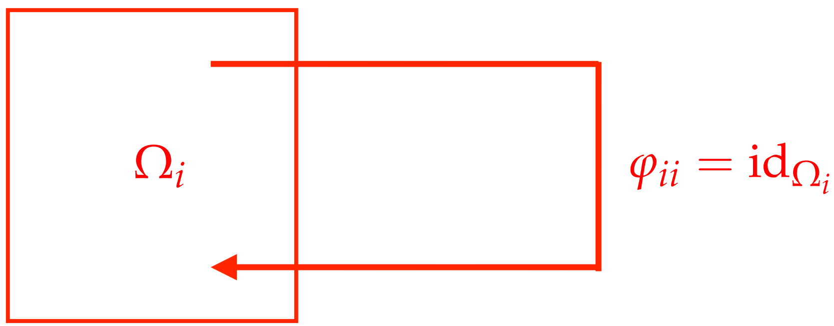
By *assumption*, the g_u maps are C^k-diffeomorphisms of the plane, for some integer k or $k = \infty$. Since rotations and reflections are C^∞-diffeomorphisms of the plane, the composition of these maps yield a C^k-diffeomorphic transition map. So, condition 3 holds.

Parametric Pseudo-Manifolds

Gluing Data

Checking...

(a) $\varphi_{ii} = \mathrm{id}_{\Omega_i}$, for all $i \in I$,



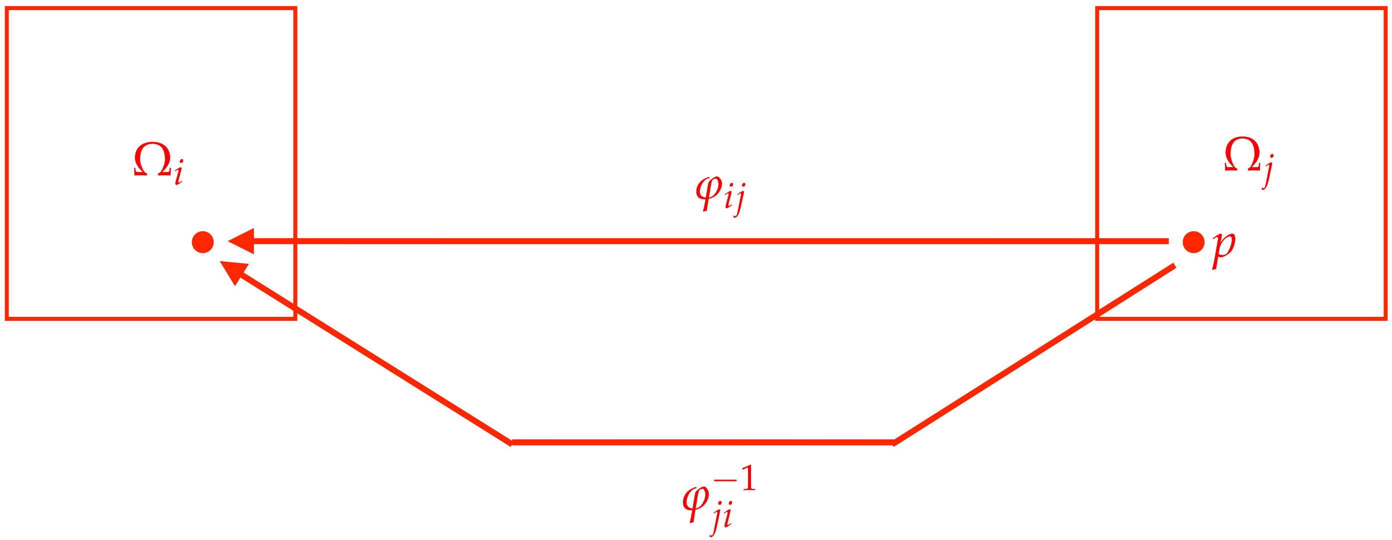
By definition, $\varphi_{uw}(x) = x$, for every $x \in \Omega_{uw}$, whenever $u = w$. So, condition 3(a) holds.

Parametric Pseudo-Manifolds

Gluing Data

Checking...

(b) $\varphi_{ij} = \varphi_{ji}^{-1}$, for all $(i, j) \in K$, and



Parametric Pseudo-Manifolds

Gluing Data

Checking...

If $u = w$ then condition 3(b) is trivially true.

So, let us assume that $u \neq w$. Then, for every point $p \in \Omega_{wu}$, we have (by definition) that

$$\begin{aligned}
\varphi_{wu}^{-1}(p) &= (r_{wu}^{-1} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw})^{-1})(p) \\
&= \left(r_{uw}^{-1} \circ g_u^{-1} \circ h^{-1} \circ (g_w^{-1})^{-1} \circ (r_{wu}^{-1})^{-1}\right)(p) \\
&= (r_{uw}^{-1} \circ g_u^{-1} \circ h \circ g_w \circ r_{wu})(p) \\
&= \varphi_{uw}(p) \,.
\end{aligned}$$

This implies that condition 3(b) holds when $u \neq w$ too.

Parametric Pseudo-Manifolds

Gluing Data

What about condition 3(c)?

(c) For all i, j, k, if

$$\Omega_{ji} \cap \Omega_{jk} \neq \emptyset \,,$$

then

$$\varphi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik} \qquad \text{and} \qquad \varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x) \,,$$

