Splines, Subdivision & Manifolds

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Historical Perspective

splines *subdivision surfaces* *manifolds*

1960 1970 1980 1990 2000 2010

Bezier (60)
de Casteljau (60)

De Boor (72)
Chaikin (74)
Riesenfeld (80)

Doo-Sabin (78)
Catmull-Clark (78)
Loop (87)
Dyn (90)

Zorin (89)
Kobbelt (00)
Velho (01)
Schroeder (02)

Grimm (95)

Naveau (02)
Zorin (04)
Gu (06)
Siqueira (09)

Modeling

- Splines (Regular Surfaces)
  - Free-Form Modeling
- Subdivision (Arbitrary Surfaces)
  - Efficient Algorithms
- Manifolds (Smooth Surfaces)
  - Intrinsic Operations

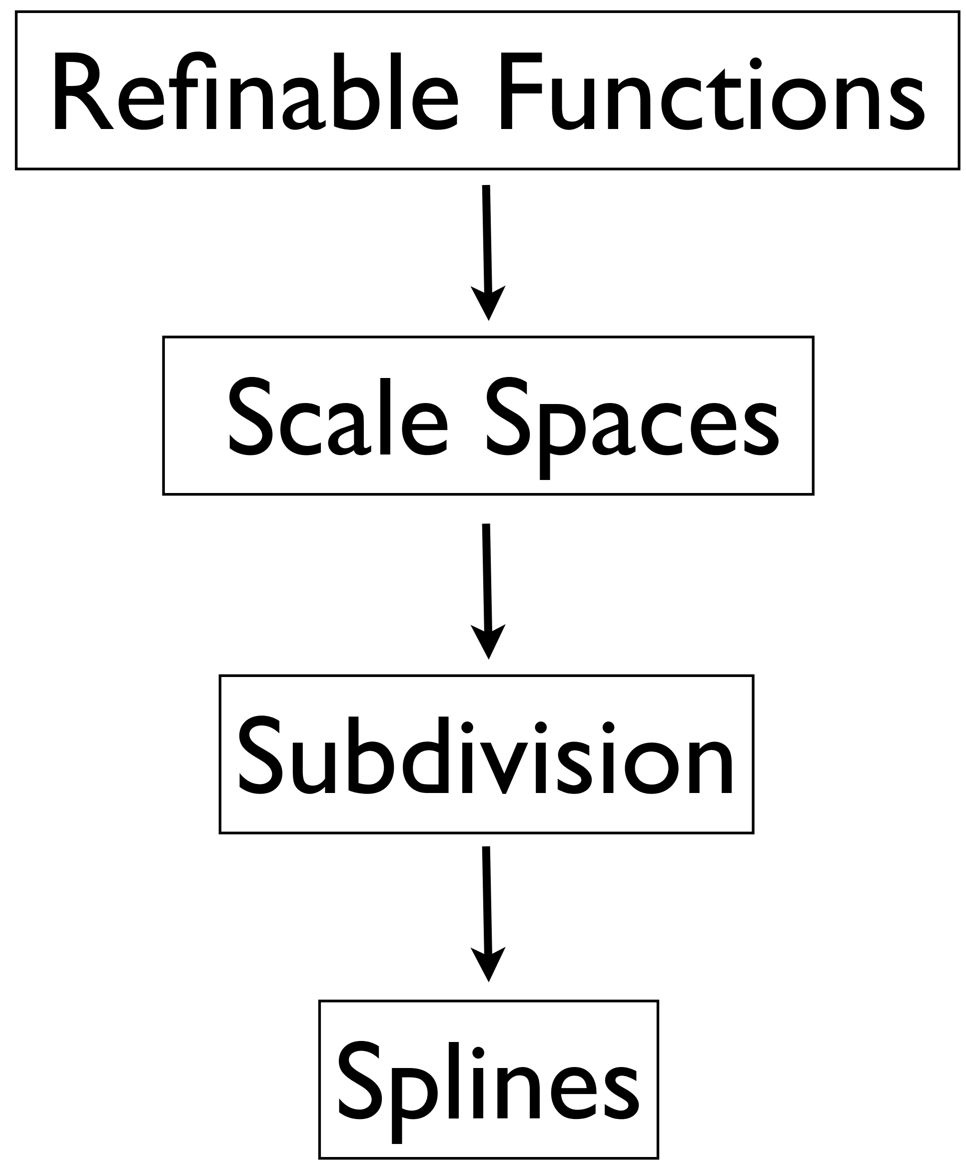
Related Developments

- Parametrizations
- Surface Deformation
- Re-meshing
- Quad Structures
- Geometry Processing
- Discrete Differential Geometry (DDG)

Scale Spaces & Wavelets

- Constructions
  - Meyer (80)
- Smoothness
  - Daubechies (88)
- Multiresolution + FWT
  - Mallat (89)
- Lifting
  - Sweldens (95)

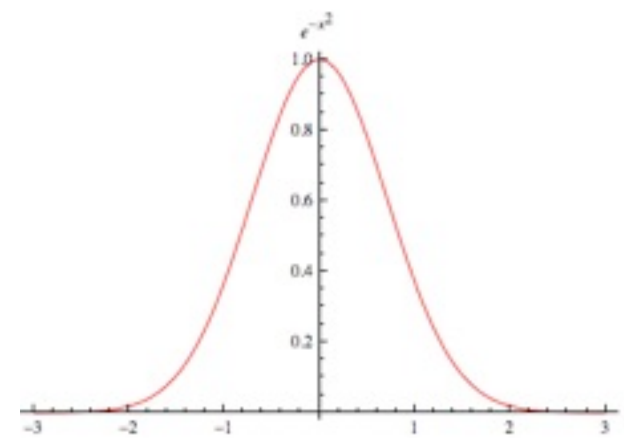
Road Map



Scaling Function

- Localization (space / frequency)

$$\phi(x)$$



- Normalization

$$\int \phi(x) dx = 1$$

Scaling Family

$$\phi_{s,t}(x) = \frac{1}{|s|^{1/2}}\phi(\frac{x}{s} - t)$$

- Two Parameters
  - Change of Scale

$$\phi_s(x) = \frac{1}{|s|^{1/2}}\phi(\frac{x}{s})$$

  - Change of Position

$$\phi_t(x) = \phi(x - t)$$

Scale Spaces

- Smoothing Operator

$$\phi_s(f(x)) = \int \phi_{s,t}(x) f(x) dt$$

- Linear Scale Space
  - Gaussian
- Physical Interpretation
  - Heat equation (Smoothing ~ Diffusion)

