Introduction to Geometric Algebra

Lecture I

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CG UFRGS
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CNPq Conselho Nacional de Desenvolvimento Científico e Tecnológico

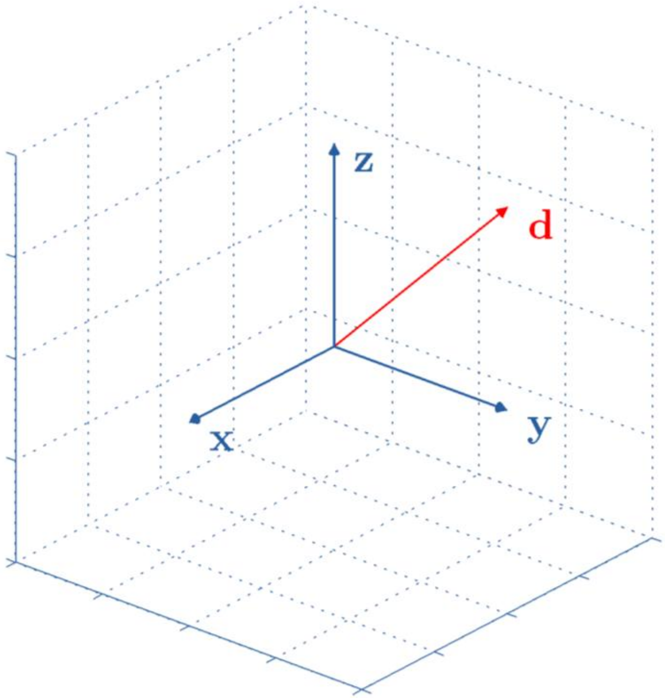
Geometric problems

- Geometric data
  - Lines, planes, circles, spheres, etc.
- Transformations
  - Rotation, translation, scaling, etc.
- Other operations
  - Intersection, basis orthogonalization, etc.