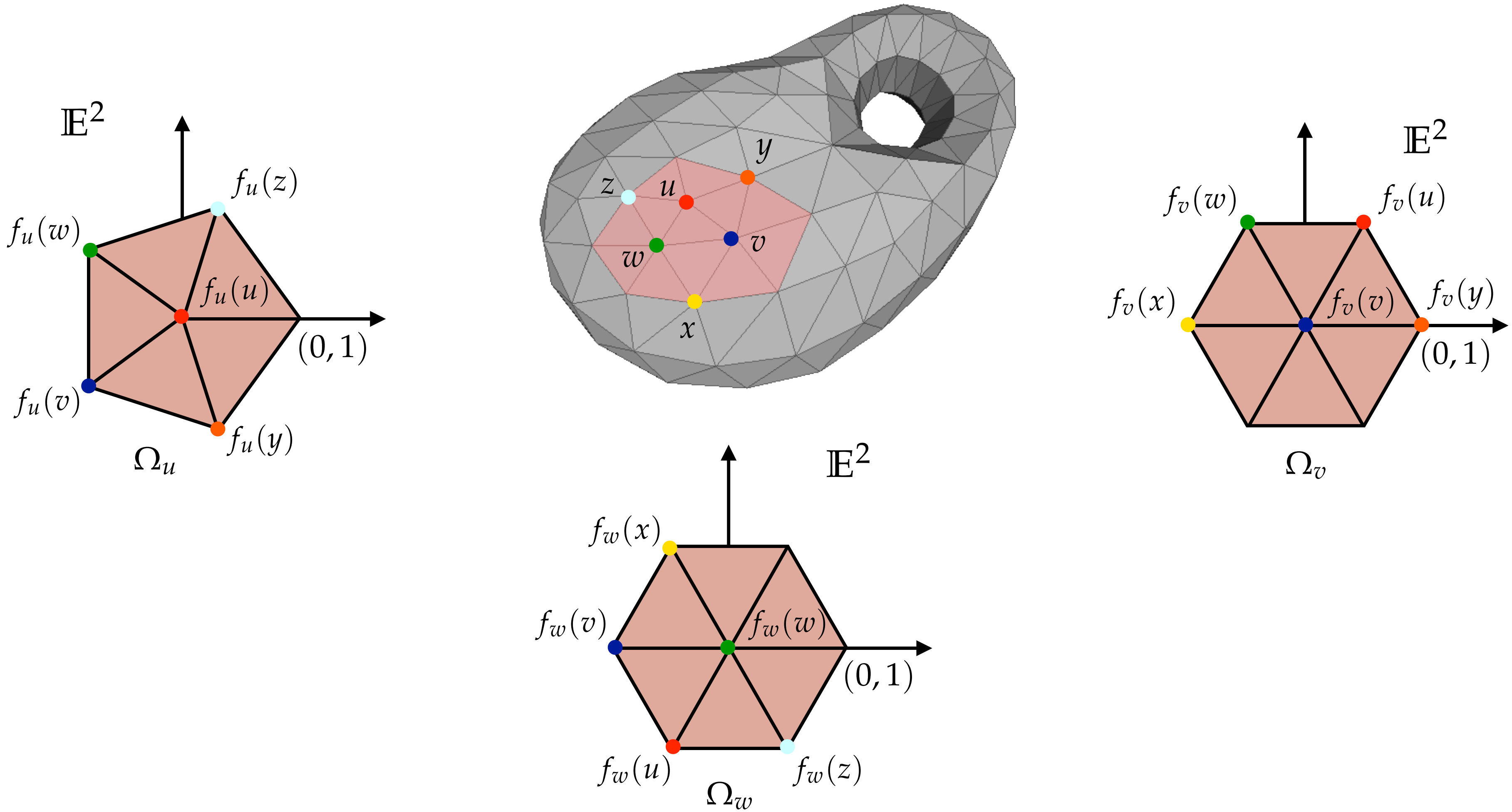
for all $x \in \Omega_{ij} \cap \Omega_{ik}$.

We cannot really verify condition 3(c) without knowing the map g_u. However, it is possible to understand what this condition requires from the map without knowing g_u.

The picture in the next slide should help us clarify the above claim.

Parametric Pseudo-Manifolds

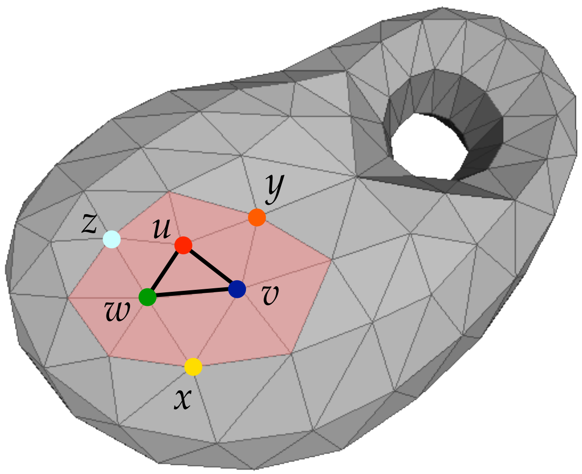
Gluing Data



Parametric Pseudo-Manifolds

Gluing Data

Note that the intersection of the stars of u, v, and w consists of exactly one triangle and its edges and vertices. This means at most 3 p-domains overlap at the same point.