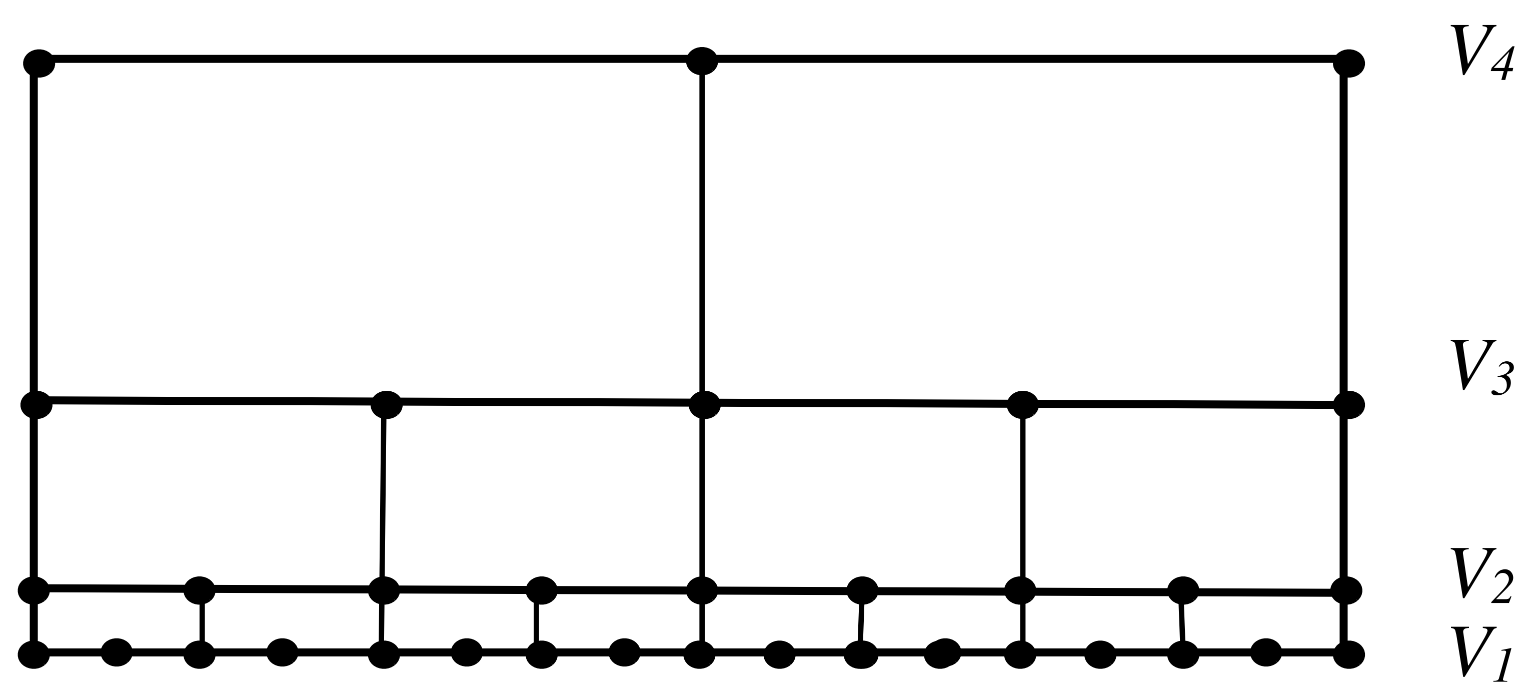
Discretization

$$\Delta_{s_0,t_0} = \{(s_0^m, ns_0^m t); m, n \in \mathbb{Z}\}$$

- Dyadic Structure

$$\Delta_{2,1} = \{(2^j, k2^j); j, k \in \mathbb{Z}\}$$

- Hyperbolic Lattice



Function Representation

- Representation Operator: $R(f) = (f_j)_{j \in \mathbb{Z}}$

$$R : L^2(\mathbb{R}) \to l^2(\mathbb{R})$$

space of functions *space of sequences*

- Approximation Spaces

$$V_j = \{\phi_{j,k}\}_{k \in \mathbb{Z}}$$

scale basis

Basis and Representation

- Orthogonal Projection ~ Basis V_j

$$\mathrm{Proj}_{v_j}(f) = \sum_k < f, \phi_{j,k} >= \sum_k f_k^j \phi_{j,k}$$

$$f = \sum_k f_k^j \phi_{j,k}$$

representation

reconstruction

$$f \mapsto (f_k^j)$$

Two-Scale Relation

- Dilation Equation
  - Scaling Basis: $\phi_0 \in V_0 \subset V_{-1}$

$$\phi_0 = \sum_k < \phi_0, \phi_{-1,k} > \phi_{-1,k} = \sum_k h_k \phi_{-1,k}$$

- Reflexive Definition of Basis

$$\phi(x) = \sqrt{2} \sum_k h_k \phi(2x - k)$$

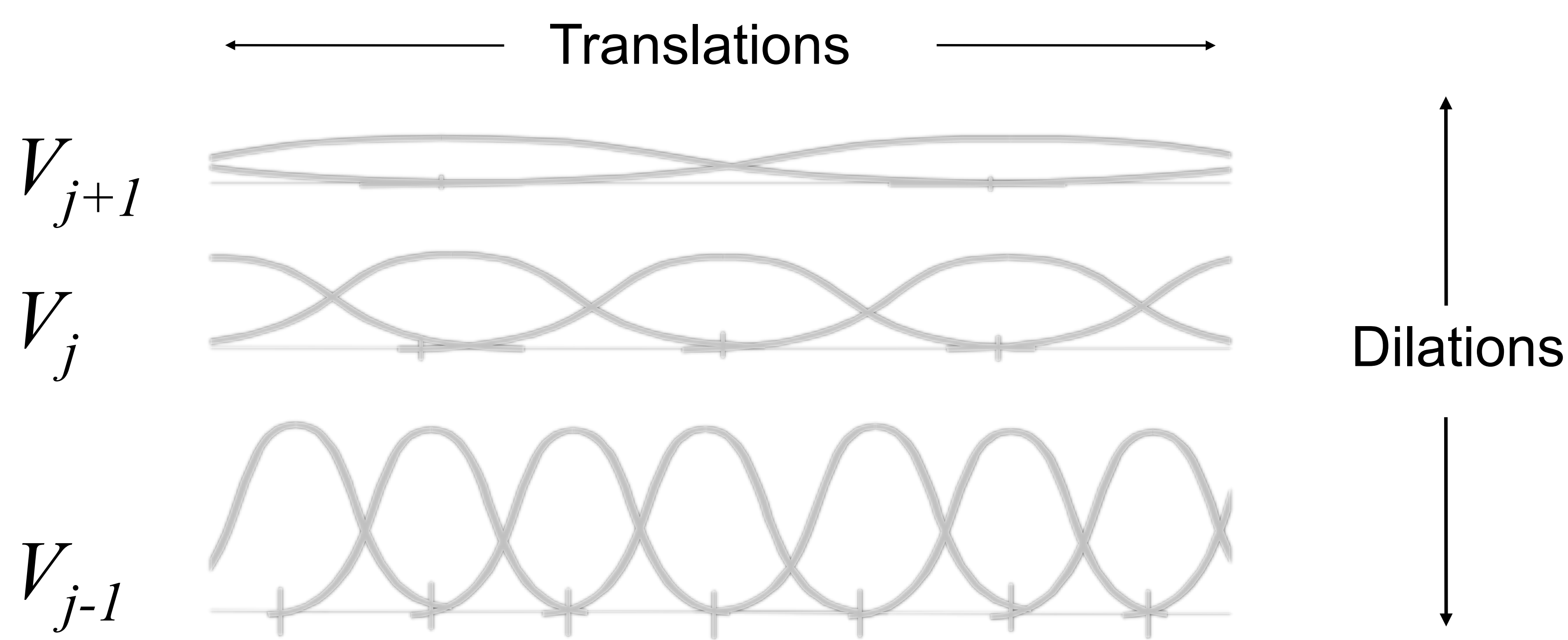
⇓

Refinable Function

Multiresolution Analysis

- Nested Approximation Spaces

$$\{0\} \subset \cdots \subset V_{j+1} \subset V_j \subset V_{j-1} \subset \cdots \subset L^2(\mathbb{R})$$



$$\phi_{j,k}(x) = 2^{-2/j}\phi(2^{-j}x - k)$$

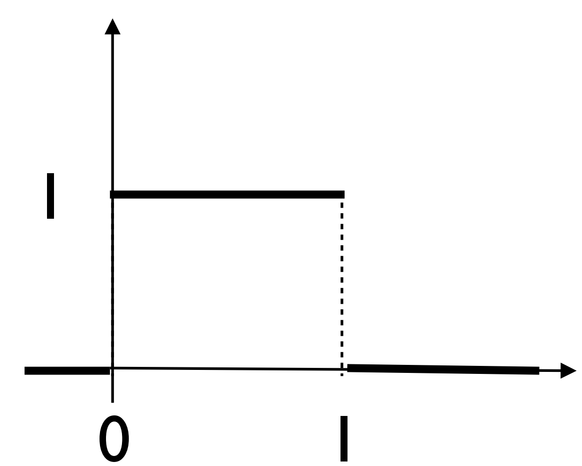
B-Spline Basis

- Refinable Scaling Functions
- Basis of P^m
  - Uniform Piecewise Polynomials
- Properties
  - Smoothness
  - Compact Support
  - Normalization
  - Partition of Unity

B-Splines

- Def: (*repeated integration*)
  - B-Spline of order 1 (Haar)

$$n^1(x) = \begin{cases} 1 & 0 \geq x > 1; \\ 0 & \text{otherwise.} \end{cases}$$



