Trimester Program on
Computational Manifolds and Applications

Introduction to Computational Manifolds and Applications

Manifold Harmonics

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Summary

Today (Tuesday): Differential Operators on Surfaces

- Differential operators in the parametric domain
- Cotangent formula
- Belkin's approach
- SPH-based scheme

Thursday: Manifold Harmonics and Applications

- Some theoretical background
- Mesh Filtering
- Embedding in high-dimension
- Fiedler tree
- Heat Trace

Spectral Mesh Processing

Although relatively recent in the context of Geometry Processing, spectral methods have already experienced a large development in the field of spectral graph theory.

Spectral Mesh Processing

Although relatively recent in the context of Geometry Processing, spectral methods have already experienced a large development in the field of spectral graph theory.

Those techniques rely on spectrum of a Laplacian-like matrix.

Laplacian Matrices

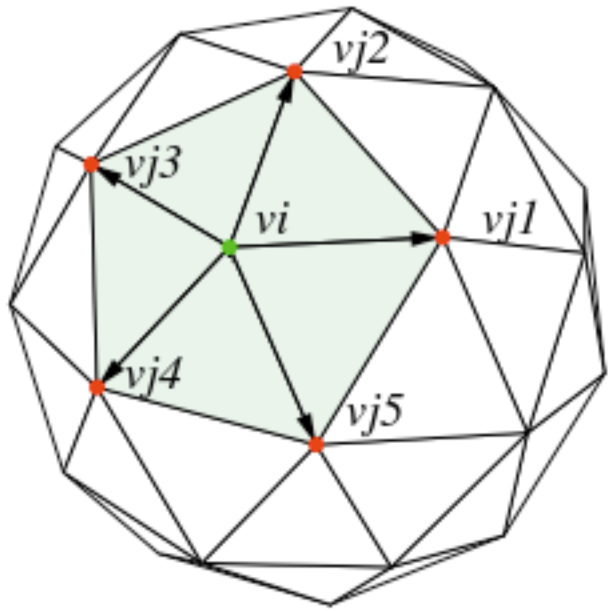
Given a surface mesh $M = V, E$ a matrix L can be built as follows:

$$l_{ij} = \begin{cases} w_{ij} & \text{if } i \neq j \text{ and } e_{ij} \in E \\ -\sum_{e_{ij} \in E} w_{ij} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

where w_{ij} is a weight assigned to each edge in E.

Laplacian Matrices

$$i \rightarrow \left[0 \cdots w_{ij1} \cdots 0 \cdots w_{ij2} \cdots 0 \cdots -\sum_{e_{ij} \in E} w_{ij} \cdots 0 \cdots w_{ij3} \cdots 0 \cdots w_{ij4} \cdots 0 \cdots w_{ij5} \cdots 0 \right]$$

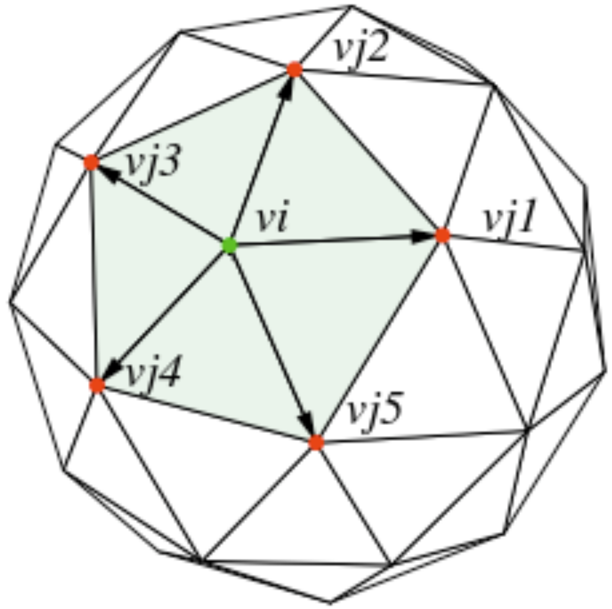


Laplacian Matrices

$$i \rightarrow \begin{bmatrix} 0 \cdots w_{ij1} \cdots 0 \cdots w_{ij2} \cdots 0 \cdots -\sum_{e_{ij} \in E} w_{ij} \cdots 0 \cdots w_{ij3} \cdots 0 \cdots w_{ij4} \cdots 0 \cdots w_{ij5} \cdots 0 \end{bmatrix}$$

$$w_{ij} = \frac{\cot(\beta_{ij}) + \cot(\beta'_{ij})}{2}$$

Cotangent Formula !!



Short Review of Eigenvalues and Eigenvectors

$$L\mathrm{v} = \lambda\,\mathrm{v}$$

Short Review of Eigenvalues and Eigenvectors

eigenvalue

$$L\mathrm{v} = \lambda \mathrm{v}$$

eigenvector

Short Review of Eigenvalues and Eigenvectors

$$L\mathrm{v} = \lambda \mathrm{v}$$

ARPACK – Large sparse matrices

Lanczos algorithm (derived from the power method)

Short Review of Eigenvalues and Eigenvectors

$$L\mathrm{v} = \lambda \mathrm{v}$$

Let $\mathrm{V}_i = \{\alpha \mathrm{v}_i \,|\, \alpha \in \mathbb{R}\}$, where v_i is an eigenvector of L, and

$$\mathrm{V} = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

then the subspaces V_i are invariant under $L : \mathrm{V} \subset \mathbb{R}^n \to \mathrm{V}$.

Short Review of Eigenvalues and Eigenvectors

If L is symmetric then

- The eigenvalues of L are real
- The eigenvectors $\{v_1, \ldots, v_n\}$ of L forms an orthonormal basis

$$x \in \mathbb{R}^n \Rightarrow x = \sum_i < x, v_i > v_i$$

Spectral Mesh Processing

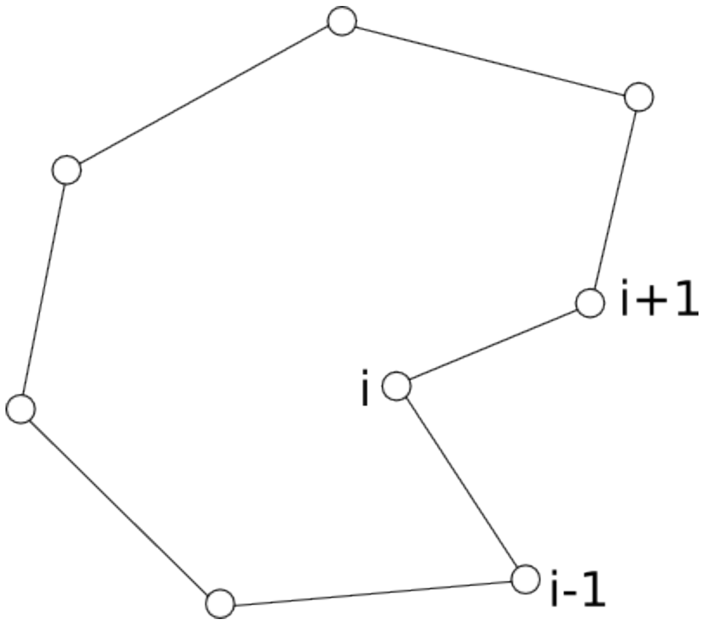
There are three main steps involved in most spectral mesh processing methods:

1. Construction of the matrix L

2. Eigendecomposition of L.

3. Handling the eigendecomposition towards obtaining the desired results.

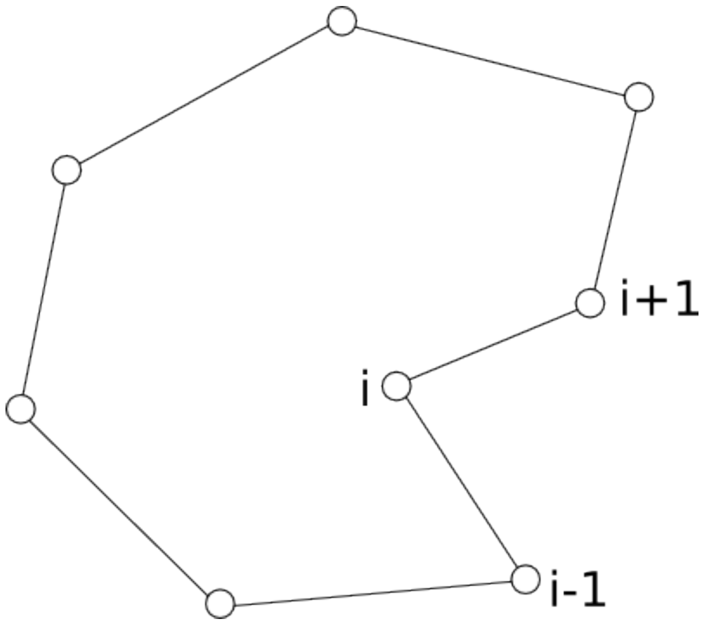
Spectral Mesh Processing



$$P = \{V, E\}$$

$$f : V \to \mathbb{R} \quad f = (f_1, \ldots, f_n)$$

Spectral Mesh Processing



$$P = \{V, E\}$$

$$f : V \to \mathbb{R} \quad f = (f_1, \ldots, f_n)$$

$$w_{ij} = \frac{1}{N_i} = 1/2$$

$$(Lf)_i = \frac{1}{2}(f_{i-1} - f_i) + \frac{1}{2}(f_{i+1} - f_i)$$

$$L = \frac{1}{2}\begin{bmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ \vdots & & & & \\ -1 & & & -1 & 2 \end{bmatrix}$$

Spectral Mesh Processing

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eigenvalues

$$\lambda_i = 1 - \cos(2\pi\lfloor i/2 \rfloor/n)$$

$$0 \leq \lambda_1 \leq \cdots \leq \lambda_n$$

eigenvectors

$$(\mathrm{v}_i)_j = \begin{cases} \sqrt{1/n} & i = 1 \\ \sqrt{2/n}\sin(2\pi j\lfloor i/2 \rfloor/n) & i \text{ even} \\ \sqrt{2/n}\cos(2\pi j\lfloor i/2 \rfloor/n) & i \text{ odd} \end{cases}$$