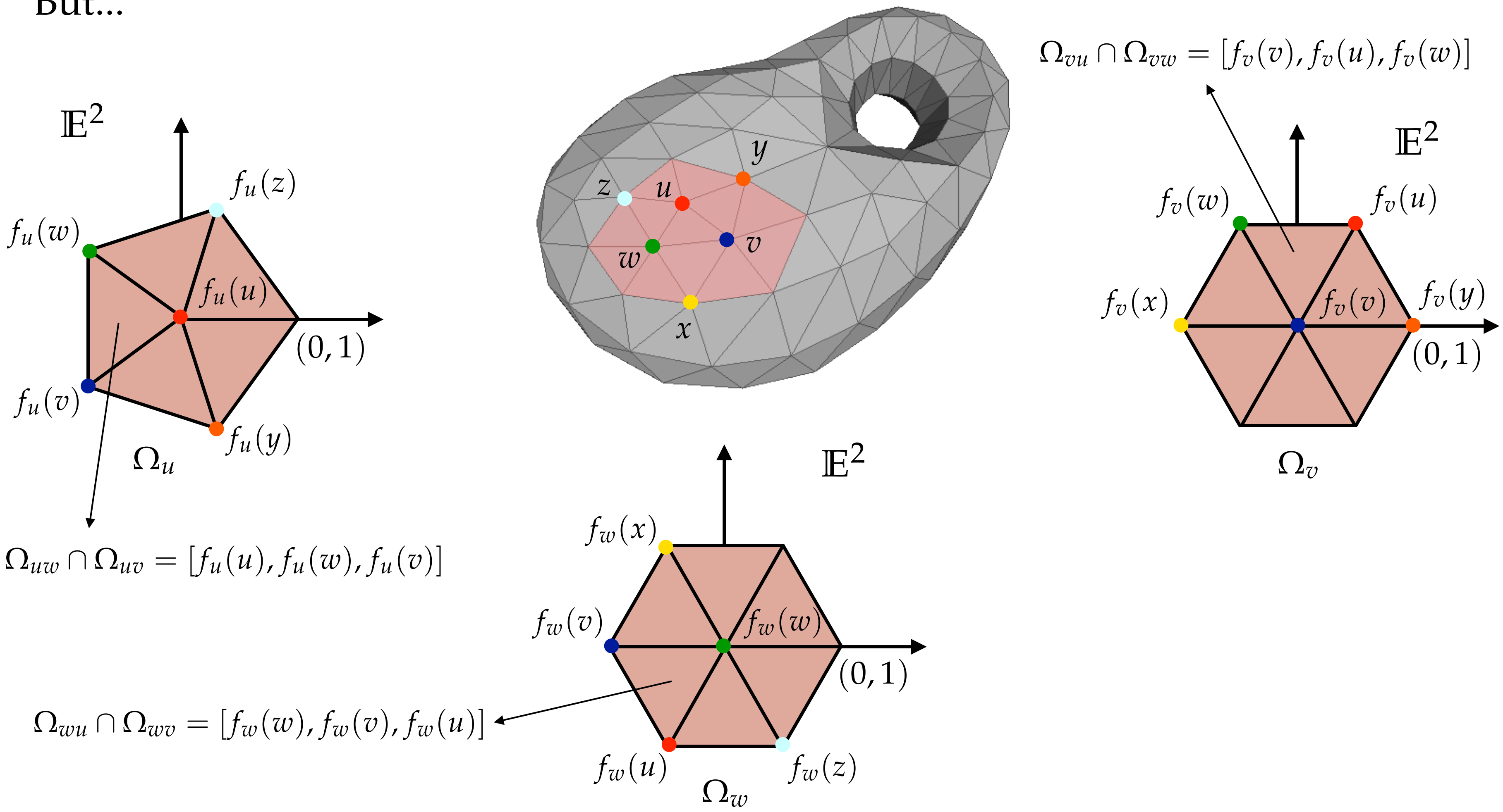


So, the cocycle condition must hold for points that belong to the triangles $\Omega_{uw} \cap \Omega_{uv}$, $\Omega_{vu} \cap \Omega_{vw}$, and $\Omega_{wu} \cap \Omega_{wv}$, which are pairwise identified by the transition functions.

Parametric Pseudo-Manifolds

Gluing Data

But...



Parametric Pseudo-Manifolds

Gluing Data

For the cocycle condition...

(c) For all i, j, k, if

$$\Omega_{ji} \cap \Omega_{jk} \neq \emptyset \,,$$

then

$$\varphi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik} \qquad \text{and} \qquad \varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x) \,,$$

for all $x \in \Omega_{ij} \cap \Omega_{ik}$.

let us set $i = u$, $w = j$, and $v = k$.

Parametric Pseudo-Manifolds

Gluing Data

So, we must show that

(c) for all u, v, and w such that $[u, v, w]$ is a triangle of $\mathcal{K}$, if

$$\Omega_{wu} \cap \Omega_{wv} \neq \emptyset$$

then

$$\varphi_{uw}(\Omega_{wu} \cap \Omega_{wv}) = \Omega_{uv} \cap \Omega_{uw} \quad \text{and} \quad \varphi_{vu}(p) = (\varphi_{vw} \circ \varphi_{wu})(p)\,,$$

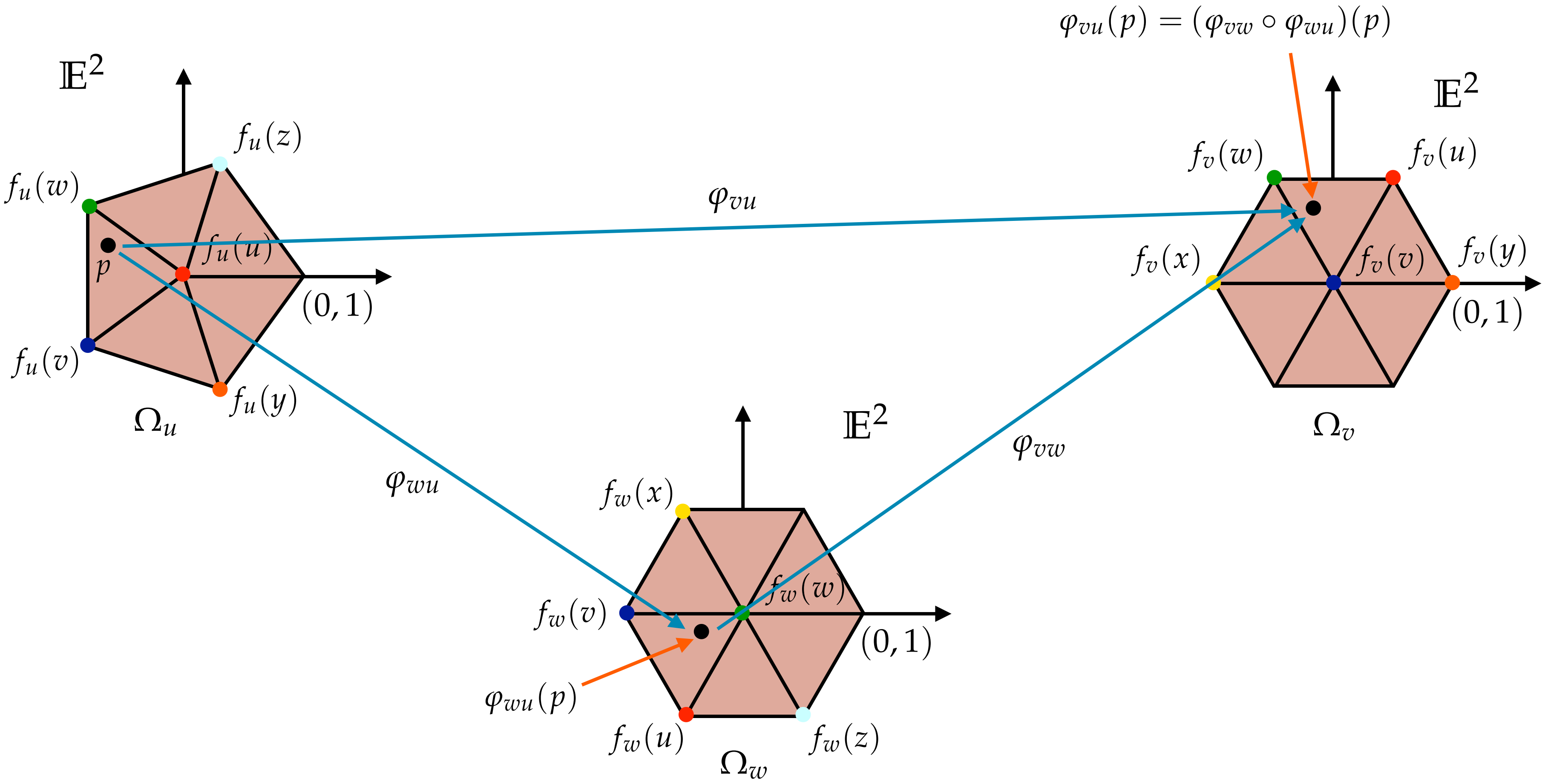
for every $p \in \Omega_{uv} \cap \Omega_{uw}$.