  - B-Spline of order $m > 1$

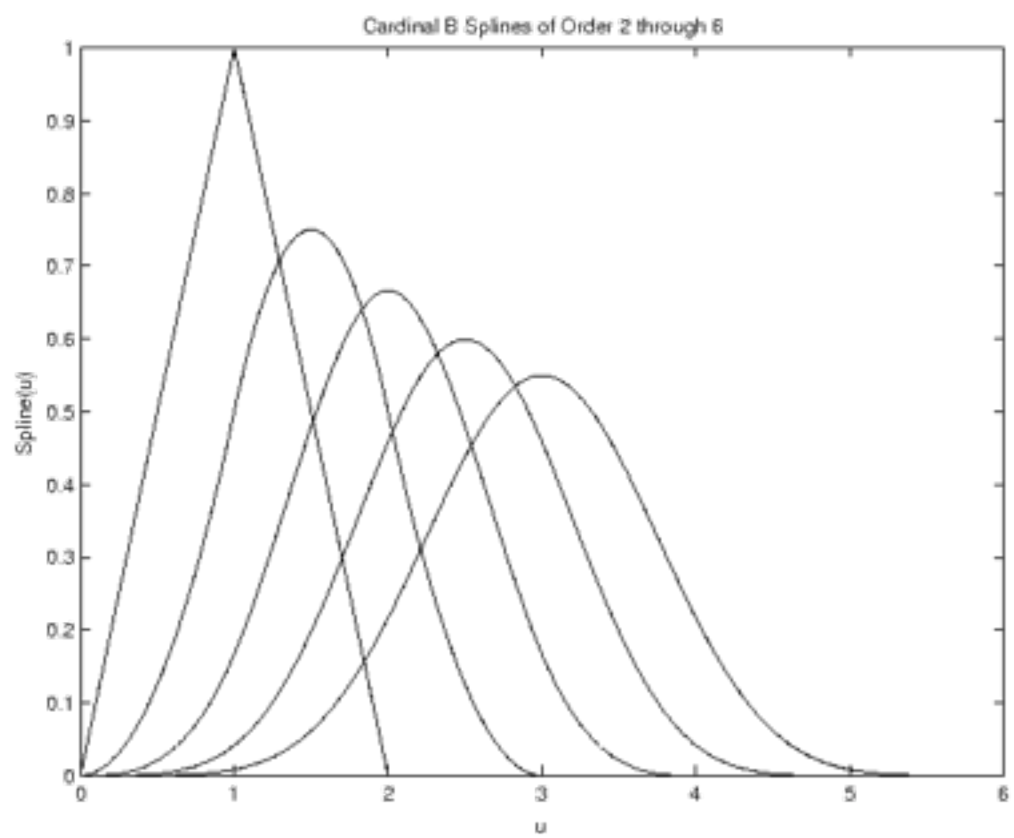
$$n^m(x) = \int n^{m-1}(x-t)dt$$

Obs: recurrence relation

B-Splines & Gaussian

- Theorem:

$$\lim_{m \to \infty} n^m(x) = G(x)$$



B-Spline Subdivision

- Refinement Relation

$$n^m(x) = \sum_{k=0}^{m} S_k^m n^m(2x - k)$$

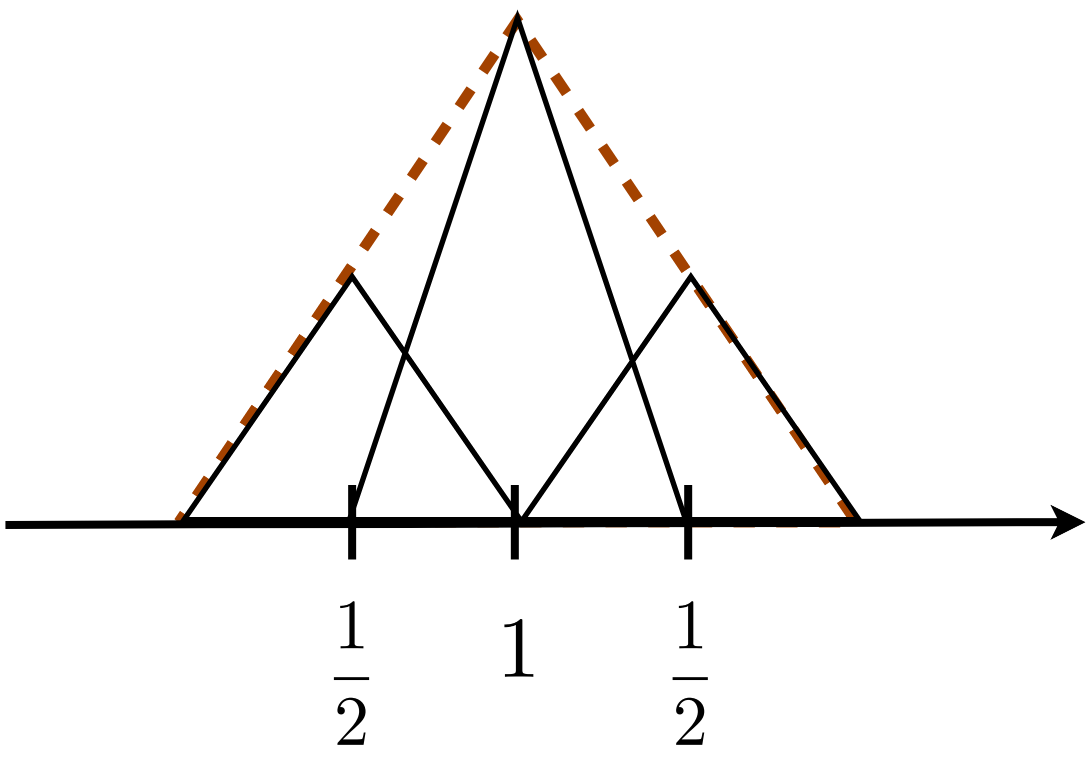
- Subdivision Mask

$$S_k^m = \frac{1}{2^{m-1}} \binom{m}{k}$$

Refinable Funcions

- Example: Linear Spline

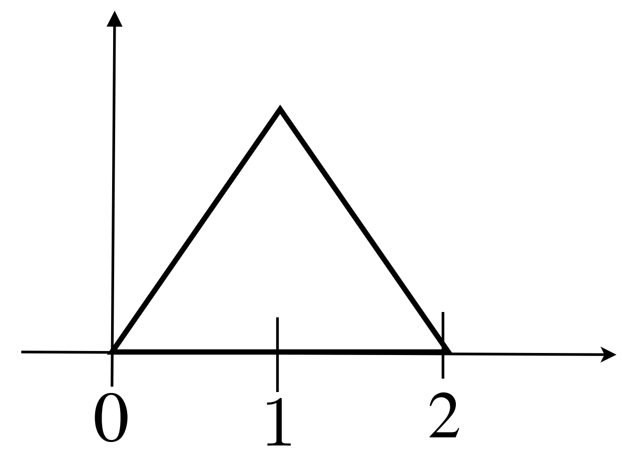
$$\phi(x) = \sqrt{2}[\frac{1}{2}\phi(2x+1) + \phi(2x) + \frac{1}{2}\phi(2x-1)]$$



Example

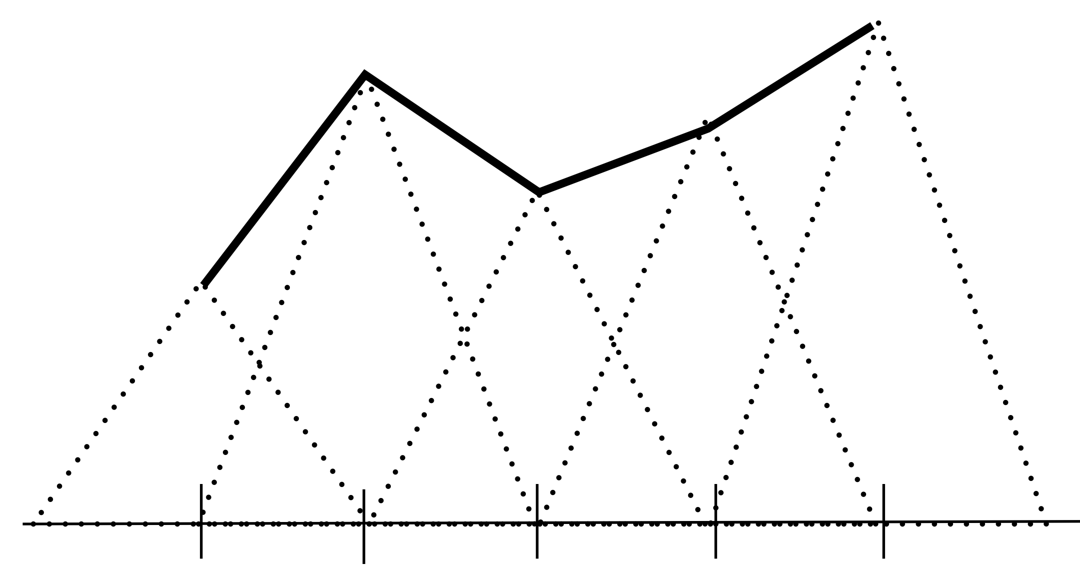
- Linear B-Spline

$$n^2(x) = \begin{cases} x & \text{if } 0 < x \geq 1; \\ 2 - x & \text{if } 1 < x \geq 2; \\ 0 & \text{otherwise} \end{cases}$$