Spectral Mesh Processing

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Fourier Basis

Spectral Mesh Processing

$$f = \sum_i < f, \mathrm{v}_i > \mathrm{v}_i$$

$$f^F = \sum_i F(< f, \mathrm{v}_i >)\mathrm{v}_i$$

filter

Spectral Mesh Processing

$$f = \sum_i < f, \mathrm{v}_i > \mathrm{v}_i$$

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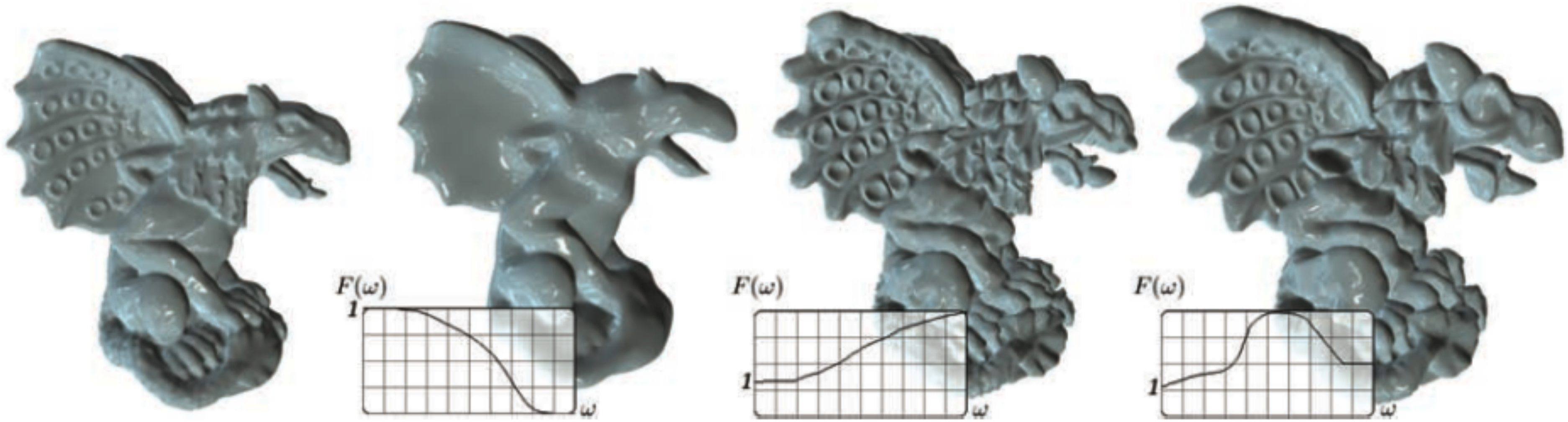
filter

For surfaces, the spectrum of the Laplace operator behaves quite similarly to a Fourier basis, allowing for filtering functions defined on the surface.

Spectral Mesh Processing

In particular, if the coordinates of the vertices of surface mesh are seem as functions defined on the surface, band-pass filtering can be performed. [Vallet and Levy, SGP'08]

$$\hat{x}^F = \sum F(\omega)\alpha_i v_i = \sum F(\sqrt{\lambda_i})\alpha_i v_i$$



Spectral Mesh Processing

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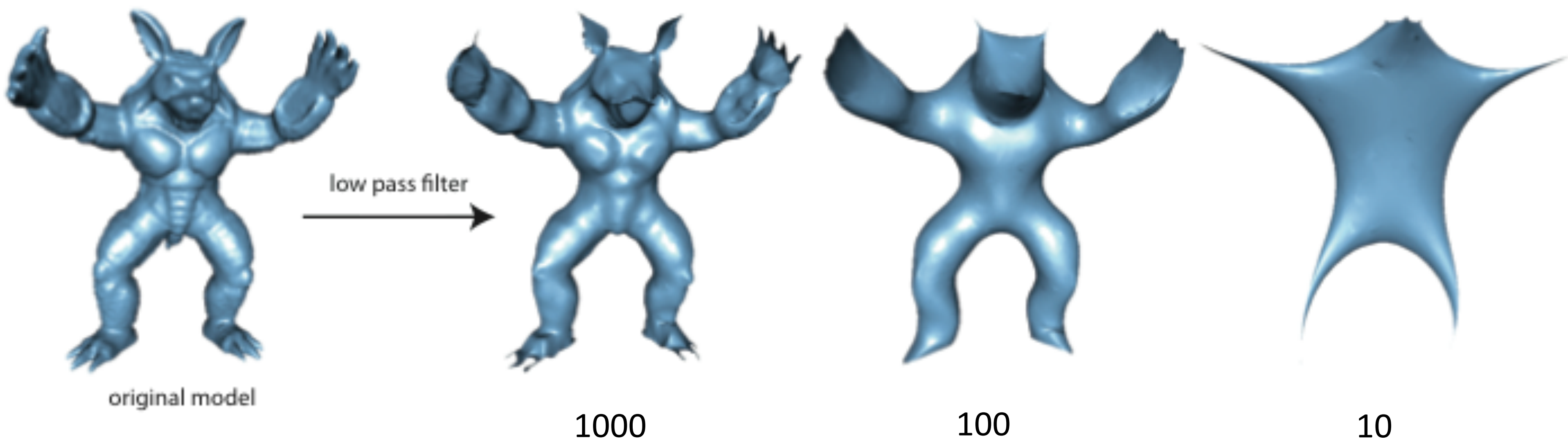


Spectral Mesh Processing

[Taubin, Siggraph'95]

$$x = \sum_i < x, \mathrm{v}_i > \mathrm{v}_i$$

$$Lx = \sum_i \lambda_i < x, \mathrm{v}_i > \mathrm{v}_i$$

Spectral Mesh Processing

[Taubin, Siggraph'95]

$$x = \sum_i < x, \mathrm{v}_i > \mathrm{v}_i$$

$$Lx = \sum_i \lambda_i < x, \mathrm{v}_i > \mathrm{v}_i$$

$$L^k x = \sum_i \lambda_i^k < x, \mathrm{v}_i > \mathrm{v}_i$$

Spectral Mesh Processing

[Taubin, Siggraph'95]

$$F^k(Lx) = \sum_i F^k(\lambda_i) < x, \mathrm{v}_i > \mathrm{v}_i$$

$$F(\lambda_i) \sim 1 \text{ for low frequencies}$$

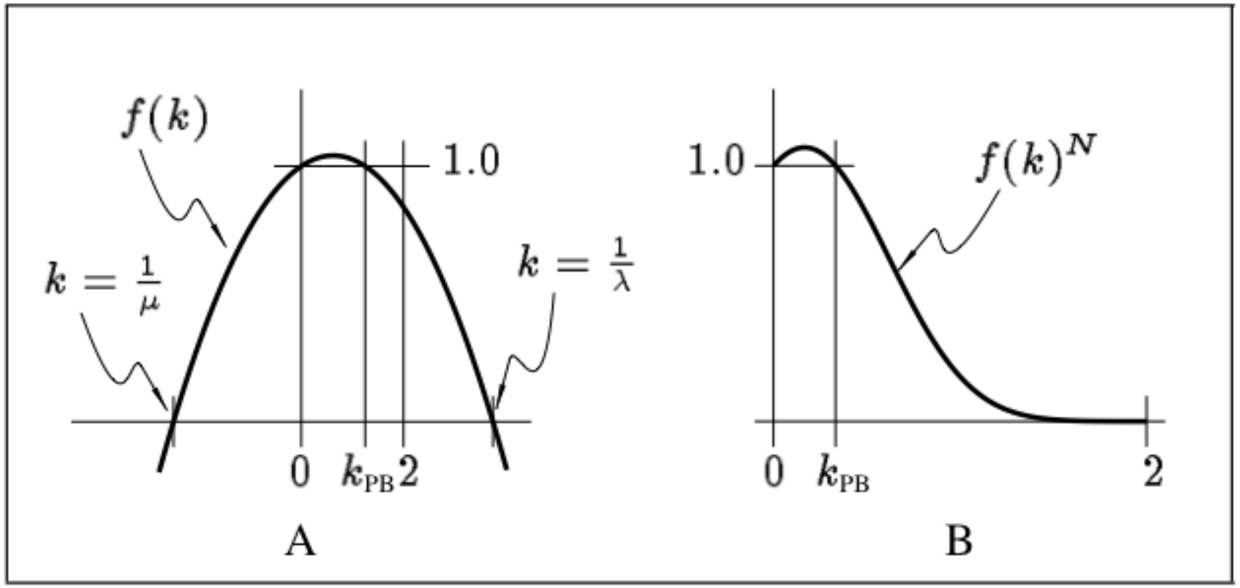
$$F(\lambda_i) \sim 0 \text{ for high frequencies}$$

Spectral Mesh Processing

[Taubin, Siggraph'95]

$$F(\lambda) = (1 - \alpha\lambda)(1 - \mu\lambda)$$

where $\alpha > 0$ and $\mu < -\alpha$



Spectral Mesh Processing

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$$x^k = x^{k-1} + \alpha \Delta x$$