Assume that the statement "if $\Omega_{wu} \cap \Omega_{wv} \neq \emptyset$ then $\varphi_{uw}(\Omega_{wu} \cap \Omega_{wv}) = \Omega_{uv} \cap \Omega_{uw}$" holds, and consider the condition "$\varphi_{vu}(p) = (\varphi_{vw} \circ \varphi_{wu})(p)$, for every $p \in \Omega_{uv} \cap \Omega_{uw}$".

Parametric Pseudo-Manifolds

Gluing Data

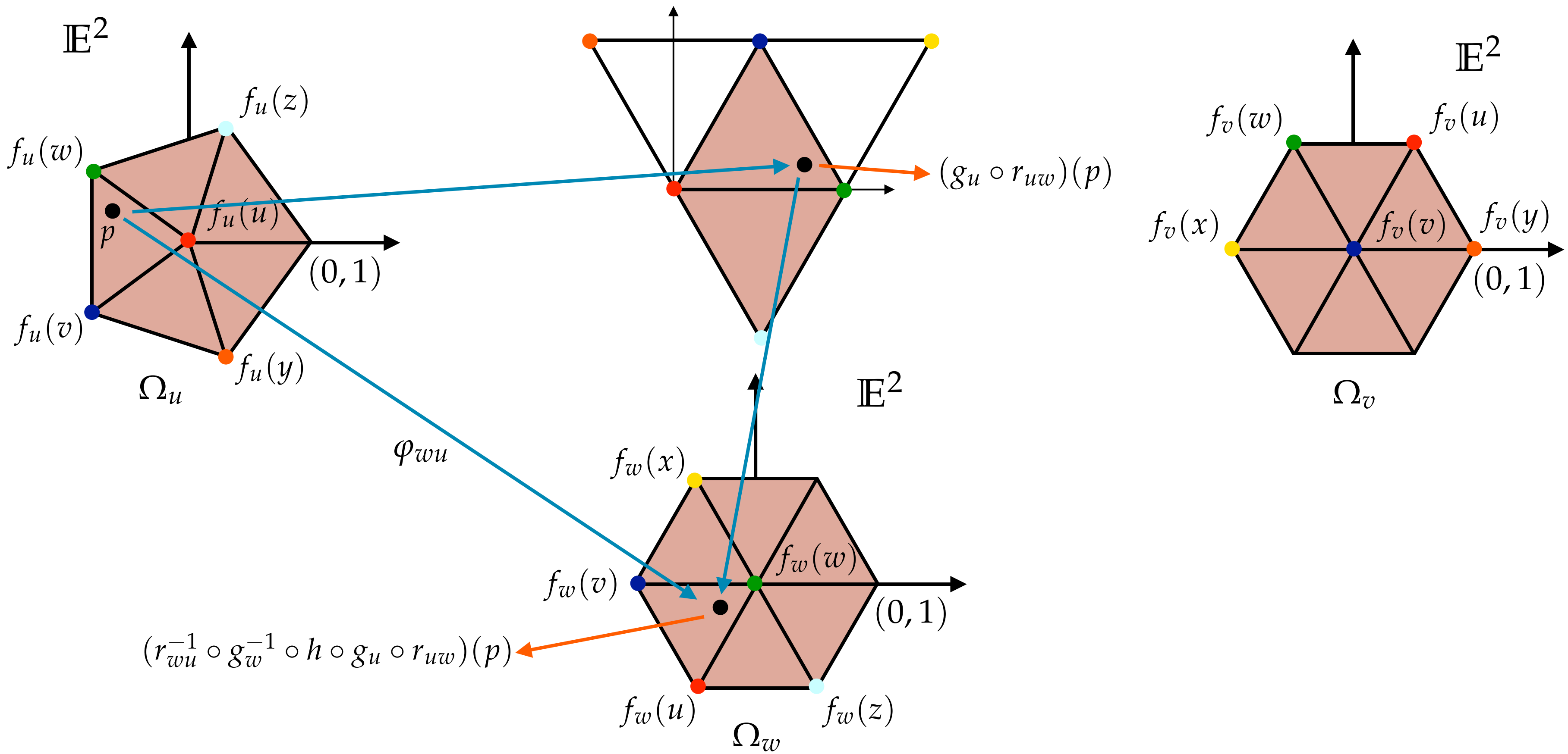
An illustration:



Parametric Pseudo-Manifolds

Gluing Data

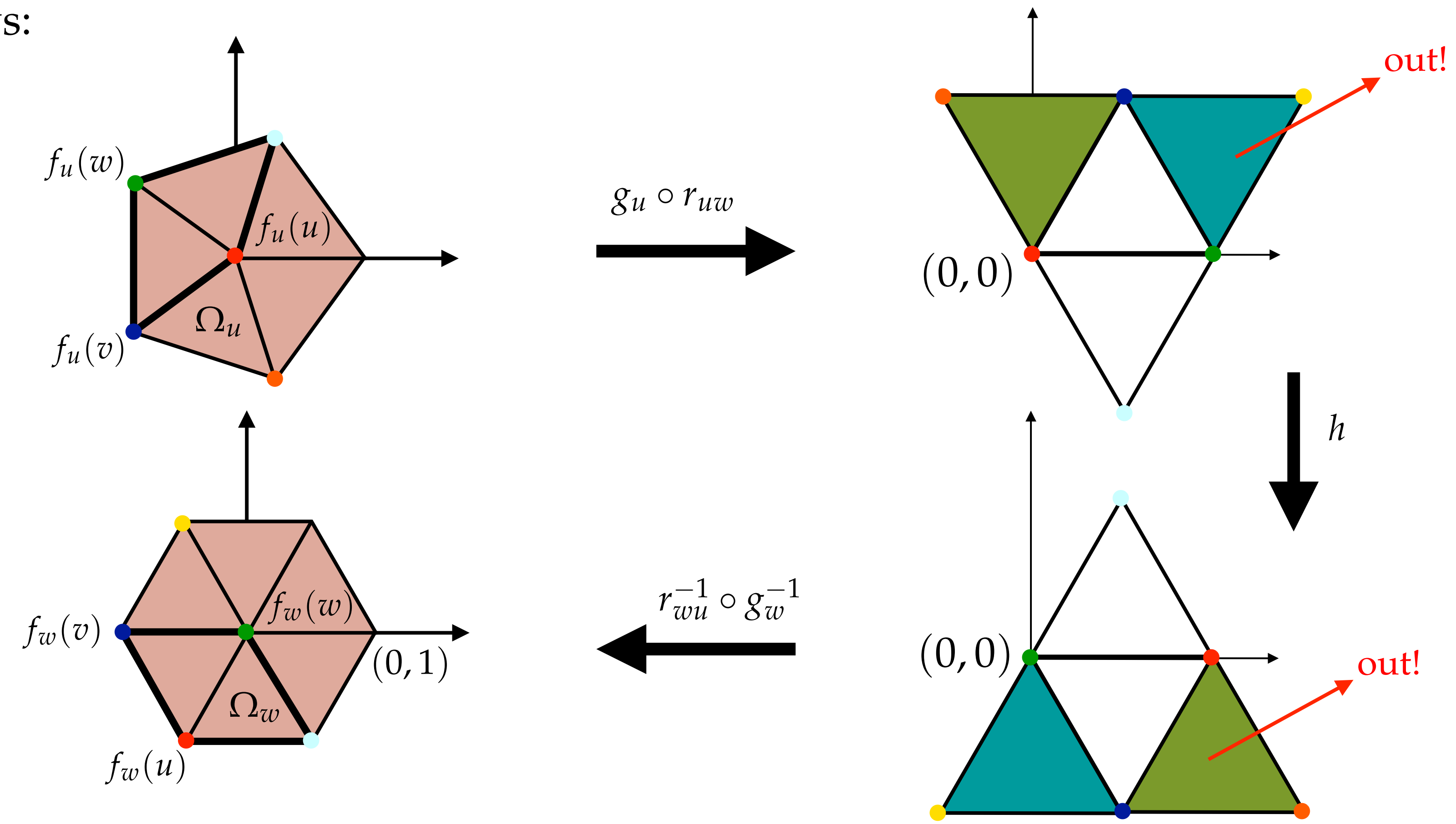
What is the "anatomy" of this composition?