- Rep of $p(x) = \{p_i\}$

$$p(x) = \sum p_i n^m(x - i)$$



Scaling and Refinement

- Given f in V_j

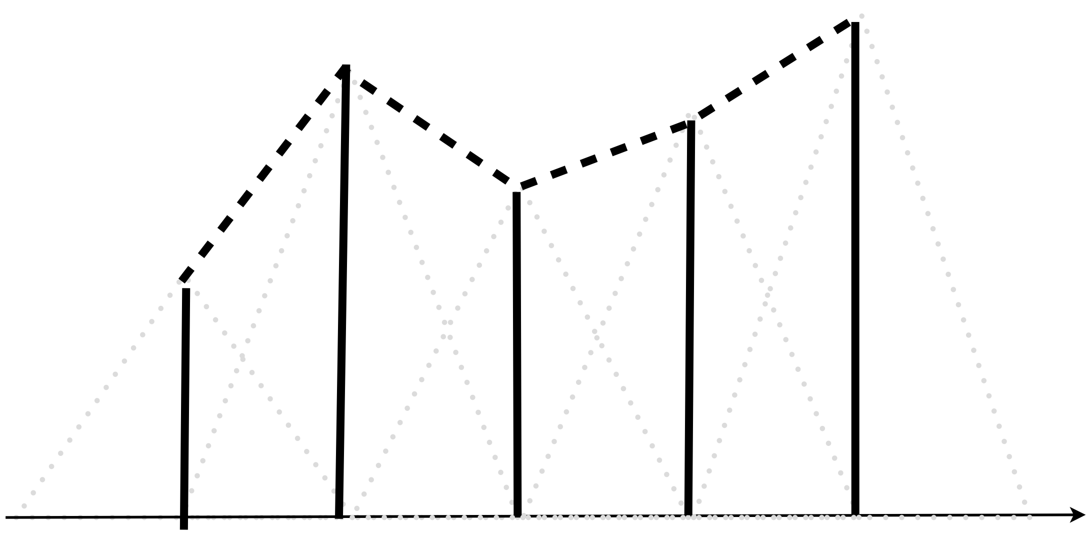
$$f(x) = \sum_k f_k^j(x)$$

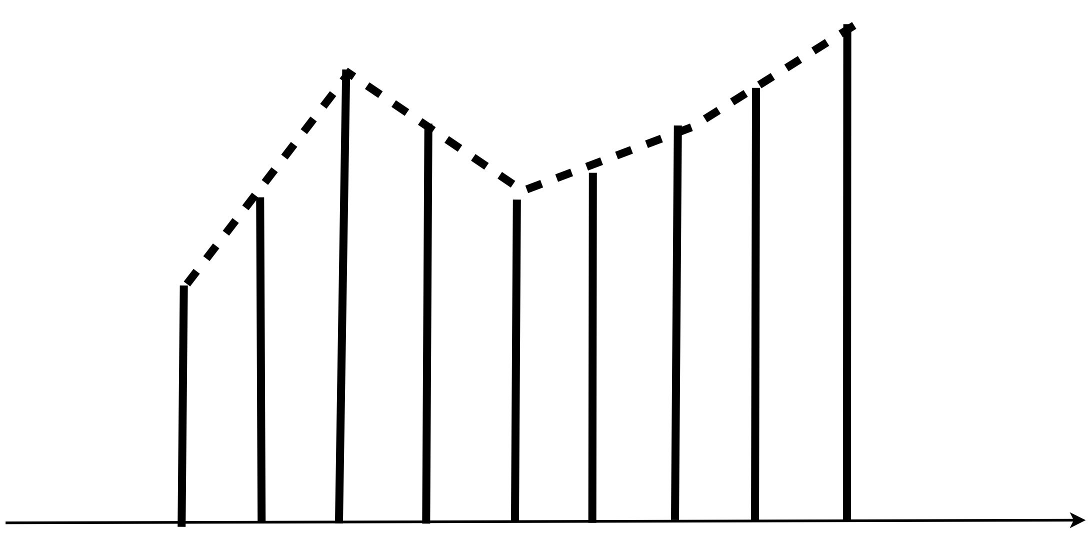
- Compute Rep of f in V_{j-1}

$$\begin{aligned} f_k^{j-1} &= < f, \phi_{j-1,k} > \\ &= < \sum_k f_k^j \phi_k^j, \phi_{j-1,k} > \\ &= \sum_k f_k^j < \phi_k^j, \phi_{j-1,k} > \\ &= \sum_k f_k^j h_k \end{aligned}$$

Refinement & Reconstruction

- Decrease Scale = Increase Resolution





Subdivision

- Reconstruction of f by Refinement *(limit process)*
  - start with $\{f_k^j\}_{k\in\mathbb{Z}}$
  - iterate $j \to -\infty$

$$f_k^{j-1} = \sum f_k^j h_k$$

subdivision operator

Elements of Subdivision

$$P_\infty = S^\infty P_0$$

- Base Shape (control points)

  $$P_0$$

- Limit Shape

  $$P_\infty$$

- Subdivision Scheme

  $$S$$

Subdivision Process

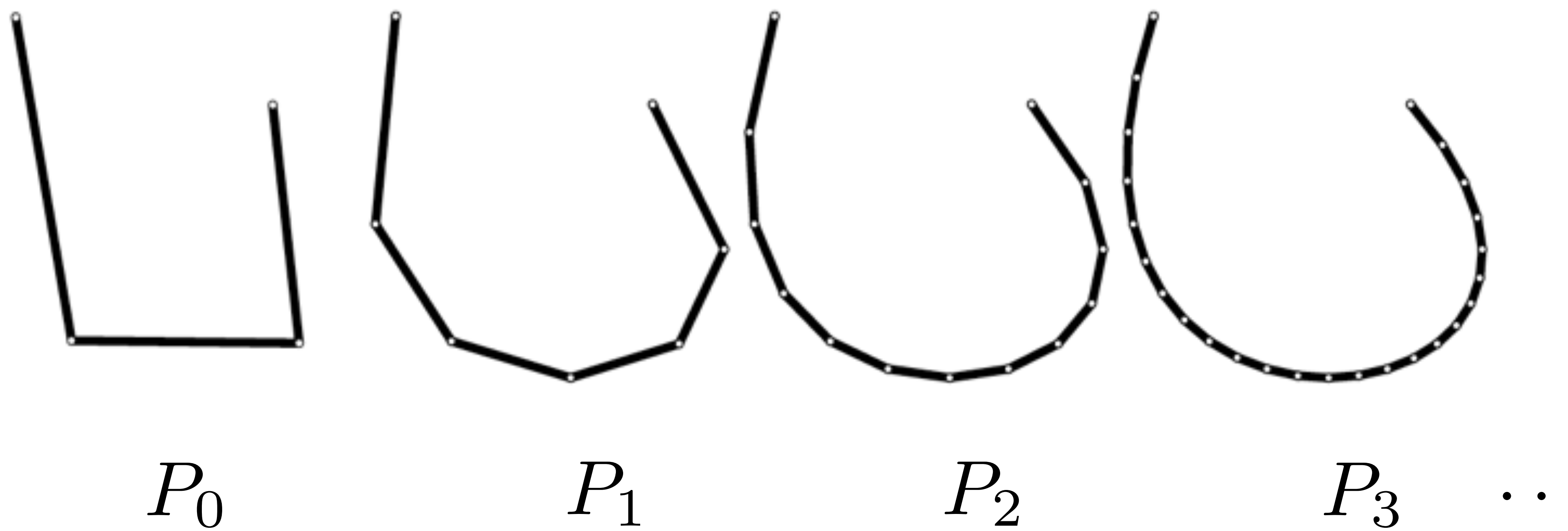
- Subdivision Iteration

$$P_k = SP_{k-1}$$

- Limit Shape

$$P_\infty = S^\infty P_0$$

Graphical Example



Subdivision Schemes

- Anatomy of Subdivision

$$S = (R, G)$$

  - R : Refinement Operator
  - G : Smoothing Operator
- Issues
  - Representation (multiresolution basis)
  - Convergence