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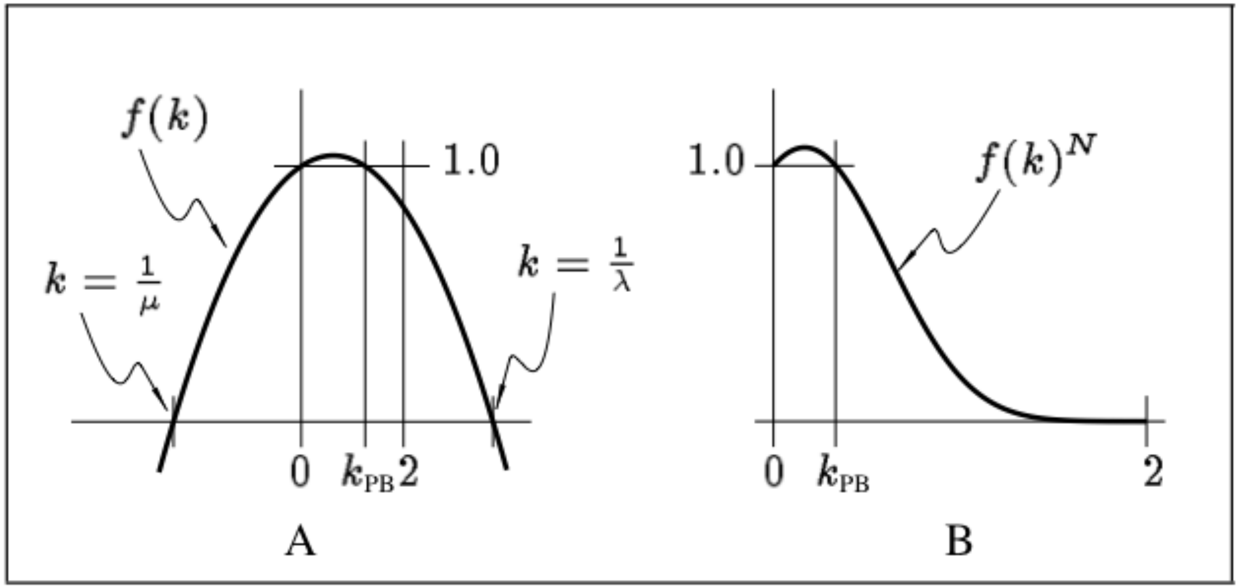

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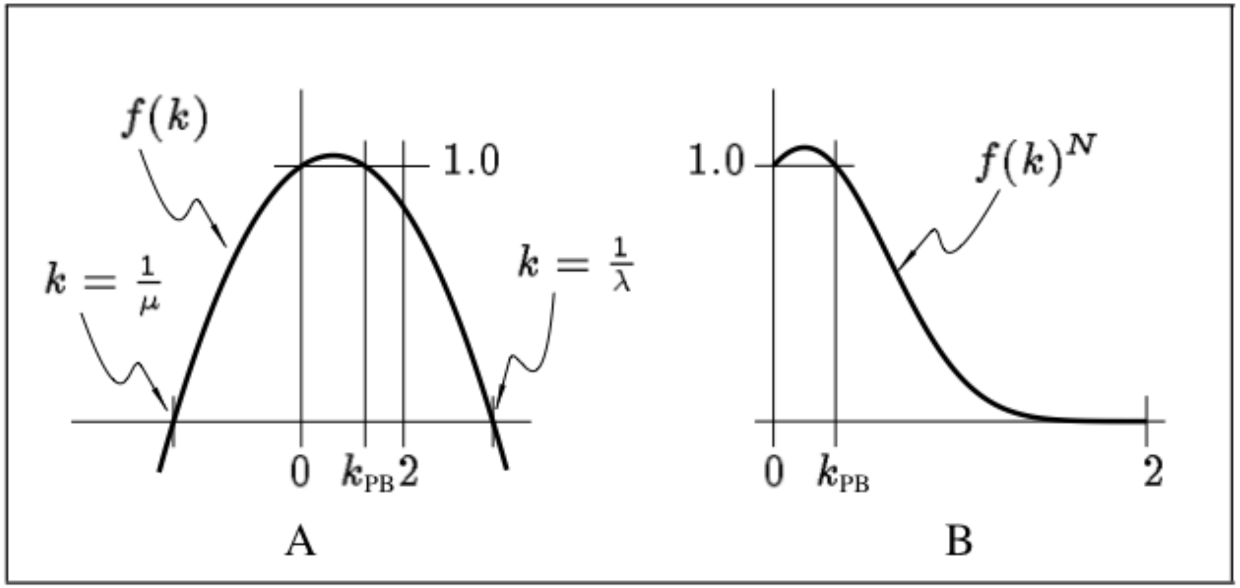
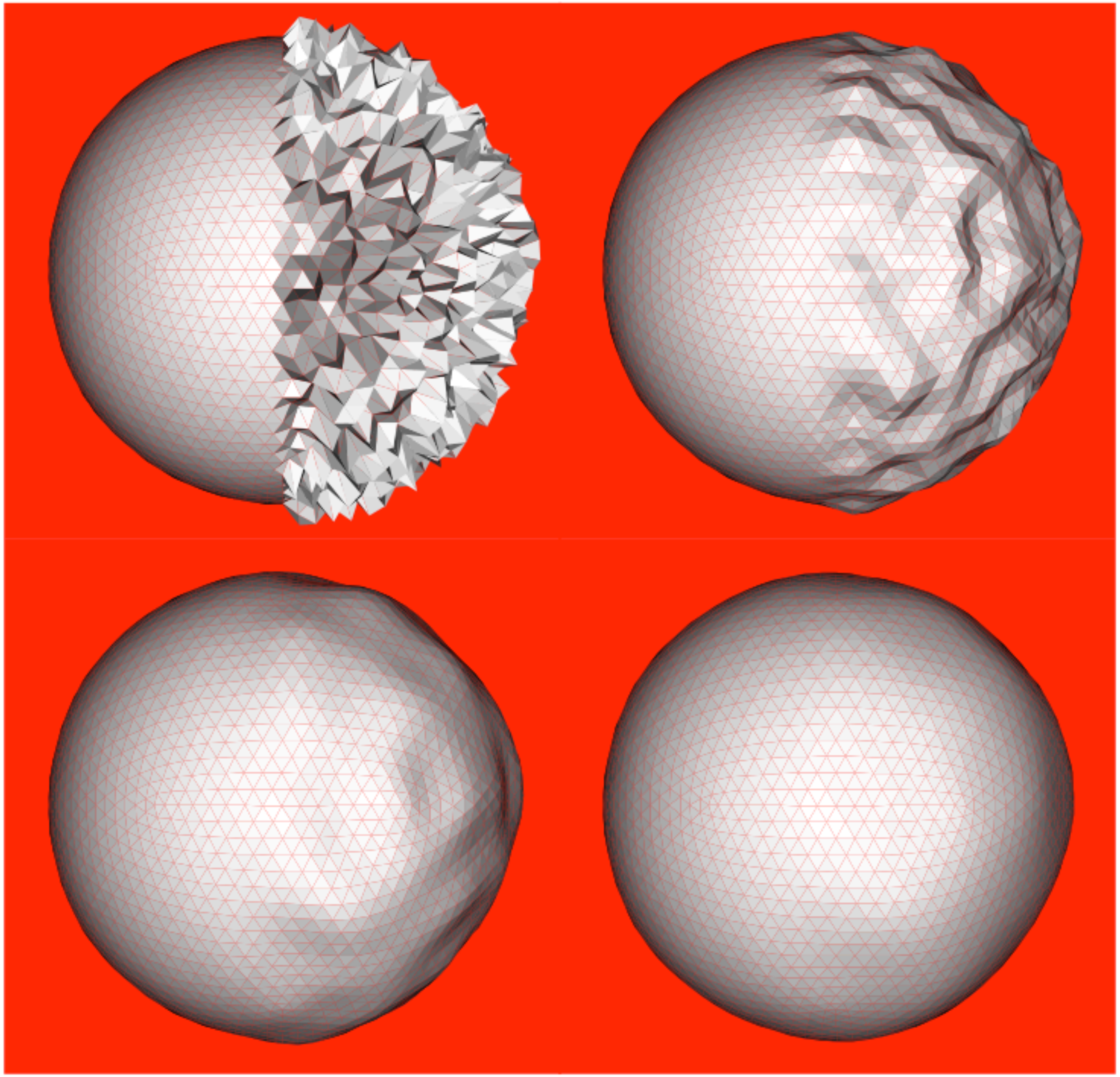


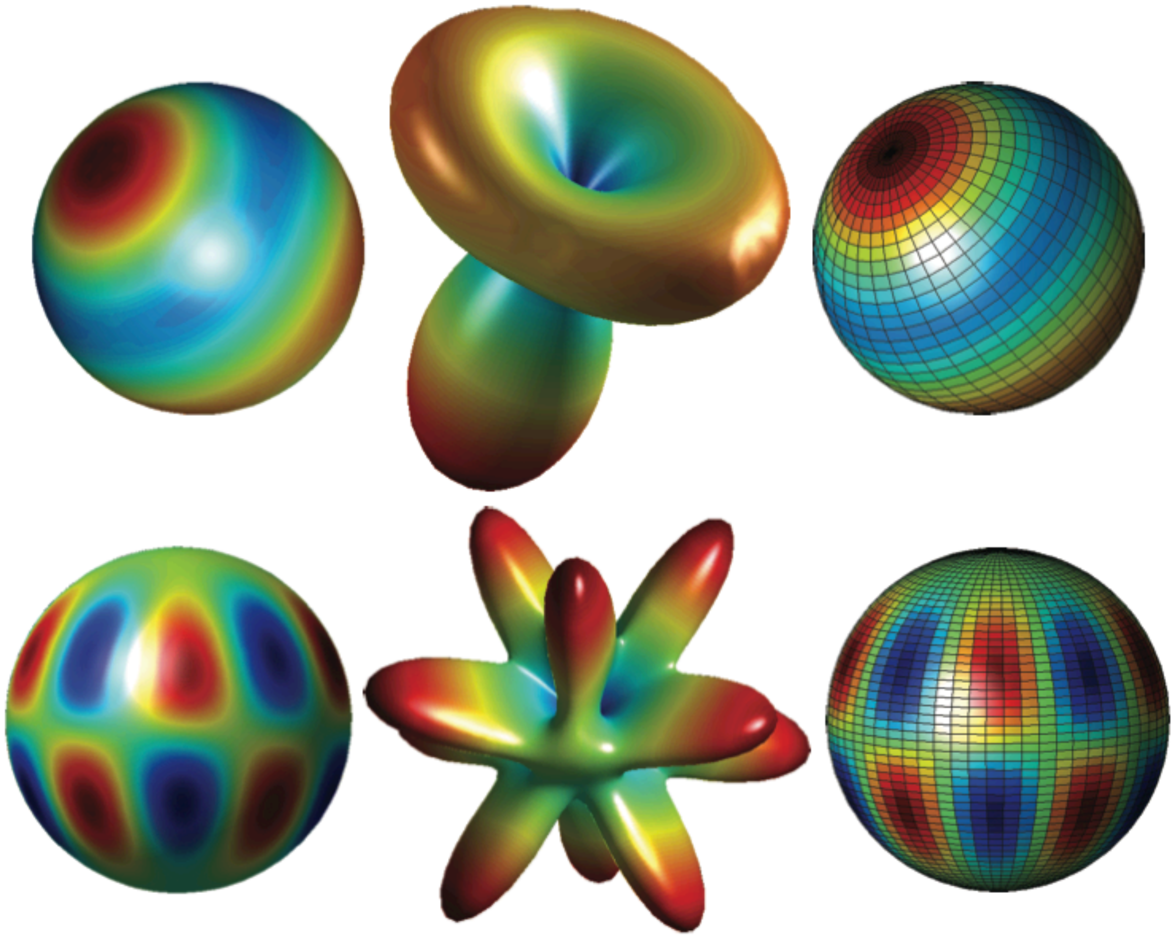
Avoid to compute the spectrum

[Taubin, Siggraph'95]