Parametric Pseudo-Manifolds

Gluing Data

So, the composition $r_{wu}^{-1} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw}$ applied to Ω_{uw} can be visualized as follows:



Parametric Pseudo-Manifolds

Gluing Data

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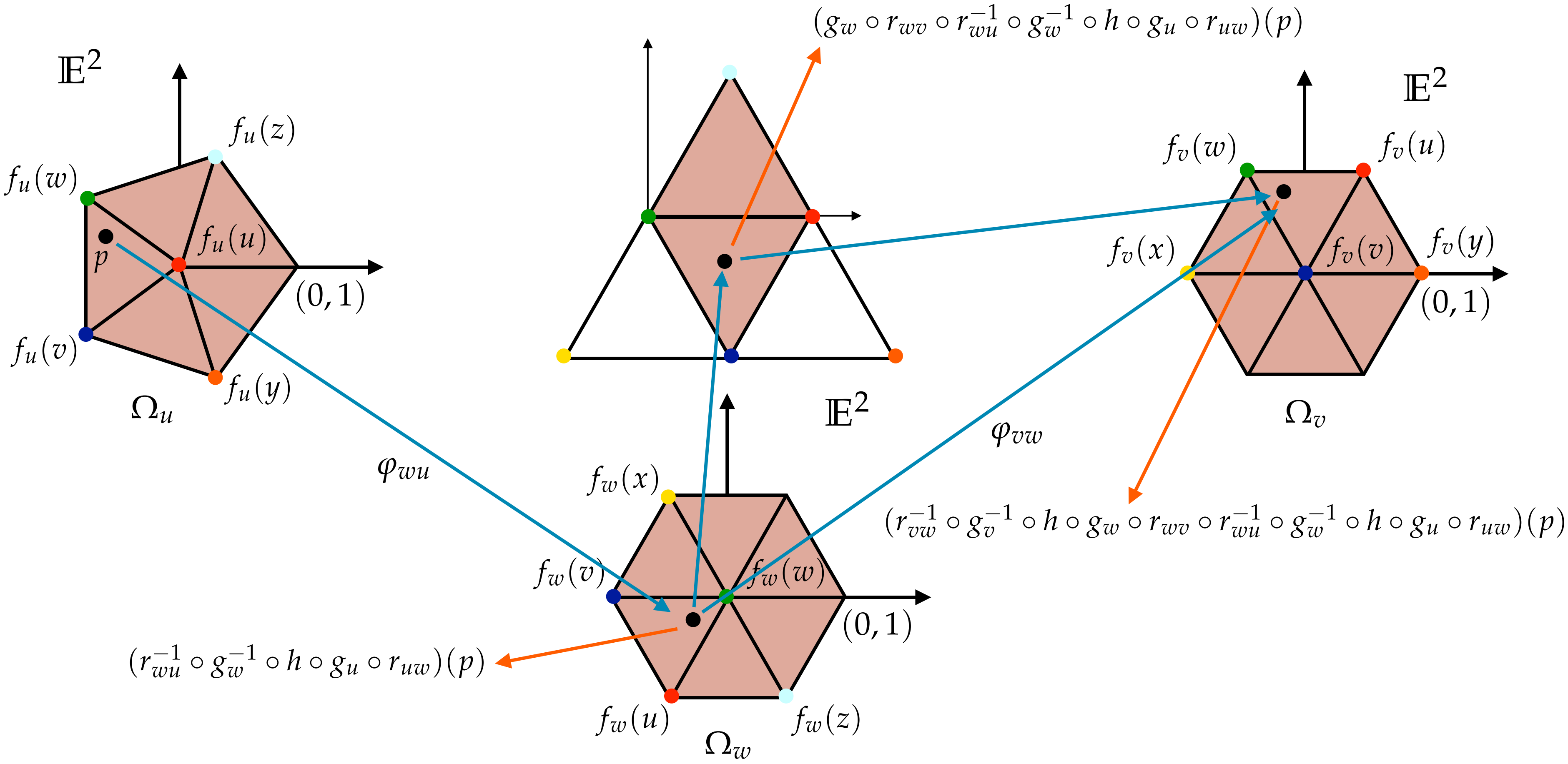
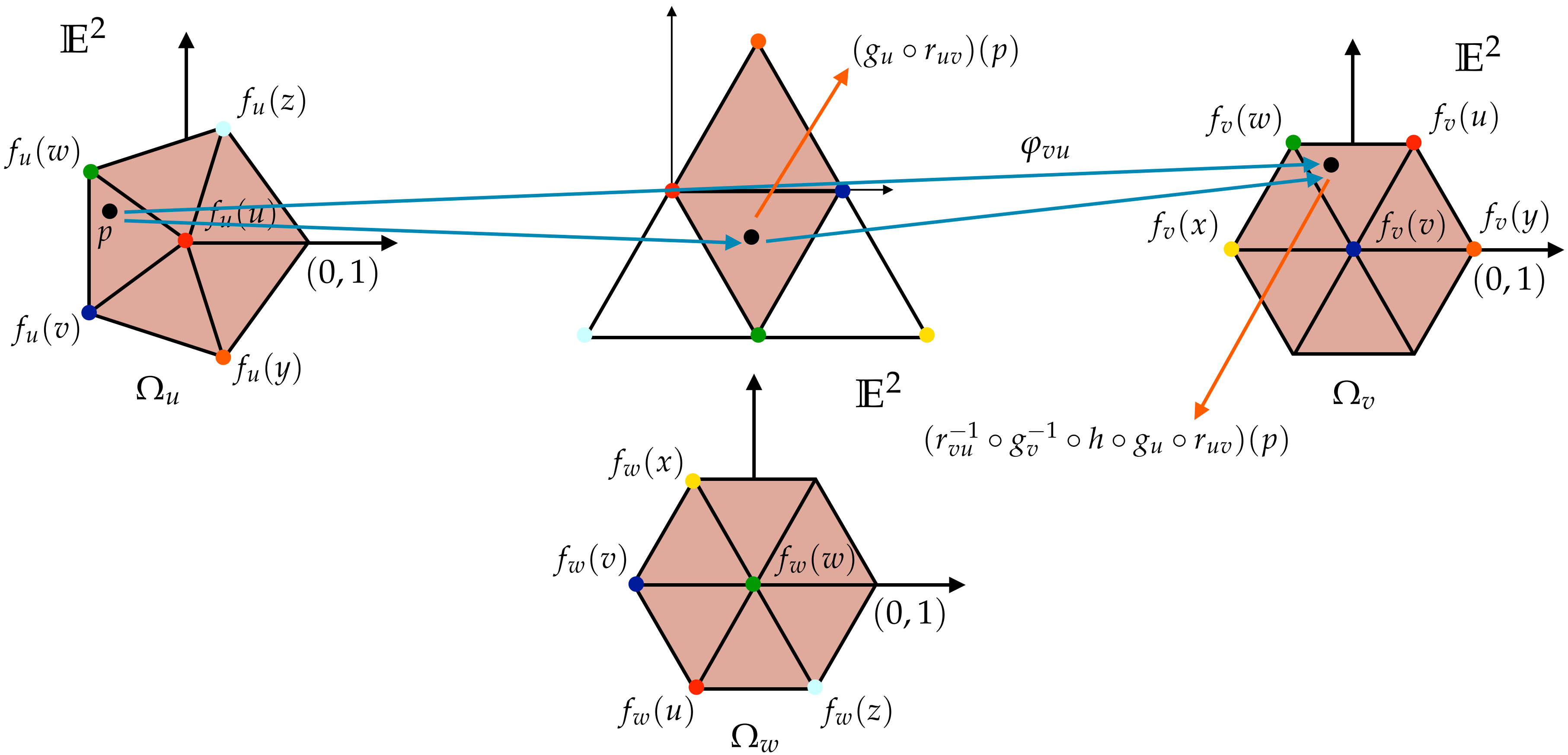