- Linear Algebra is the standard framework

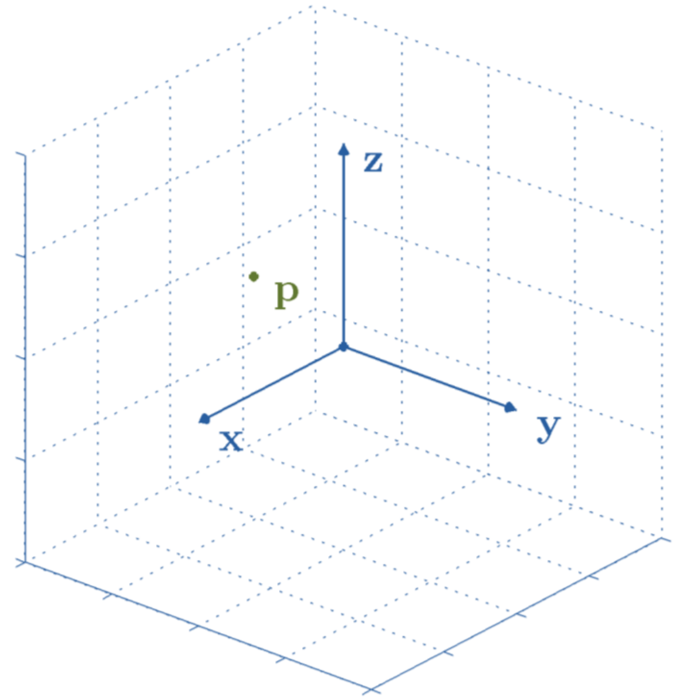
Using vectors to encode geometric data

- Directions



$$\mathbf{d} = -0.8\,x + 0.3\,y + 0.5\,z$$

- Points



$$\mathbf{p} = 0.5\,x - 0.1\,y + 0.5\,z$$

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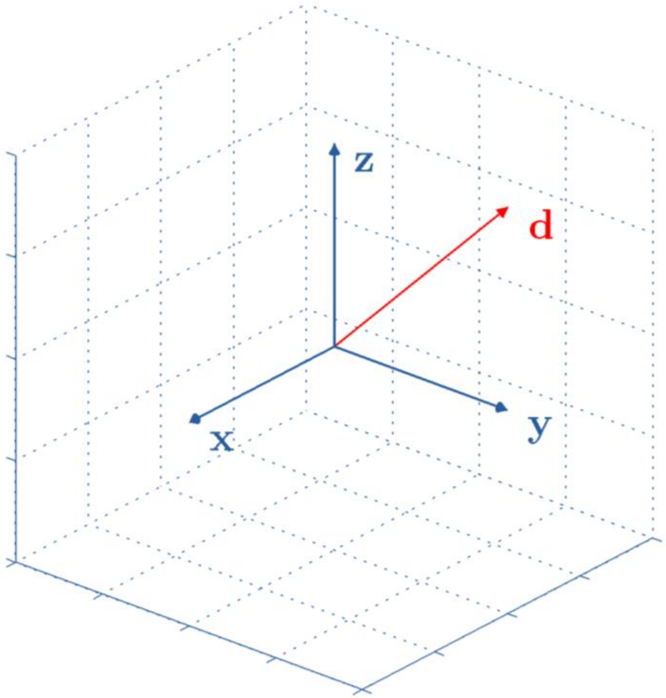
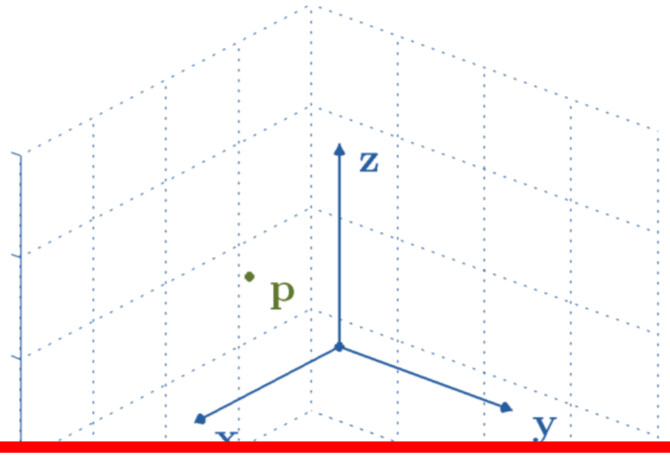
Using vectors to encode geometric data

- Directions



$\mathbf{d} = -0.8\,x + 0.3\,y + 0.5\,z$

- Points



Drawback

The semantic difference between a direction vector and a point vector is not encoded in the vector type itself.

R. N. Goldman (1985), Illicit expressions in vector algebra, ACM Trans. Graph., vol. 4, no. 3, pp. 223–243.

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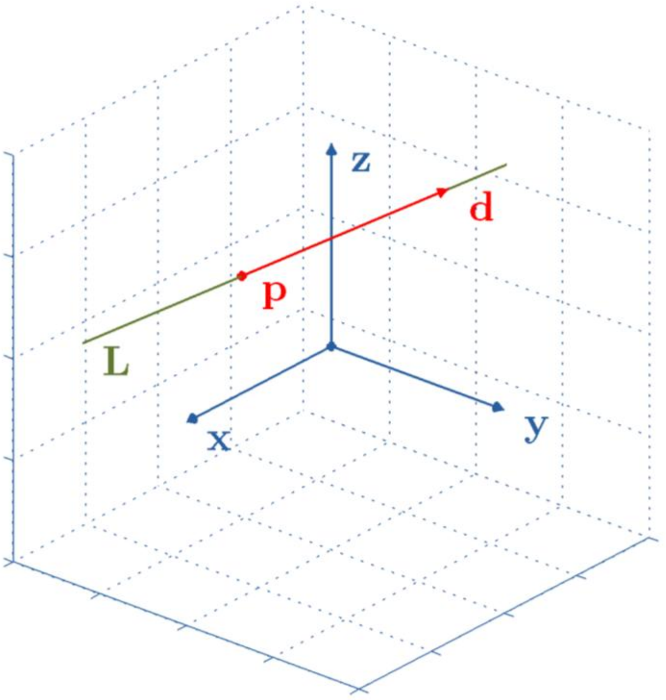
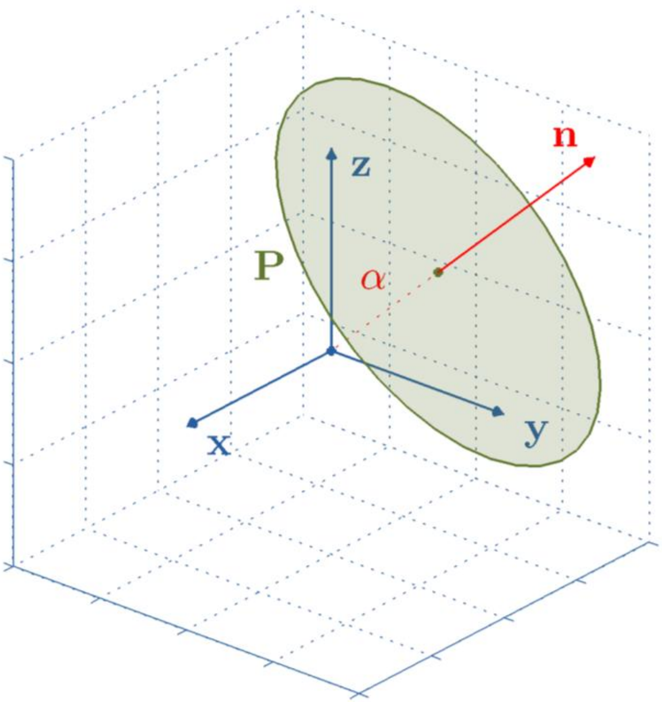
Assembling geometric data