Spectral Mesh Processing

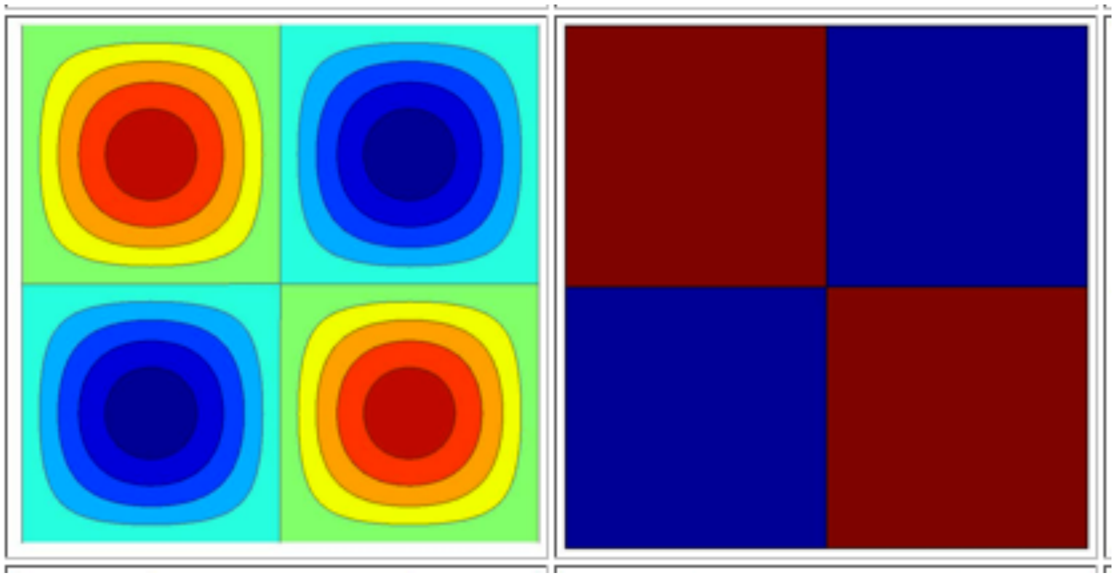
What about eigenvectors ?



Spectral Mesh Processing

Nodal Domain: The *nodal set* of an eigenfunction is the set of points where the eigenfunction is zero.

The regions bounded by the nodal set are called *nodal domains*.



Spectral Mesh Processing

Nodal Domain: The *nodal set* of an eigenfunction is the set of points where the eigenfunction is zero.

The regions bounded by the nodal set are called *nodal domains*.

An eigenfunction is built by interpolating the values of an eigenvector (defined on the vertices of a mesh) in each point of the surface.

Spectral Mesh Processing

Courant's Nodal Theorem: Let the eigenvectors of the Laplace operator be labeled in ascending order according to the corresponding eigenvalues. Then, the *k-th* eigenfunction has at most *k* nodal domains, that is, the *k-th* eigenfunction can separate the surface into at most *k* connected components.

Spectral Mesh Processing

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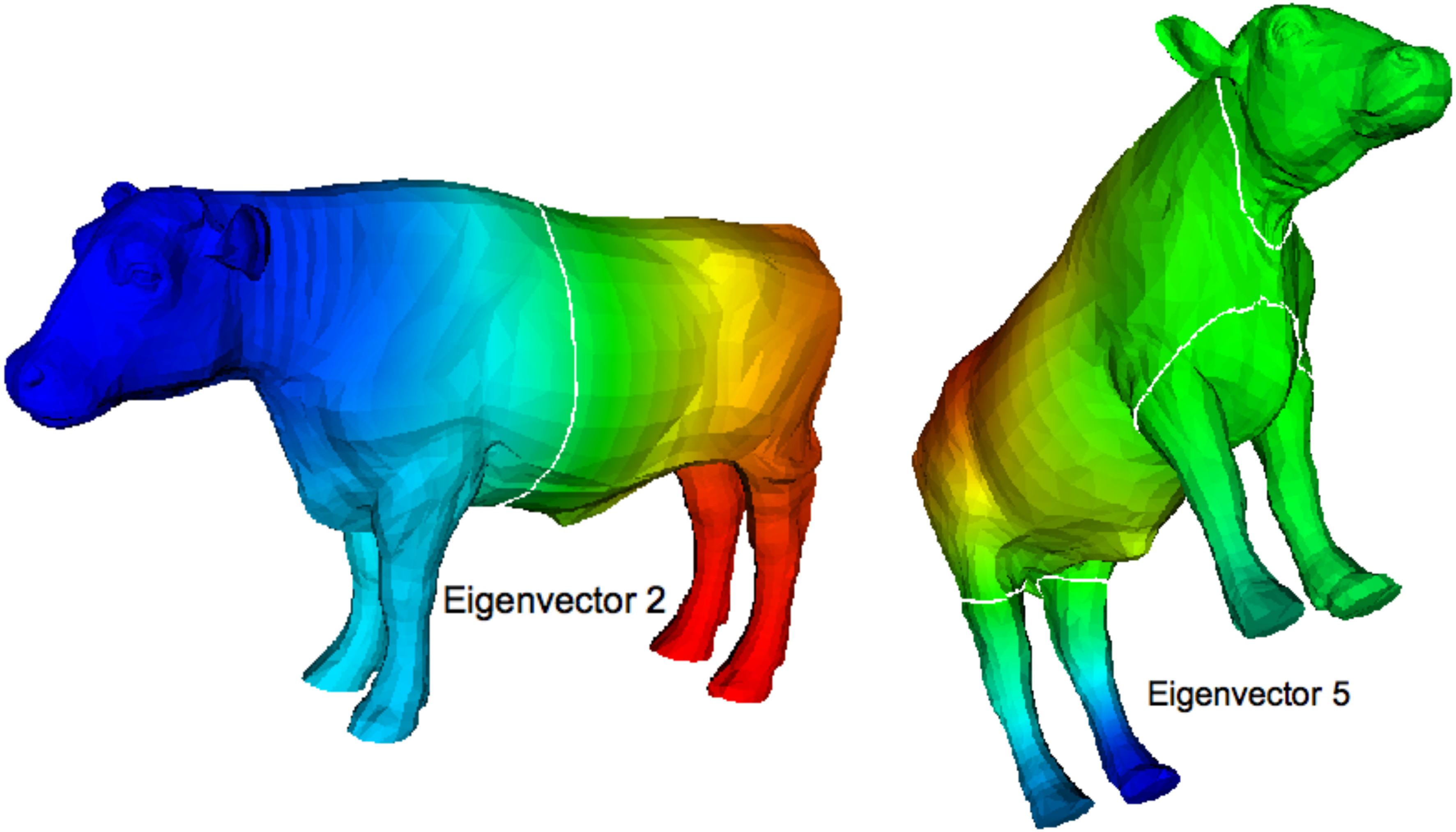


Zero is an eigenvalue of the Laplace operator with a constant corresponding eigenvector.

Spectral Mesh Processing

- Eigenvectors capture symmetries of the model;
- Invariant by isometric transformation;
- Not sensitive to small topological and geometrical changes