Parametric Pseudo-Manifolds

Gluing Data

What is the "anatomy" of this composition?



Parametric Pseudo-Manifolds

Gluing Data

So, we must show that

$$(r_{vu}^{-1} \circ g_v^{-1} \circ h \circ g_u \circ r_{uv})(p) = (r_{vw}^{-1} \circ g_v^{-1} \circ h \circ g_w \circ r_{wv} \circ r_{wu}^{-1} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw})(p) \, .$$

This means that if you have a *candidate* for the g map, you must verify if the above expression holds for it. Obviously, this expression may be too complicated to be useful.

Nevertheless, it can help us derive a simple sufficient condition for testing candidate maps.

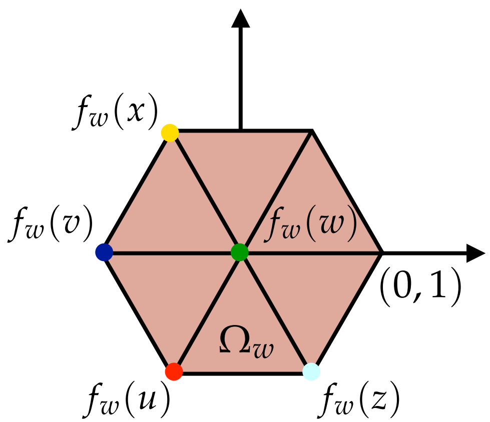
Parametric Pseudo-Manifolds

Gluing Data

We want to show that

$$(r_{vu}^{-1} \circ g_v^{-1} \circ h \circ g_u \circ r_{uv})(p) = (r_{vw}^{-1} \circ g_v^{-1} \circ h \circ g_w \circ r_{wv} \circ r_{wu}^{-1} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw})(p)\,.$$

First, note that $r_{wv} \circ r_{wu}^{-1}$ is equal to a rotation of $\frac{2\pi}{n_w}$ around $(0,0)$.



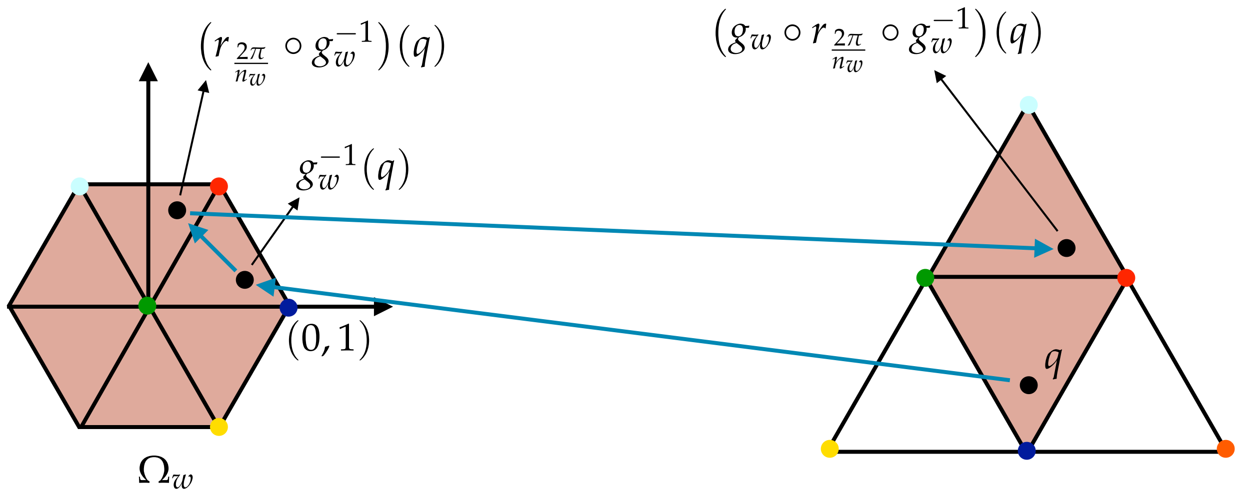
So,

$$r_{vw}^{-1} \circ g_v^{-1} \circ h \circ g_w \circ r_{wv} \circ r_{wu}^{-1} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw} = r_{vw}^{-1} \circ g_v^{-1} \circ h \circ g_w \circ r_{\frac{2\pi}{n_w}} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw}\,.$$

Parametric Pseudo-Manifolds

Gluing Data

Now, consider the composition $(g_w \circ r_{\frac{2\pi}{n_w}} \circ g_w^{-1})(q)$, where q is a point in $g_w(\Omega_{uw} \cap \Omega_{uv})$.