Subdivision Algorithm

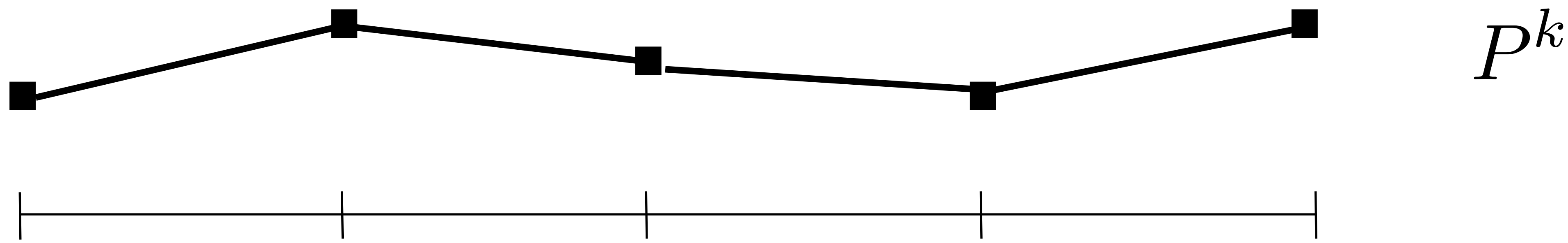
- Start with P_0
- Repeat:
  - Upsample and Change Level

$$P_j(x) = \uparrow P_{j-1}(x)$$

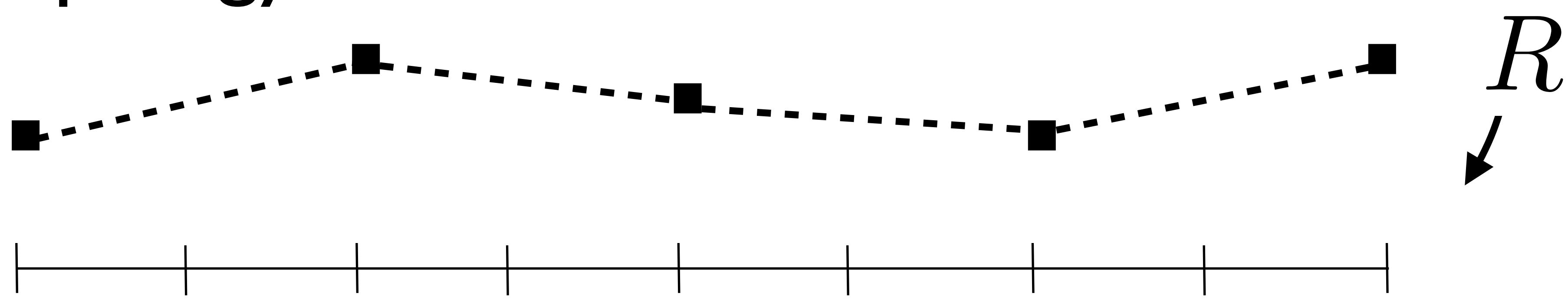
  - Update

$$P_j(x) = G P_j(x)$$

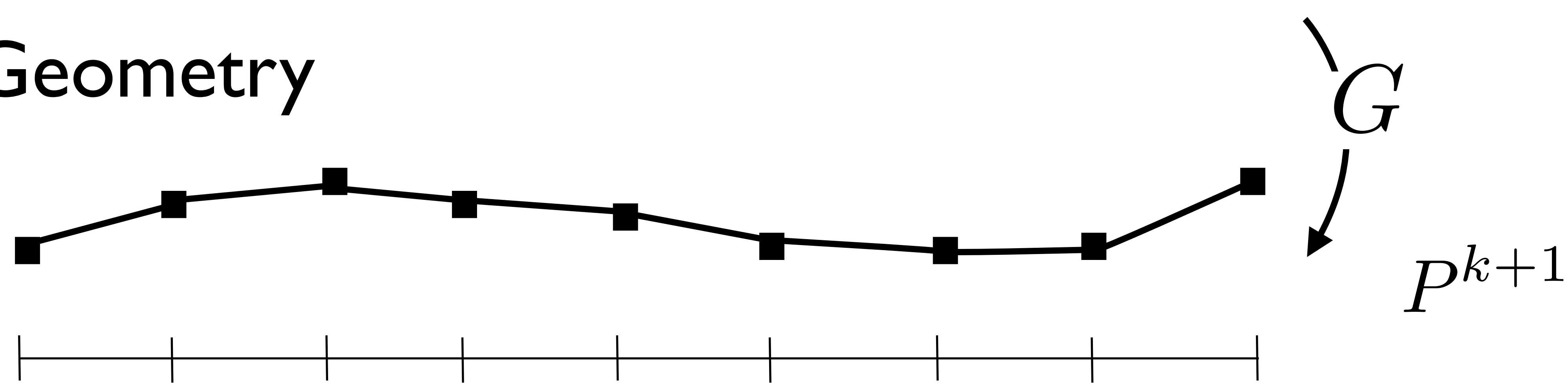
Graphical View



- Topology



- Geometry



Vector Notation

- Knot Vector

$$P_0 = (\ldots, p^0_{-1}, p^0_0, p^0_1, \ldots)$$

- Subdivision Matrix

$$\begin{pmatrix} \vdots \\ p^1_{-1} \\ p^1_0 \\ p^1_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \cdots & \begin{matrix} \cdot \\ \cdot \\ s_2 \\ \\ \\ \end{matrix} & \begin{matrix} s_{-2} \\ s_{-1} \\ s_0 \\ s_1 \\ s_2 \end{matrix} & \begin{matrix} \\ \\ s_{-2} \\ \cdot \\ \cdot \end{matrix} & \cdots \end{pmatrix} \bullet \begin{pmatrix} \vdots \\ p^0_{-1} \\ p^0_0 \\ p^0_1 \\ \vdots \end{pmatrix}$$