Spectral Mesh Processing

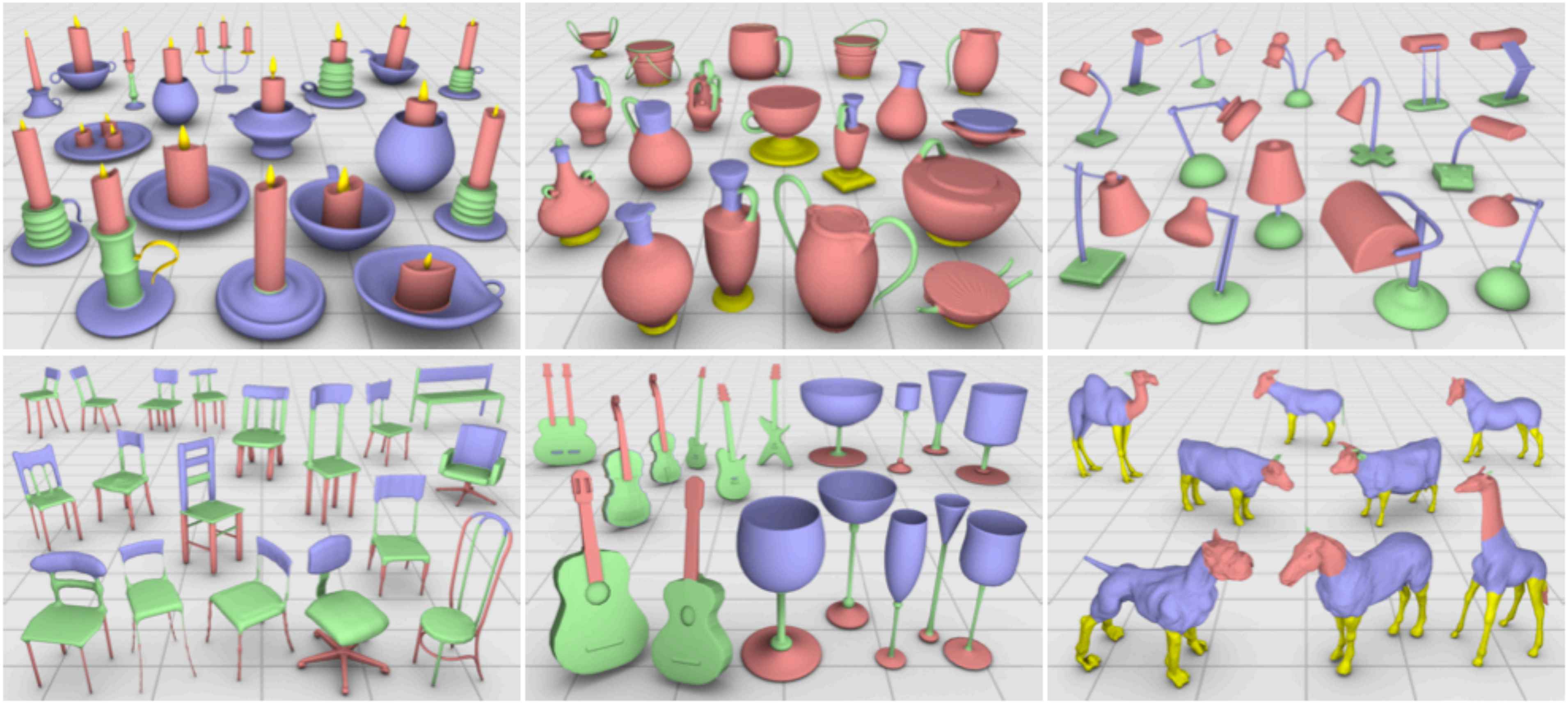
- Eigenvectors capture symmetries of the model;
- Invariant by isometric transformation;
- Not sensitive to small topological and geometrical changes

Powerful tool for many mesh processing tasks.

Spectral Mesh Processing

Mesh Segmentation

[O. Sidi et al., SigAsia'11]



Spectral Mesh Processing

Global Point Signature

[Rustamov., SGP'07]

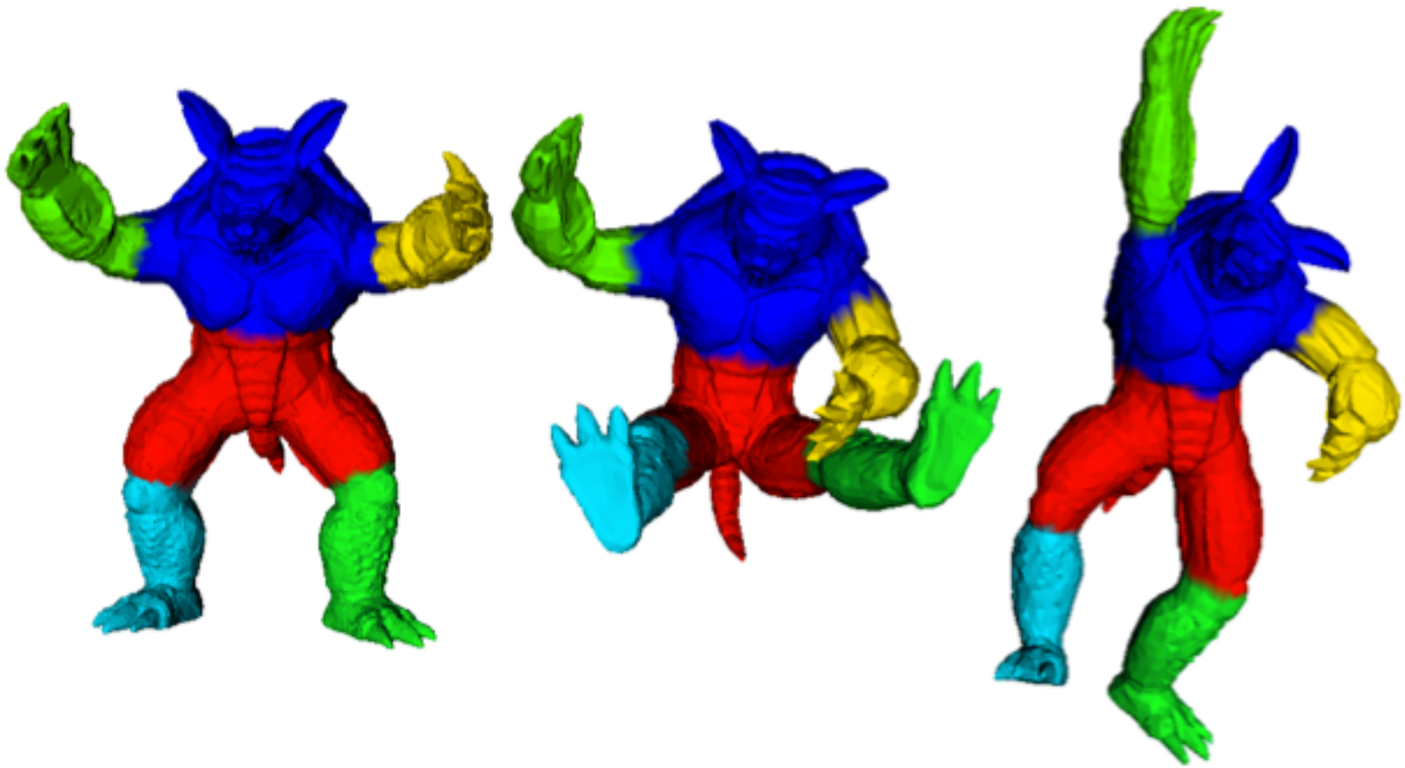
$$GPS(x) = \left(\frac{1}{\sqrt{\lambda_1}}\mathrm{v}_1(x), \frac{1}{\sqrt{\lambda_2}}\mathrm{v}_2(x), \cdots\right)$$

Euclidean distance in the GPS space is related to Green's function on the surface.

Spectral Mesh Processing

Global Point Signature

[Rustamov., SGP'07]



Spectral Mesh Processing

Global Point Signature

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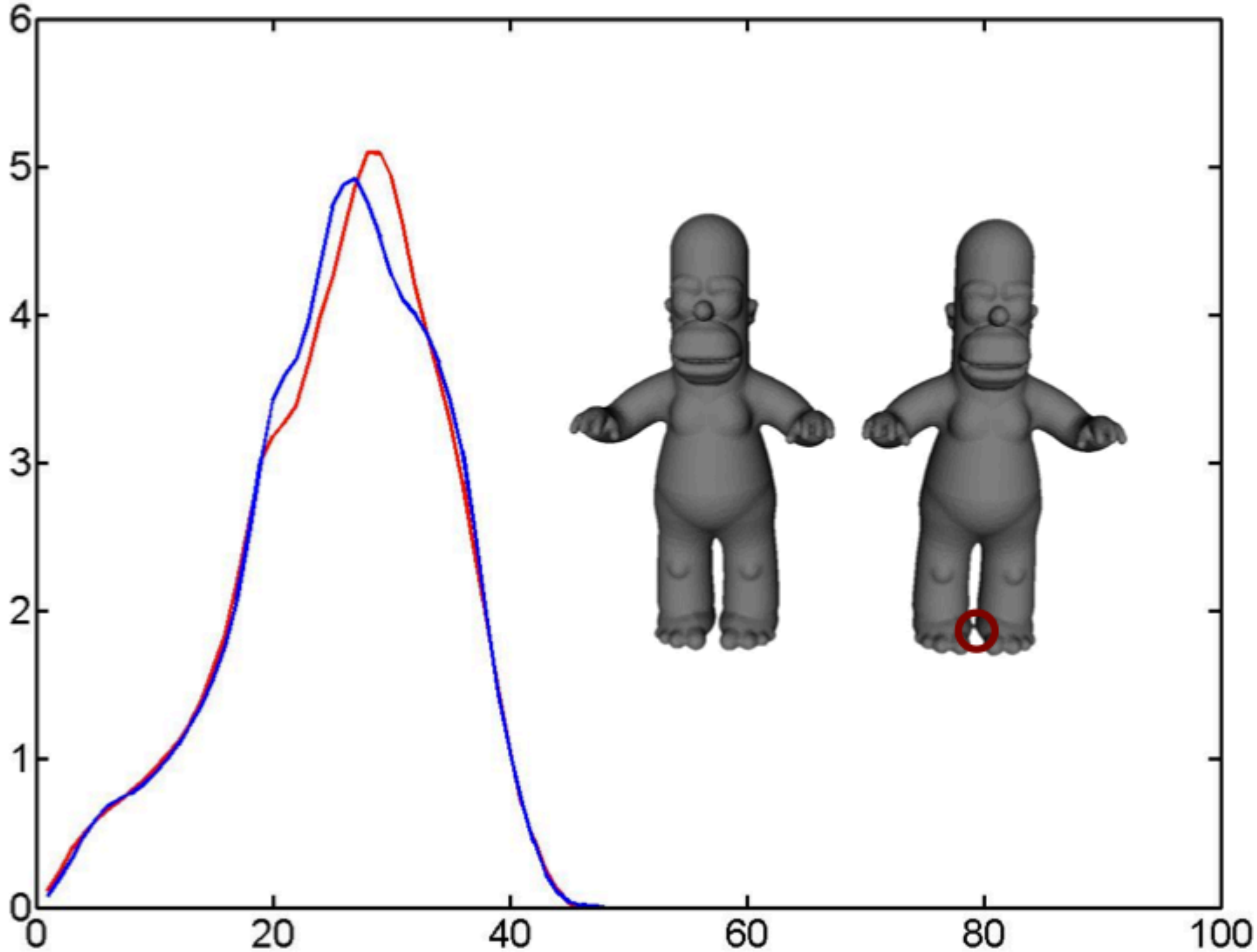