This picture suggests that $(g_w \circ r_{\frac{2\pi}{n_w}} \circ g_w^{-1})(q) = r_{\frac{\pi}{3}}(q)$ might be a reasonable choice.

Parametric Pseudo-Manifolds

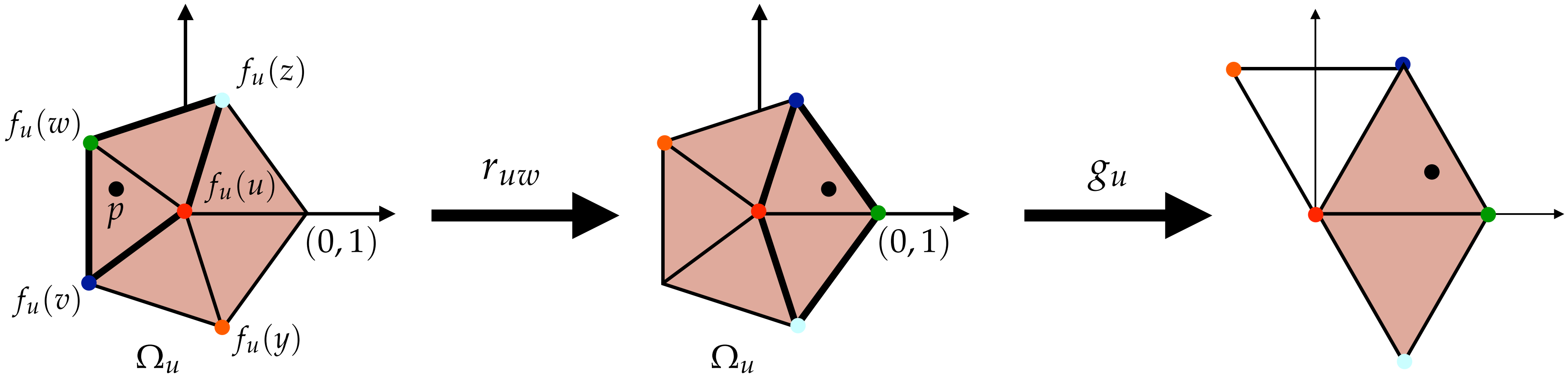
Gluing Data

So,

$$r_{vw}^{-1} \circ g_v^{-1} \circ h \circ g_w \circ r_{\frac{2\pi}{n_w}} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw} = r_{vw}^{-1} \circ g_v^{-1} \circ h \circ r_{\frac{\pi}{3}} \circ h \circ g_u \circ r_{uw} \,.$$

The same assumption we made before leads us to one more property:

$$(g_u \circ r_{uw})(p) = (r_{\frac{\pi}{3}} \circ g_u \circ r_{uv})(p), \text{ for every } p \in \Omega_{uw}.$$



Parametric Pseudo-Manifolds

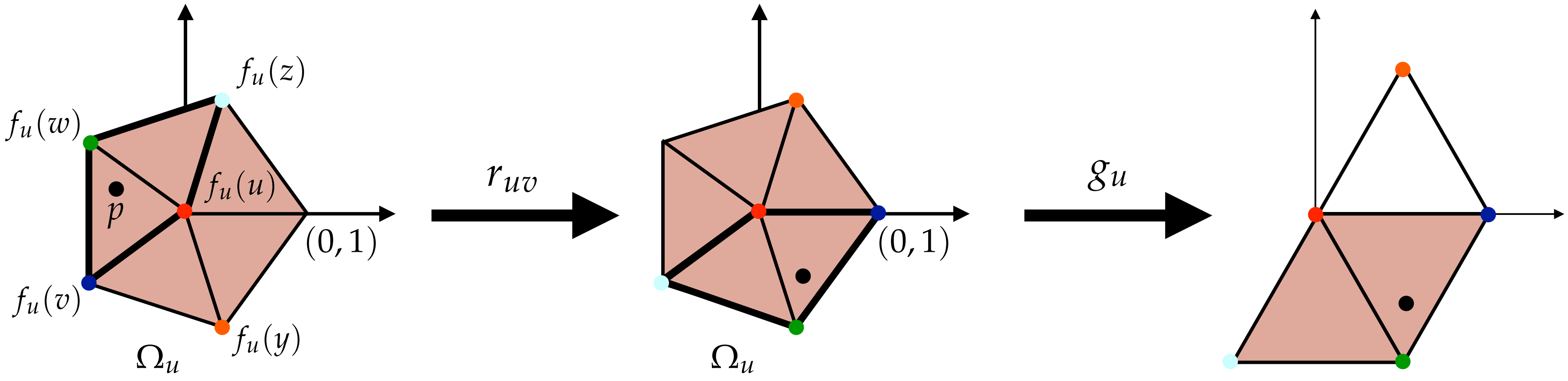
Gluing Data

So,

$$r_{vw}^{-1} \circ g_v^{-1} \circ h \circ g_w \circ r_{\frac{2\pi}{n_w}} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw} = r_{vw}^{-1} \circ g_v^{-1} \circ h \circ r_{\frac{\pi}{3}} \circ h \circ g_u \circ r_{uw} \,.$$

The same assumption we made before leads us to one more property:

$$(g_u \circ r_{uw})(p) = (r_{\frac{\pi}{3}} \circ g_u \circ r_{uv})(p), \text{ for every } p \in \Omega_{uw}.$$