refinement *smoothing*

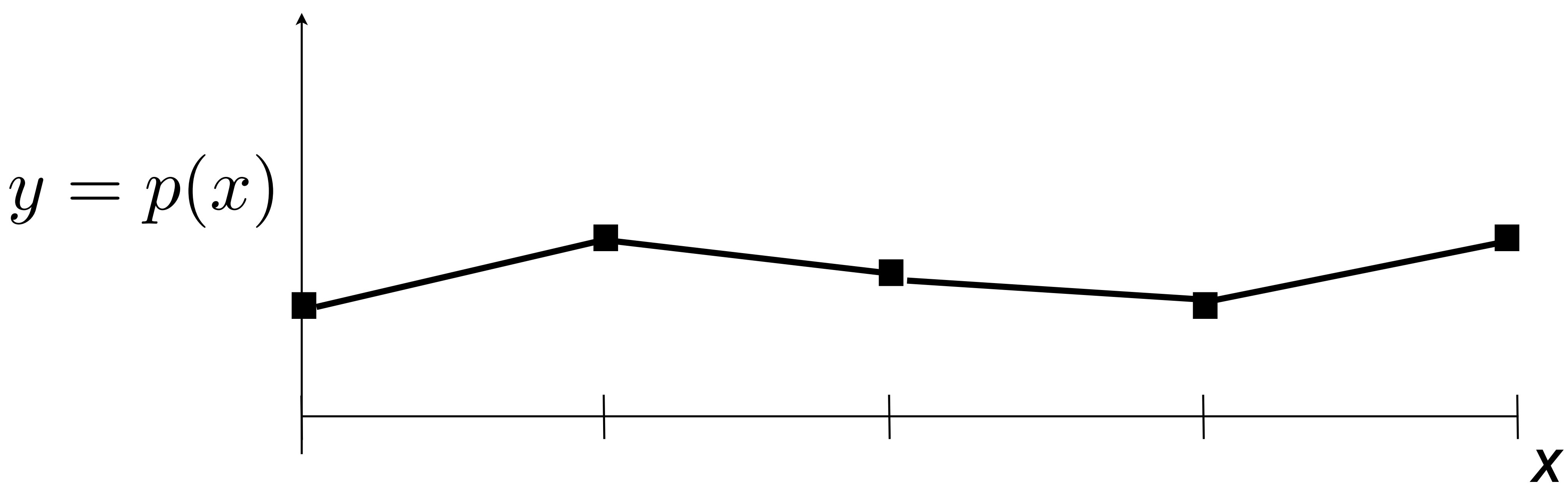
Characteristics

- Nature (level dependency)
  - Stationary $S^k = S$
  - Non-Stationary

- Domain Structure (connectivity)
  - Regular
  - Non-Regular

Functional Setting

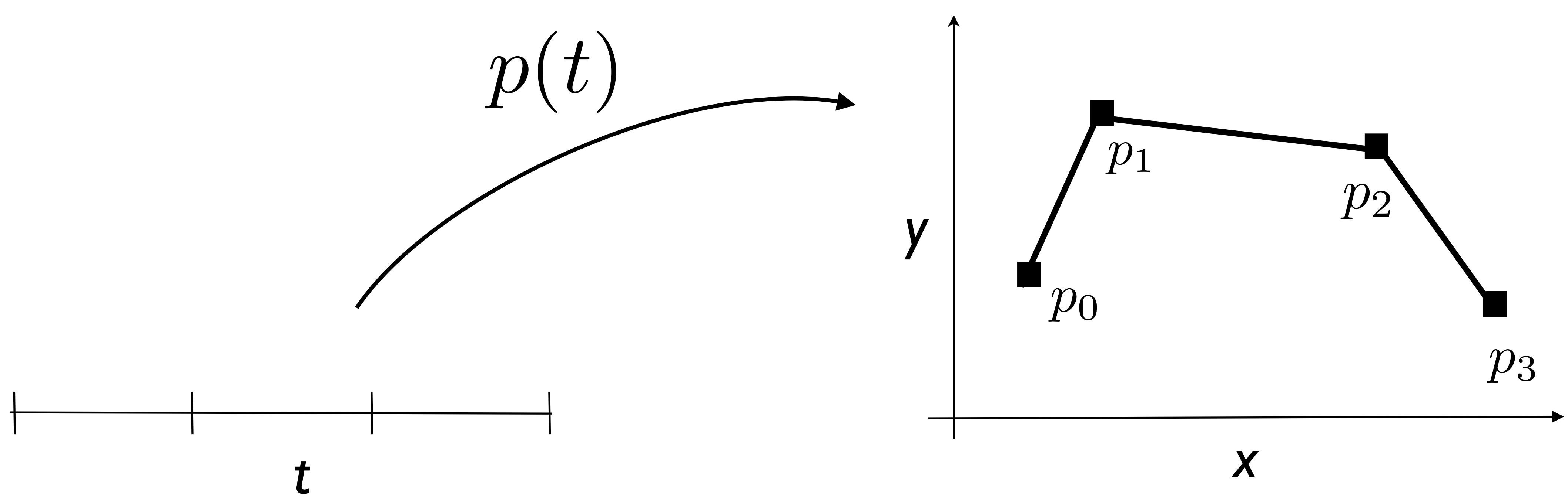
- Uniform Partition



Parametric Setting

$$(x, y) = p(t) = (x(t), y(t))$$

- Extends functional setting
  - Control Points $p_i = (x_i, y_i)$



1D Subdivision

- Example: Cubic B-Spline

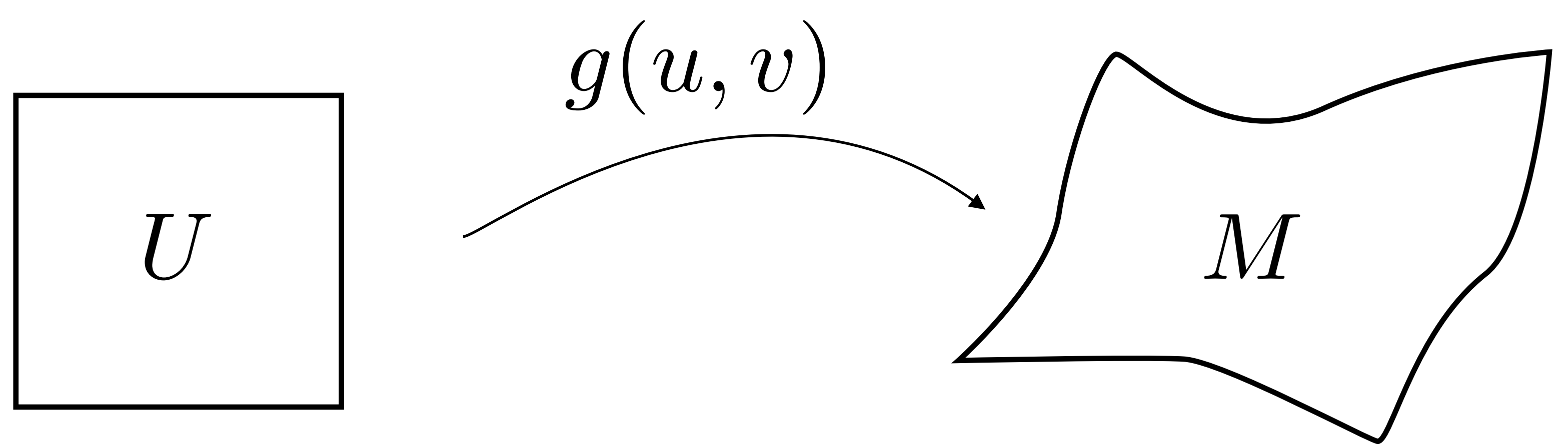
$$S(x) = \frac{1}{8}x_0 + \frac{1}{2}x_1 + \frac{3}{4}x_2 + \frac{1}{2}x_3 + \frac{1}{8}x_4$$

$$\begin{pmatrix} \vdots \\ \\ p_i^{j+1} \\ \\ \vdots \end{pmatrix} = \begin{pmatrix} & \frac{1}{2} & \frac{1}{2} & & & \\ & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & & \\ \cdots & \cdot & \frac{1}{2} & \frac{1}{2} & & \cdots \\ & & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & \\ & & \cdot & \frac{1}{2} & \frac{1}{2} & \end{pmatrix} \begin{pmatrix} \vdots \\ \\ p_k^{j} \\ \\ \vdots \end{pmatrix}$$

2D Subdivision

- Parametric Surfaces

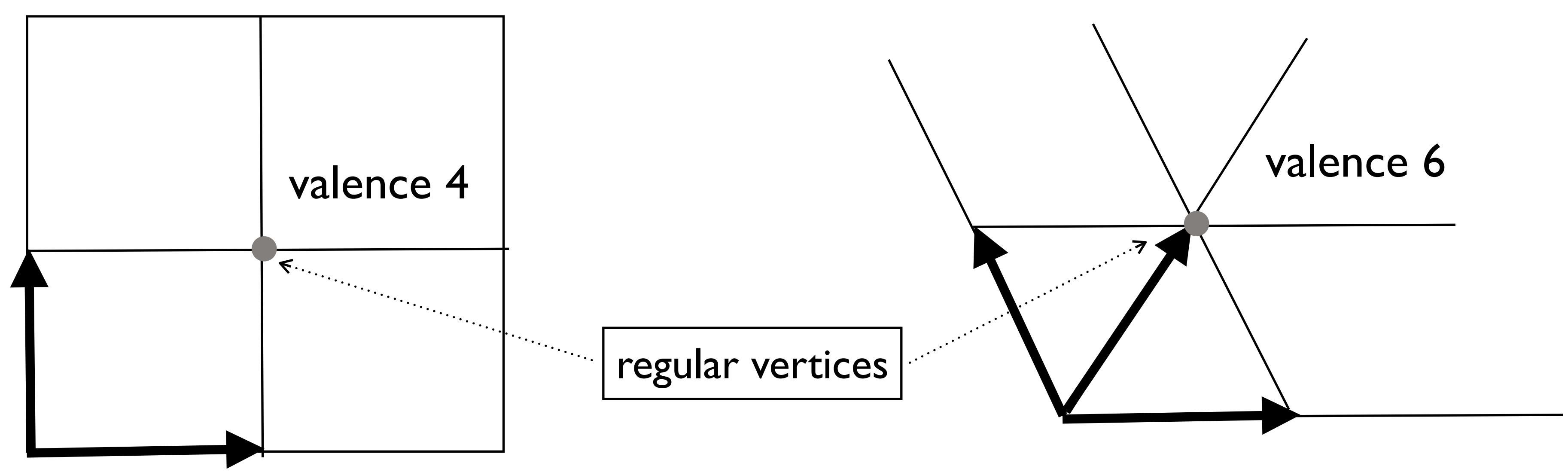
$$g : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$