Spectral Mesh Processing

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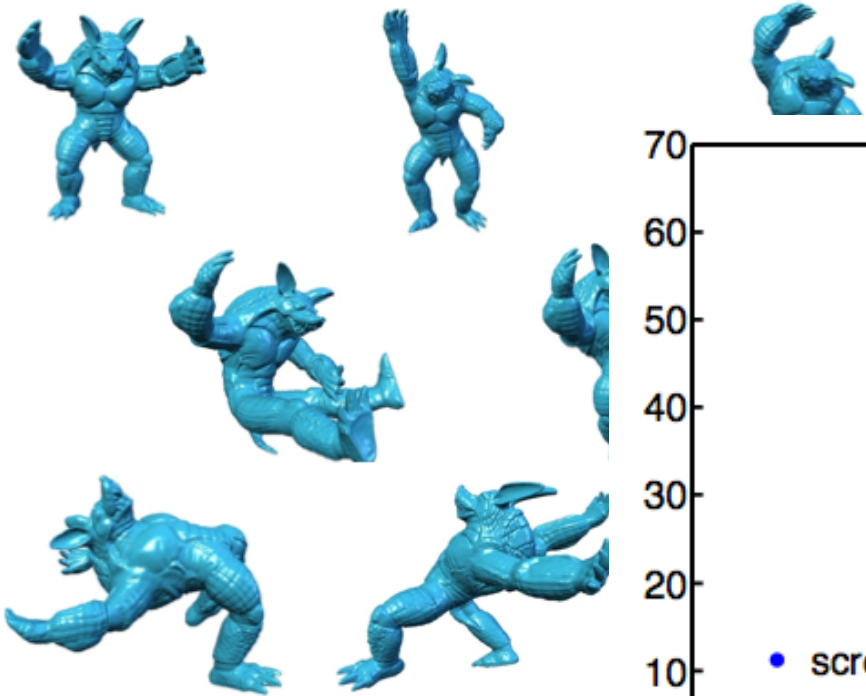
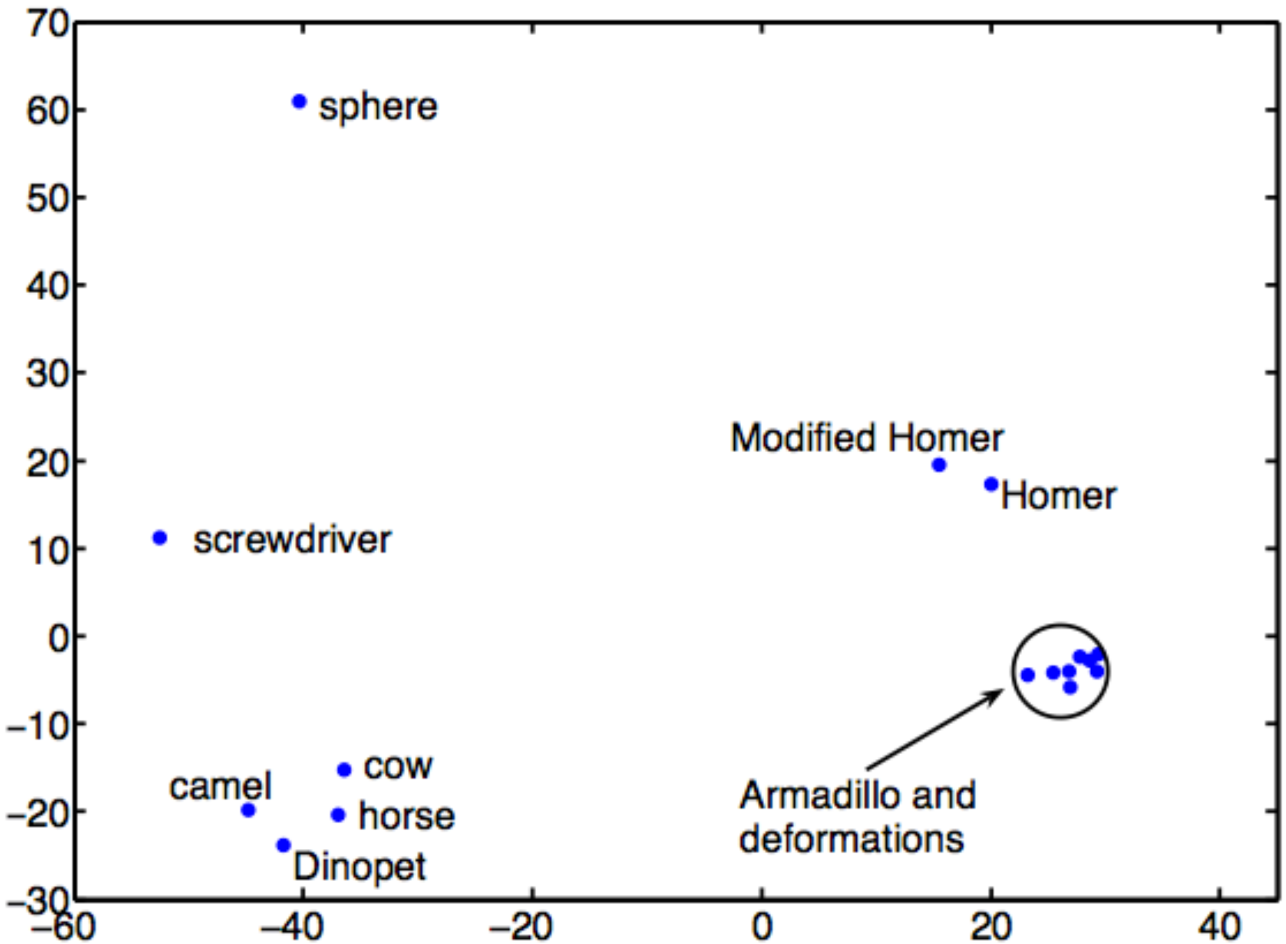




Spectral Mesh Processing

Diffusion Maps

[Goes, SGP'08]

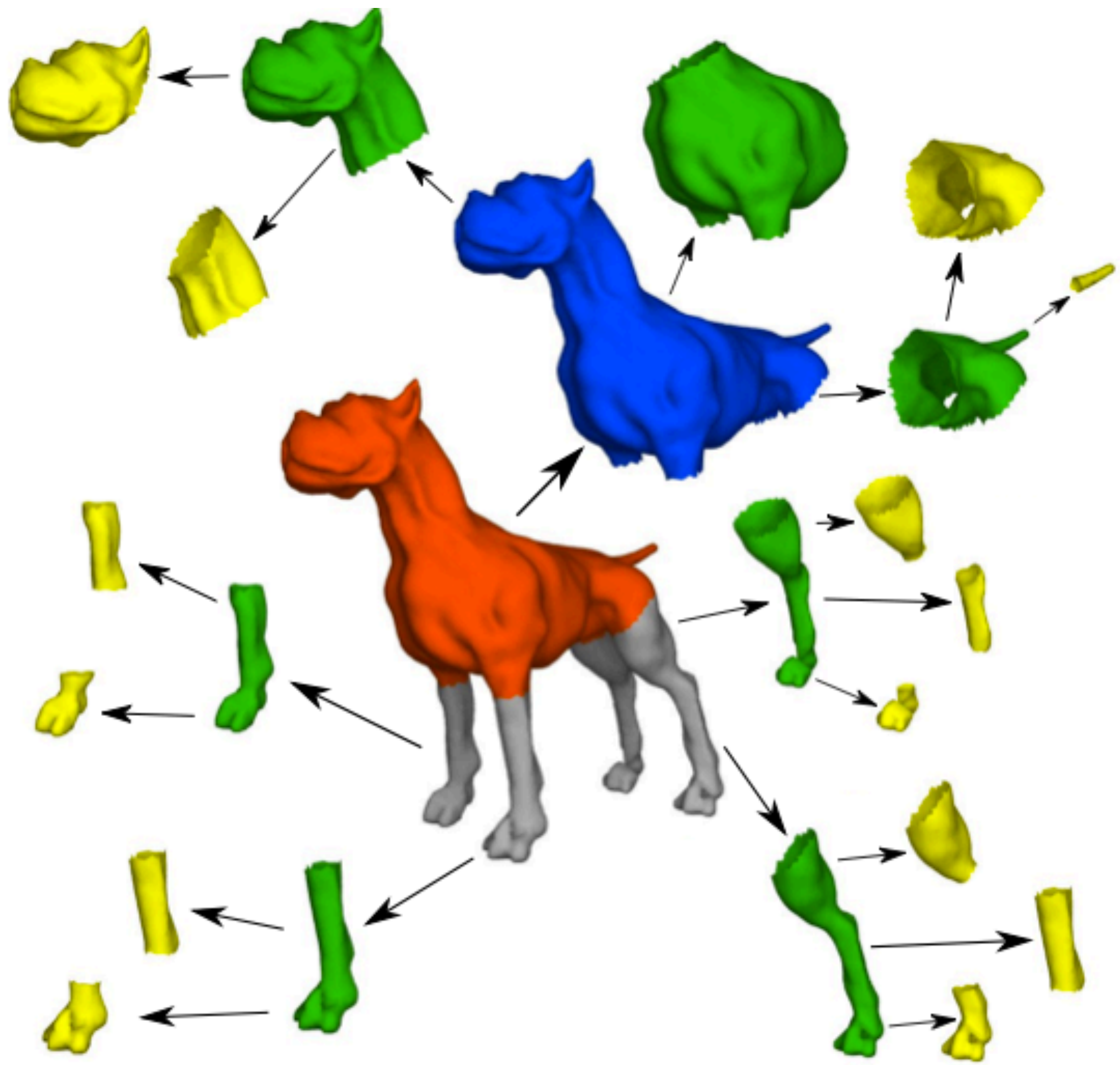
$$\Phi_t(x) = (e^{-\lambda_1 t}\mathrm{v}_1(x), e^{-\lambda_2 t}\mathrm{v}_2(x), \cdots)$$

Euclidean distance in the DM space is related to diffusion distance on the surface.