Parametric Pseudo-Manifolds

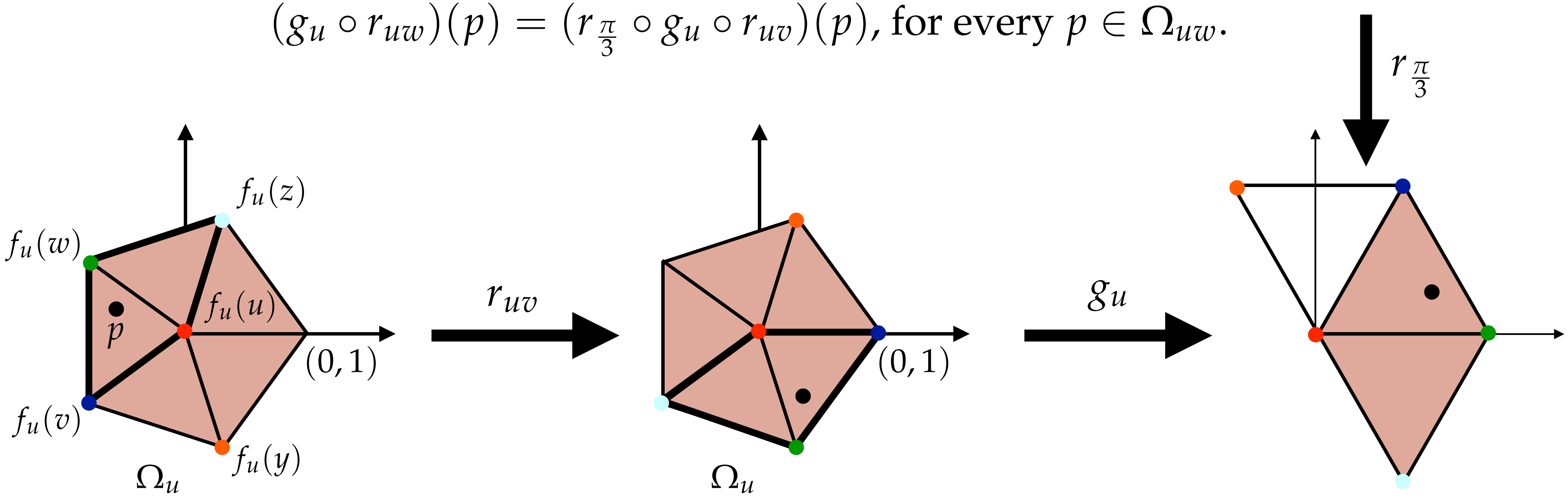
Gluing Data

So,

$$r_{vw}^{-1} \circ g_v^{-1} \circ h \circ g_w \circ r_{\frac{2\pi}{n_w}} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw} = r_{vw}^{-1} \circ g_v^{-1} \circ h \circ r_{\frac{\pi}{3}} \circ h \circ g_u \circ r_{uw} \, .$$

The same assumption we made before leads us to one more property:

$$(g_u \circ r_{uw})(p) = (r_{\frac{\pi}{3}} \circ g_u \circ r_{uv})(p), \text{ for every } p \in \Omega_{uw}.$$



Parametric Pseudo-Manifolds

Gluing Data

So,

$$r_{vw}^{-1} \circ g_v^{-1} \circ h_{wv} \circ g_w \circ r_{\frac{2\pi}{n_w}} \circ g_w^{-1} \circ h_{uw} \circ g_u \circ r_{uw} = r_{vw}^{-1} \circ g_v^{-1} \circ h_{wv} \circ r_{\frac{\pi}{3}} \circ h_{uw} \circ r_{\frac{\pi}{3}} \circ g_u \circ r_{uv} \, .$$

For the exact same reason, $g_v \circ r_{vw}(p) = r_{-\frac{\pi}{3}} \circ g_u \circ r_{vu}(p)$, for every $p \in \Omega_{vw}$. So,

$$(r_{vw}^{-1} \circ g_v^{-1})(p) = (r_{vu}^{-1} \circ g_v^{-1} \circ r_{\frac{\pi}{3}})(p) \, .$$

So,

$$r_{vw}^{-1} \circ g_v^{-1} \circ h \circ g_w \circ r_{\frac{2\pi}{n_w}} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw} = r_{vu}^{-1} \circ g_v^{-1} \circ r_{\frac{\pi}{3}} \circ h \circ r_{\frac{\pi}{3}} \circ h \circ r_{\frac{\pi}{3}} \circ g_u \circ r_{uv} \, .$$

But,

$$h = r_{\frac{\pi}{3}} \circ h \circ r_{\frac{\pi}{3}} \circ h \circ r_{\frac{\pi}{3}} \, .$$

Parametric Pseudo-Manifolds

Gluing Data

So,

$$
\begin{aligned}
r_{vw}^{-1} \circ g_v^{-1} \circ h \circ g_w \circ r_{\frac{2\pi}{n_w}} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw} &= r_{vu}^{-1} \circ g_v^{-1} \circ r_{\frac{\pi}{3}} \circ h \circ r_{\frac{\pi}{3}} \circ h \circ r_{\frac{\pi}{3}} \circ g_u \circ r_{uv} \\
&= r_{vu}^{-1} \circ g_v^{-1} \circ h \circ g_u \circ r_{uv} \\
&= \varphi_{vu} \,.
\end{aligned}
$$

The above equality holds for every point p in $\Omega_{uv} \cap \Omega_{uw}$, and hence the cocycle condition,

$$
\varphi_{vu}(p) = (\varphi_{vw} \circ \varphi_{wu})(p) \,,
$$

is satisfied (under the many assumptions we made along the way; which are they?)

Parametric Pseudo-Manifolds

Gluing Data

We made the following assumptions:

(1) The g_u map are is a C^k-diffeomorphism of the plane, for every $u \in I$.

(2) The g_u map takes $\overset{\circ}{Q}_{uw}$ onto $\overset{\circ}{Q}$, for every $(u, w) \in K$.

(3) The g_u map satisfies $(g_u \circ r_{\frac{2\pi}{n_u}} \circ g_u^{-1})(q) = r_{\frac{\pi}{3}}(q)$, where $q \in g_u(\Omega_u)$.

(4) If $f_u(w)$ precedes $f_u(v)$ in a counterclockwise enumeration of the vertices of $\mathcal{K}_u$, then $(g_u \circ r_{uw})(p) = (r_{\frac{\pi}{3}} \circ g_u \circ r_{uv})(p)$, for every point p in the gluing domain Ω_{uw}.

(5) For all u, v, w such that $[u, v, w]$ is a triangle of $\mathcal{K}$, if $\Omega_{wu} \cap \Omega_{wv} \neq \emptyset$ then

$$\varphi_{uw}(\Omega_{wu} \cap \Omega_{wv}) = \Omega_{uv} \cap \Omega_{uw} .$$

Parametric Pseudo-Manifolds

Gluing Data

It turns out that (5) can be derived from (1)-(4). We will leave that as an exercise.

So, the point of this lecture is: if we want to know if a *candidate* for the g map will satisfy condition 3 of Definition 7.1, we can test the map against conditions (1) through (4).

Conditions (3) and (4) are sufficient. Are they also necessary?

We are now fully equipped to discuss the "candidate" functions in the next lecture.

More specifically, we will go over projective and conformal transformations.