$$(x, y, z) = g(u, v) = (x(u, v), y(u, v), z(u, v))$$

Domain Discretization

- Regular Meshes



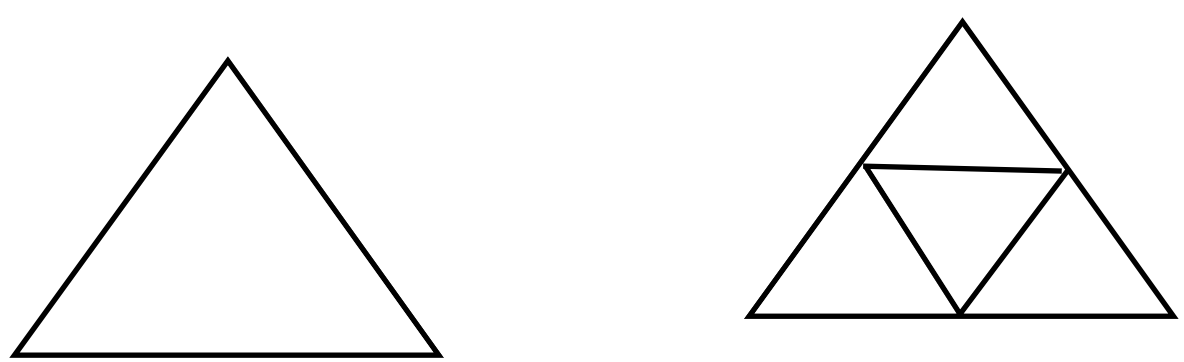
2 directions

3 directions

2D Refinement

- Quad-Mesh

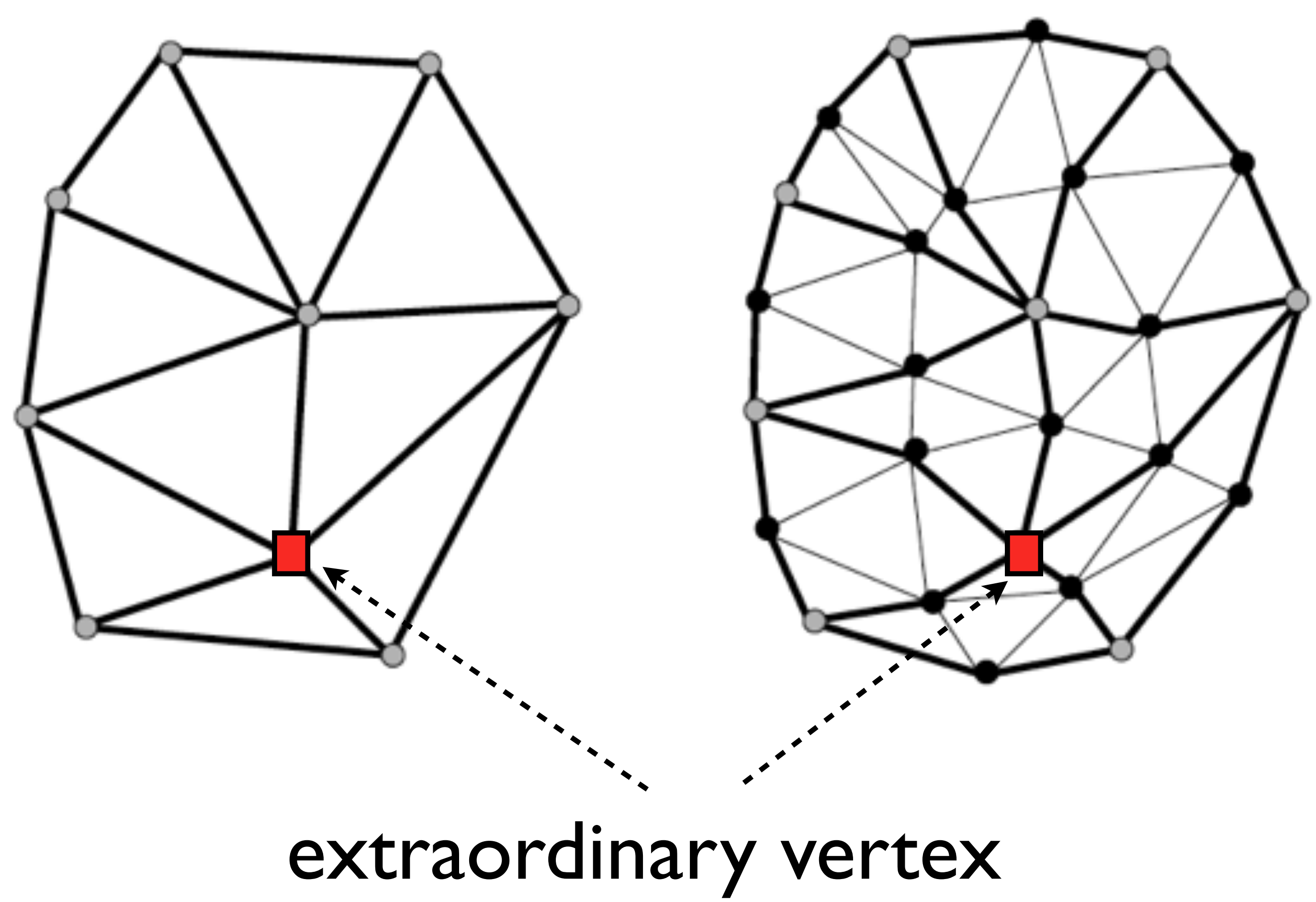
- Tri-Mesh



Obs: *preserves regularity*

Arbitrary Meshes

- Semi-Regular Meshes
  - Irregular Base Mesh
  - Regular Refinement



B-Splines & Subdivision

- Subdivision Surfaces
  - Generalize Splines to Non-Regular Connectivity

- Examples:
  - Catmull-Clark
    (tensor product bi-cubic B-spline)
  - Loop Subdivision Surface
    (three-directional quartic box spline)

Splines & Manifolds

- Uniform Splines
  - Particular case of Manifold Structure

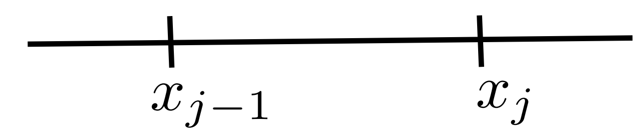
- Characterization
  - Charts: (basis functions at control vertices)
  - Transition Function: (affine transformation)
  - Partition of Unity

B-Spline Basis

- Recursive Definition

$$B_n := B_{n-1} * B_0$$

- Closed Form (at segment S_j)