Spectral Mesh Processing

Diffusion Maps

[Goes, SGP'08]



Spectral Mesh Processing

Diffusion Maps

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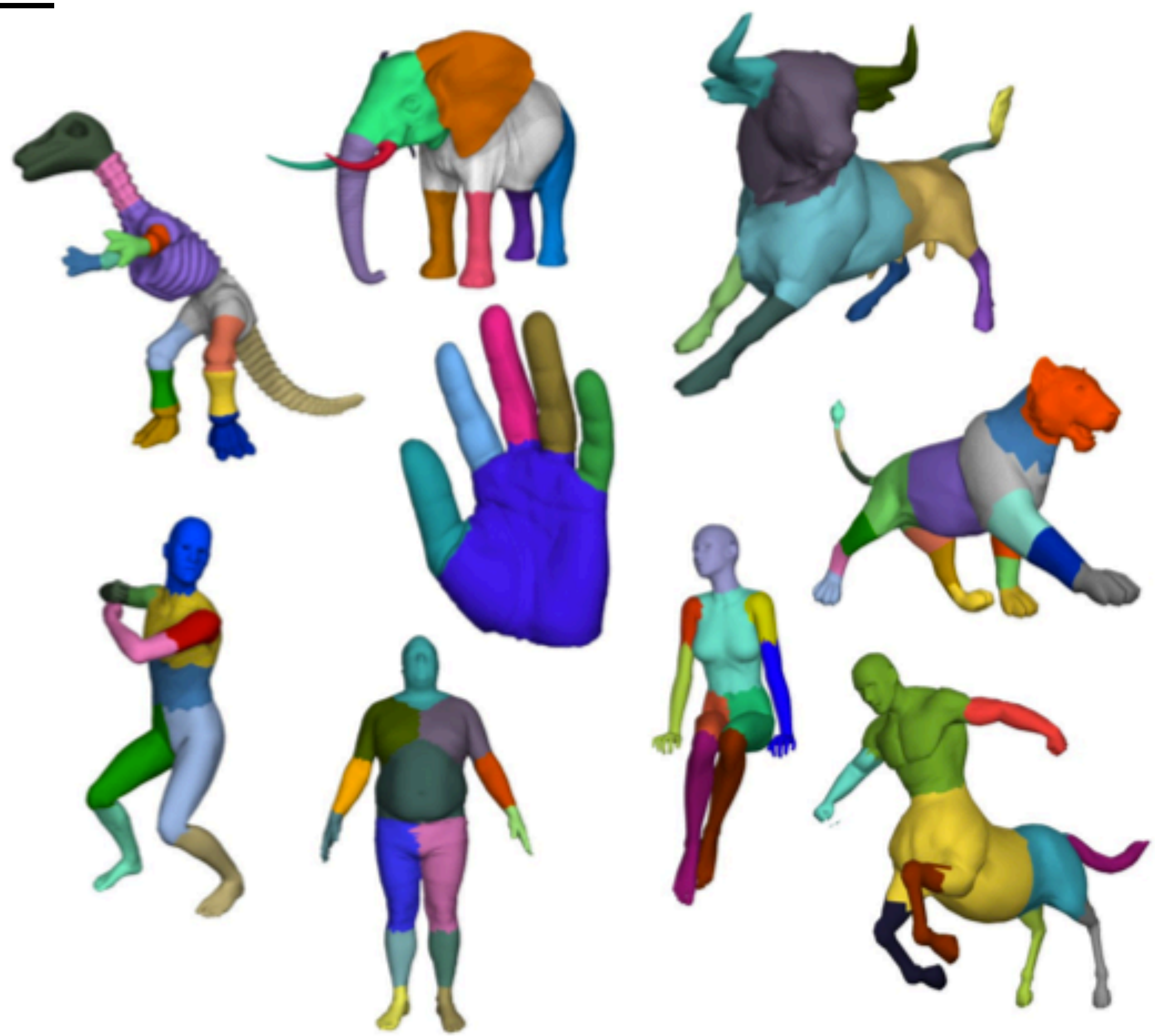


Spectral Mesh Processing

The eigenvector corresponding to the smallest non-zero eigenvalue is called Fiedler vector and it is characterized by:

$$v_2(L) = \arg\min_u \sum_{i,j} w_{ij}(u_i - u_j)^2$$

$$\text{Subject to: } \sum u_i = 0 \text{ and } \sum u_i^2 = 1$$

Spectral Mesh Processing

The eigenvector corresponding to the smallest non-zero eigenvalue is called Fiedler vector and it is characterized by:

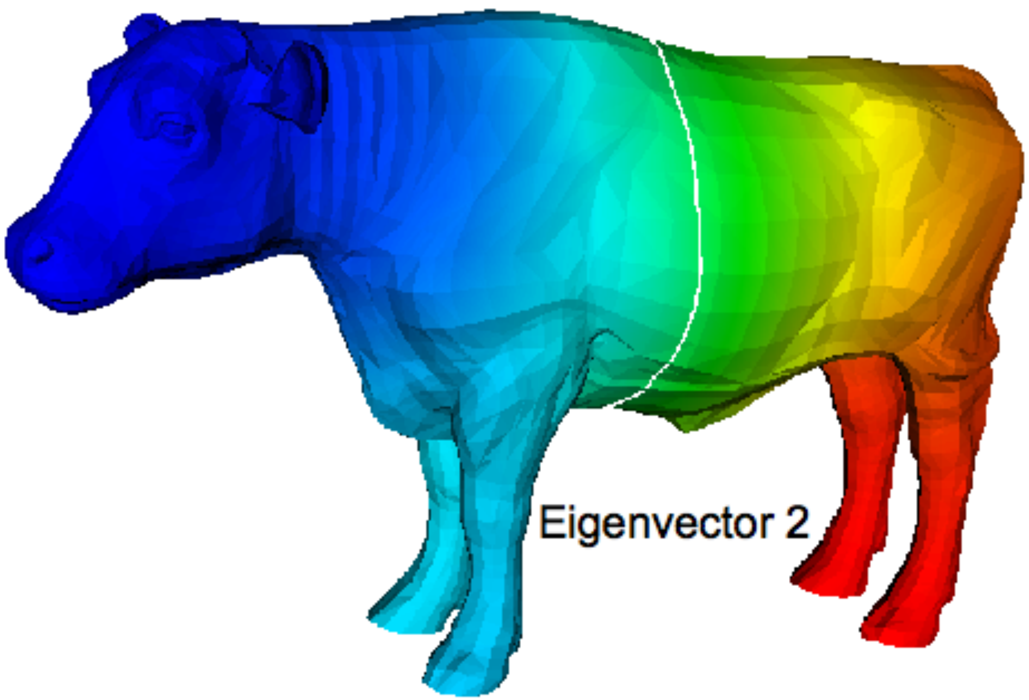
Will be minimum when adjacent vertices have similar values.

$$v_2(L) = \arg\min_u \sum_{i,j} w_{ij} (u_i - u_j)^2$$

$$\text{Subject to: } \sum u_i = 0 \text{ and } \sum u_i^2 = 1$$

Spectral Mesh Processing

The Fiedler vector also generates nodal domains with similar areas and minimal boundary curve



Spectral Mesh Processing

Fiedler Tree

[Berger, SMI'09]

Spectral Mesh Processing

Fiedler Tree

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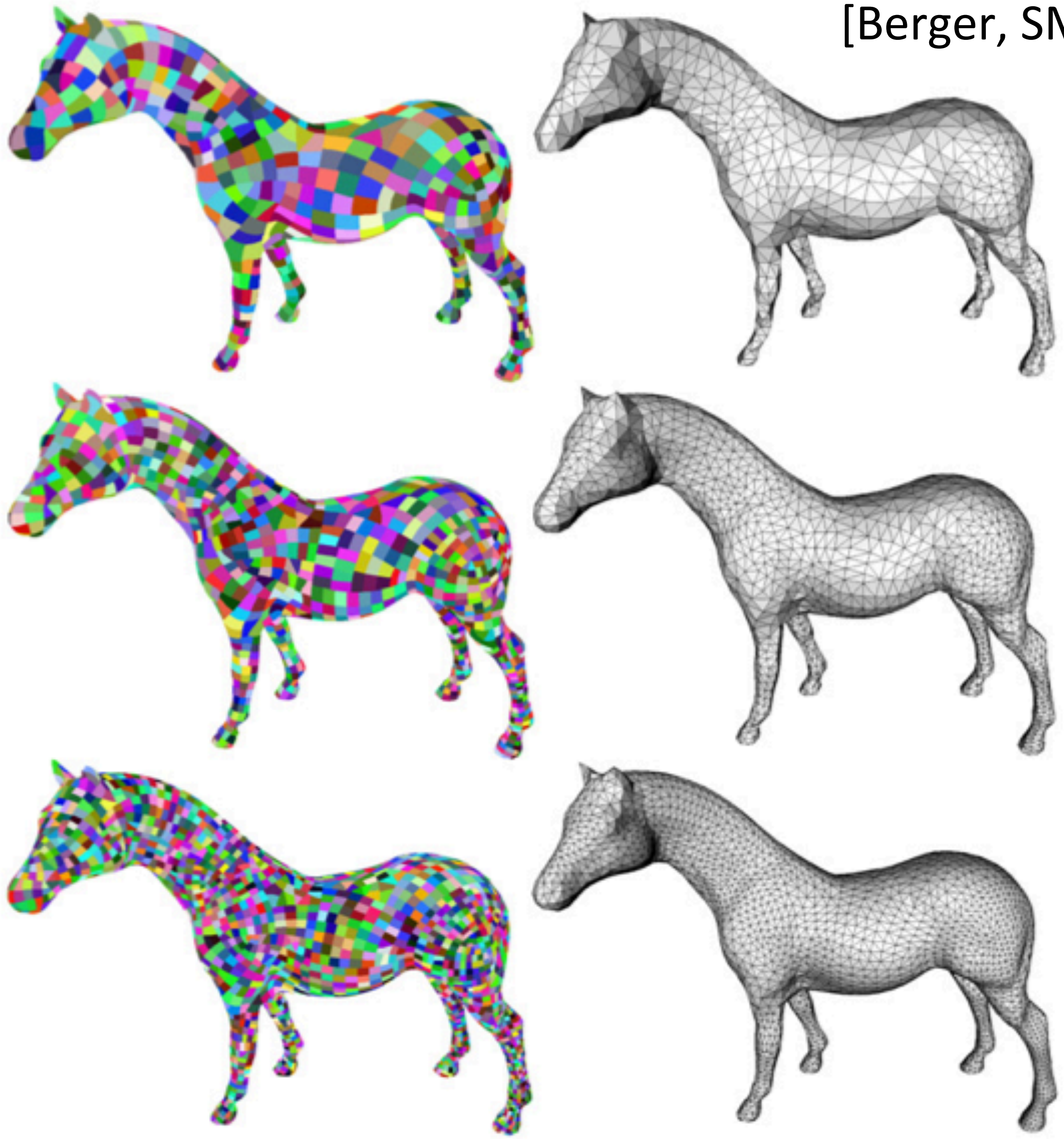