$$b_{j,n}(x) = b_n(x - x_j)$$

translation

with

$$b_n(x) := \frac{n+1}{n} \sum_{i=0}^{n+1} \omega_{i,n}(x - x_i)_+^n$$

- Truncated Power

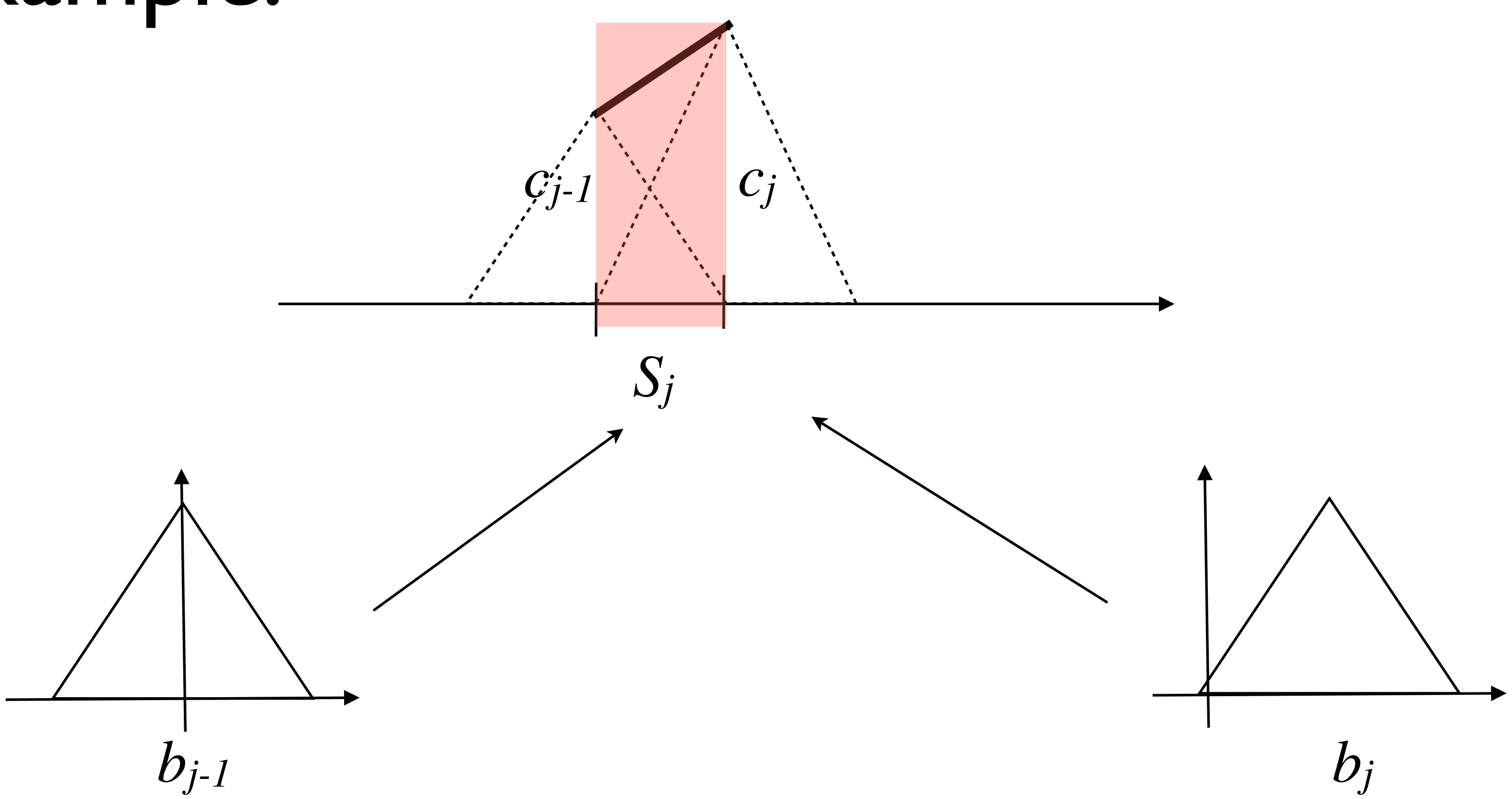
$$(x - x_i)_+^n = \begin{cases} (x - x_i)^n & \text{if } x \geq x_i; \\ 0 & \text{if } x < x_i. \end{cases}$$

B-Spline Evaluation

- At a Segment S_j

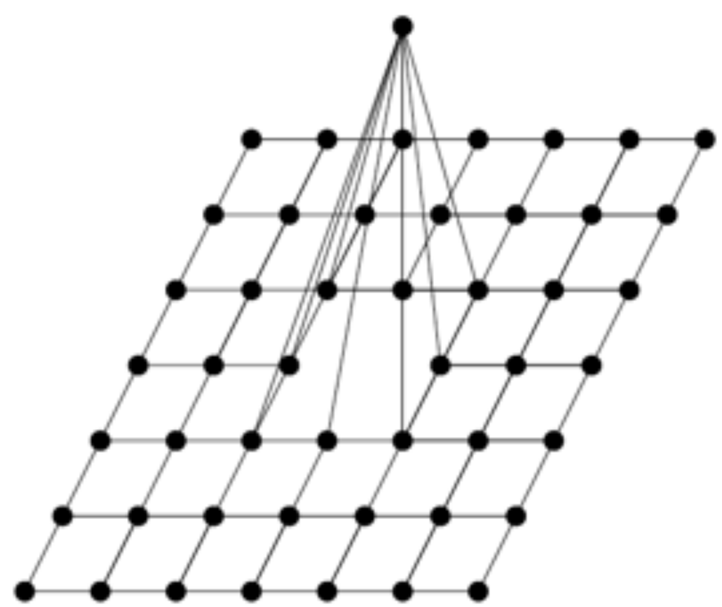
$$s_j(x) = \sum_{j=0}^{m-1} c_j b_{j,n}(x)$$

- Example:

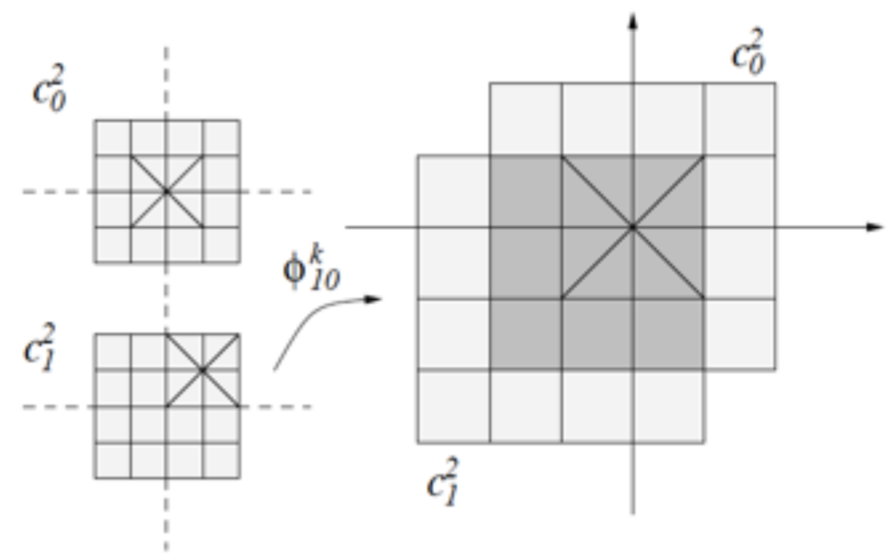


2D Scheme

- Quadrilateral Structure



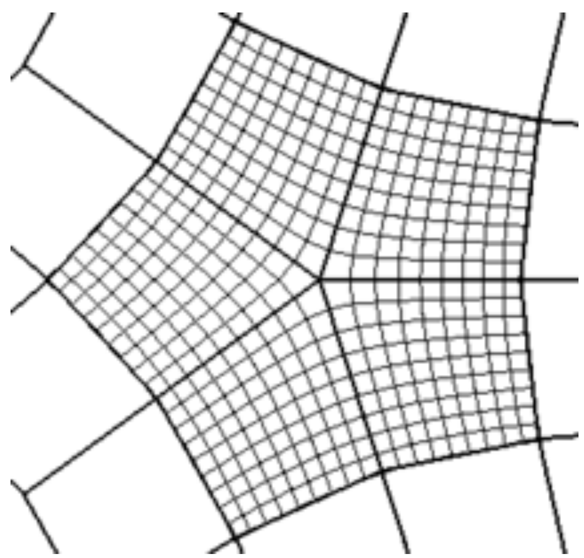
control mesh



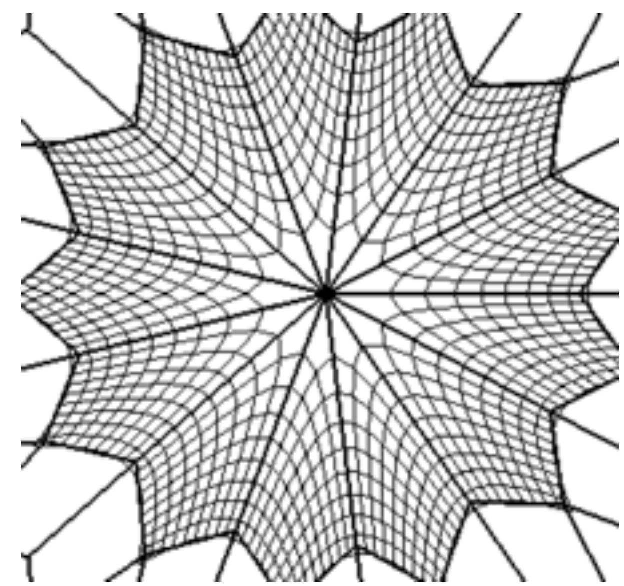
charts / transition function

Extraordinary Vertices

- Characteristic Map



k=5



k=13

▸ *Breaks good properties* (transition function, etc)

Topological Obstructions

- A closed 2-manifold M admits an affine atlas, if and only if M is a torus