Spectral Mesh Processing

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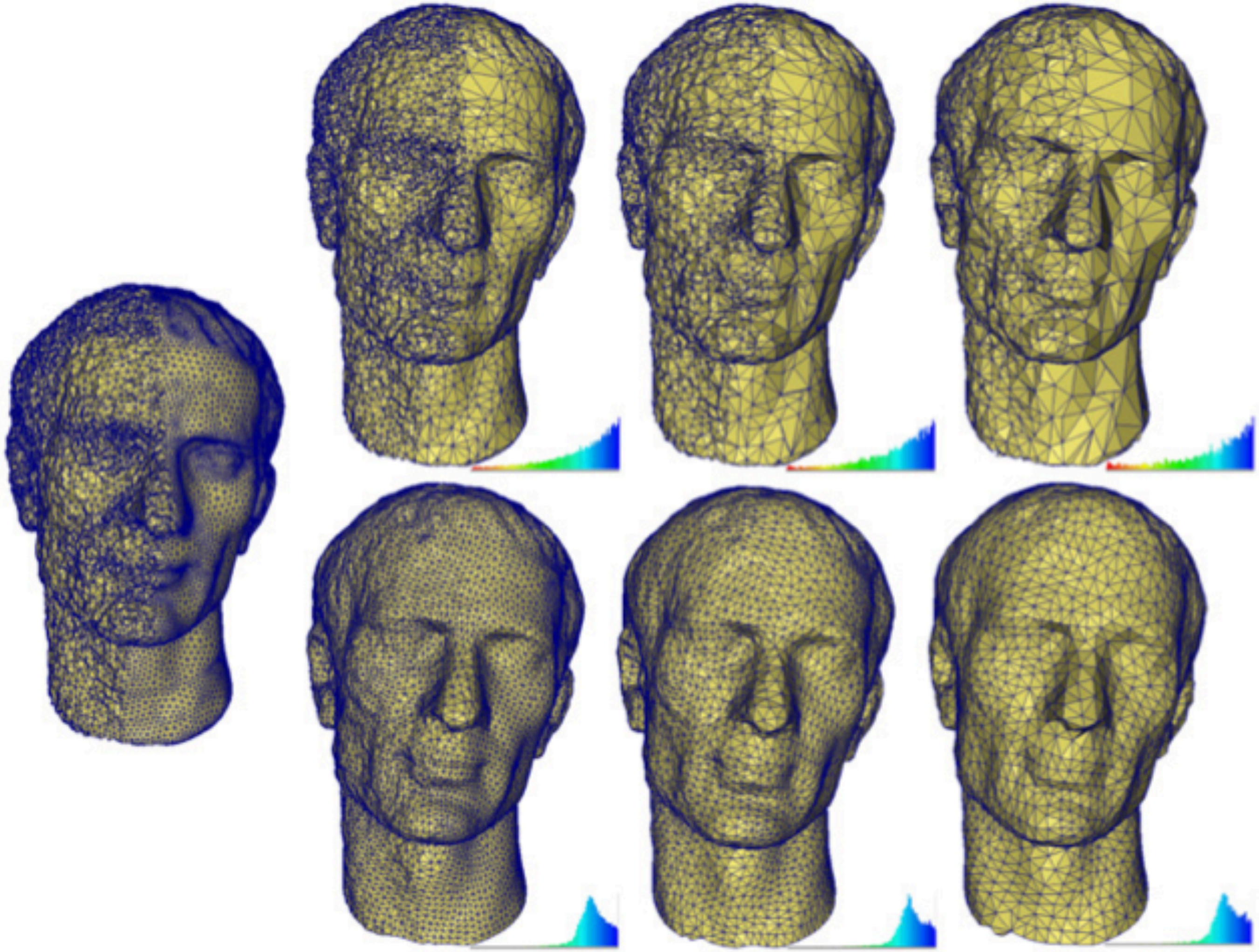

Spectral Mesh Processing

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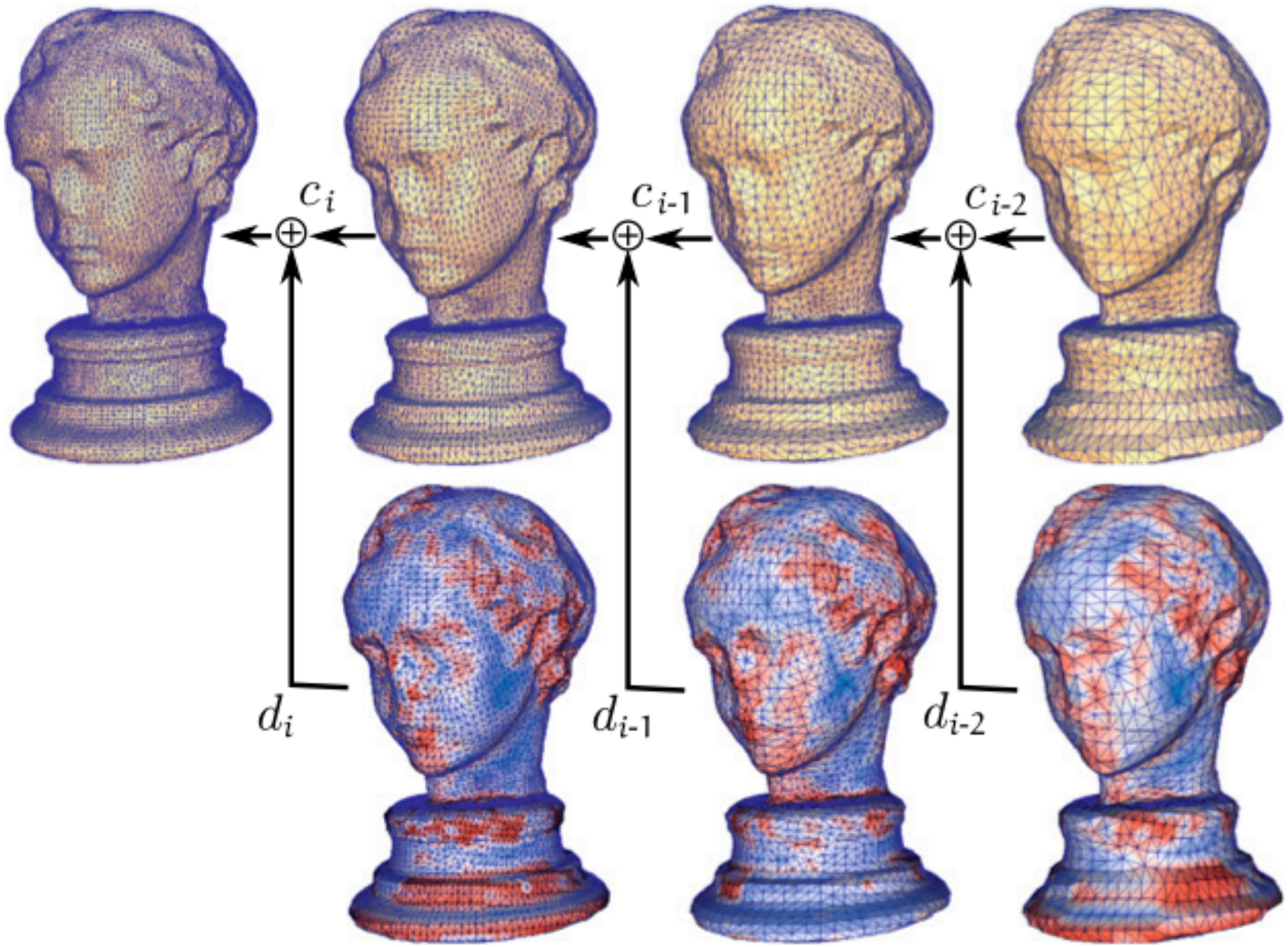

Spectral Mesh Processing

Some Interesting Results

$$\lambda_n \sim \frac{4\pi n}{\mathrm{area}(M)}, \quad \text{as} \quad n \uparrow \infty.$$

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Heat Trace

$$Z(t) = \sum_i e^{-\lambda_i t}$$

$$Z(t) \sim (4\pi T)^{-dim(M)/2} \sum_i c_i t^{i/2}$$

Spectral Mesh Processing

Some Interesting Results

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Heat Trace

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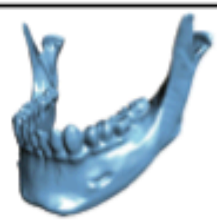
For a surface M:

$$c_0 = Area(M)$$

$$c_2 = \frac{2\pi}{3}\chi(M)$$

Spectral Mesh Processing

model	# points	estimative	surface area
	8k	heat trace triangles error	12.65 12.56 0.7%
	12k	heat trace triangles error	3.82 3.98 4.0%
	15k	heat trace triangles error	6.03 6.47 6.8%
	24k	heat trace triangles error	1.59 1.42 10.7%

That is all Folks !!