- Straight lines
  - Point vector
  - Direction vector



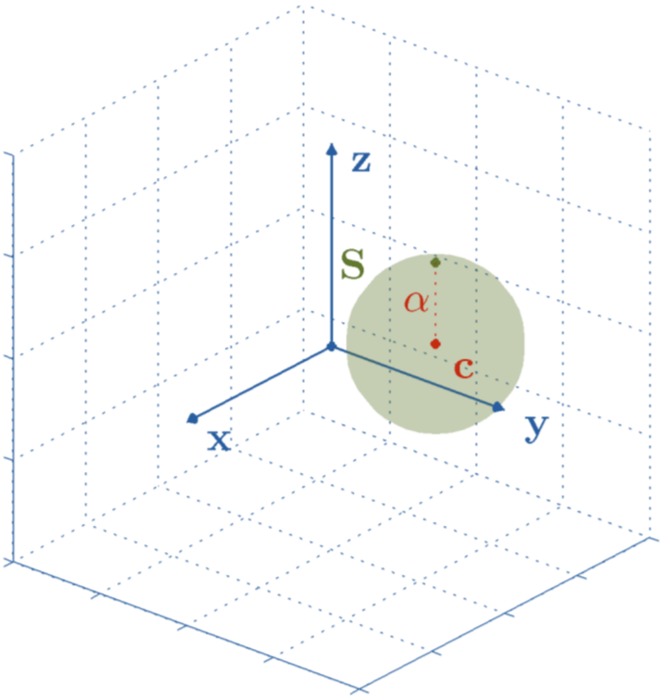
Assembling geometric data

- Straight lines
  - Point vector
  - Direction vector
- Planes
  - Normal vector
  - Distance from origin



Assembling geometric data

- Straight lines
  - Point vector
  - Direction vector
- Planes
  - Normal vector
  - Distance from origin
- Spheres
  - Center point
  - Radius



Assembling geometric data

- Straight lines
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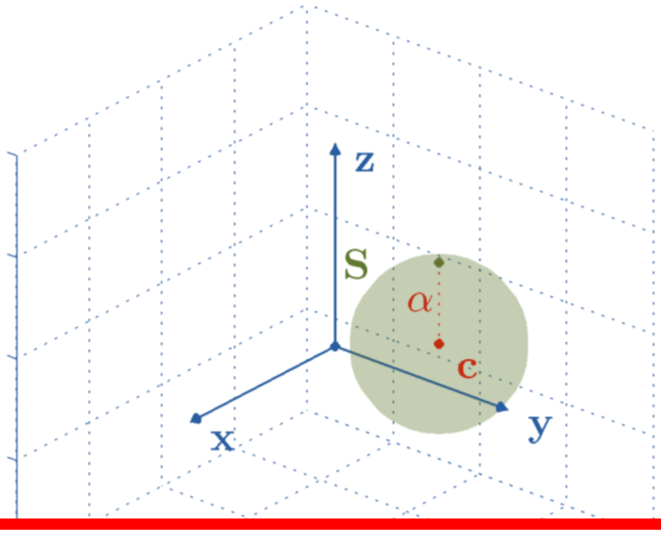


Drawback

The factorization of geometric elements prevents their use as computing primitives.

Intersection of two geometric elements

- A specialized treatment for each pair of elements
  - Straight line × Straight line
  - Straight line × Plane
  - Straight line × Sphere
  - Plane × Sphere
  - etc.
- Special cases must be handled explicitly
  - e.g., Straight line × Straight line may return
    - Empty set
    - Point
    - Straight line

Intersection of two geometric elements

- A specialized treatment for each pair of elements
  - Straight line × Straight line
  - Straight line × Plane
  - Straight line × Sphere
  - Plane × Sphere
  - etc.
- Special cases must
  - e.g., Straight line ×
    - Empty set
    - Point
    - Straight line

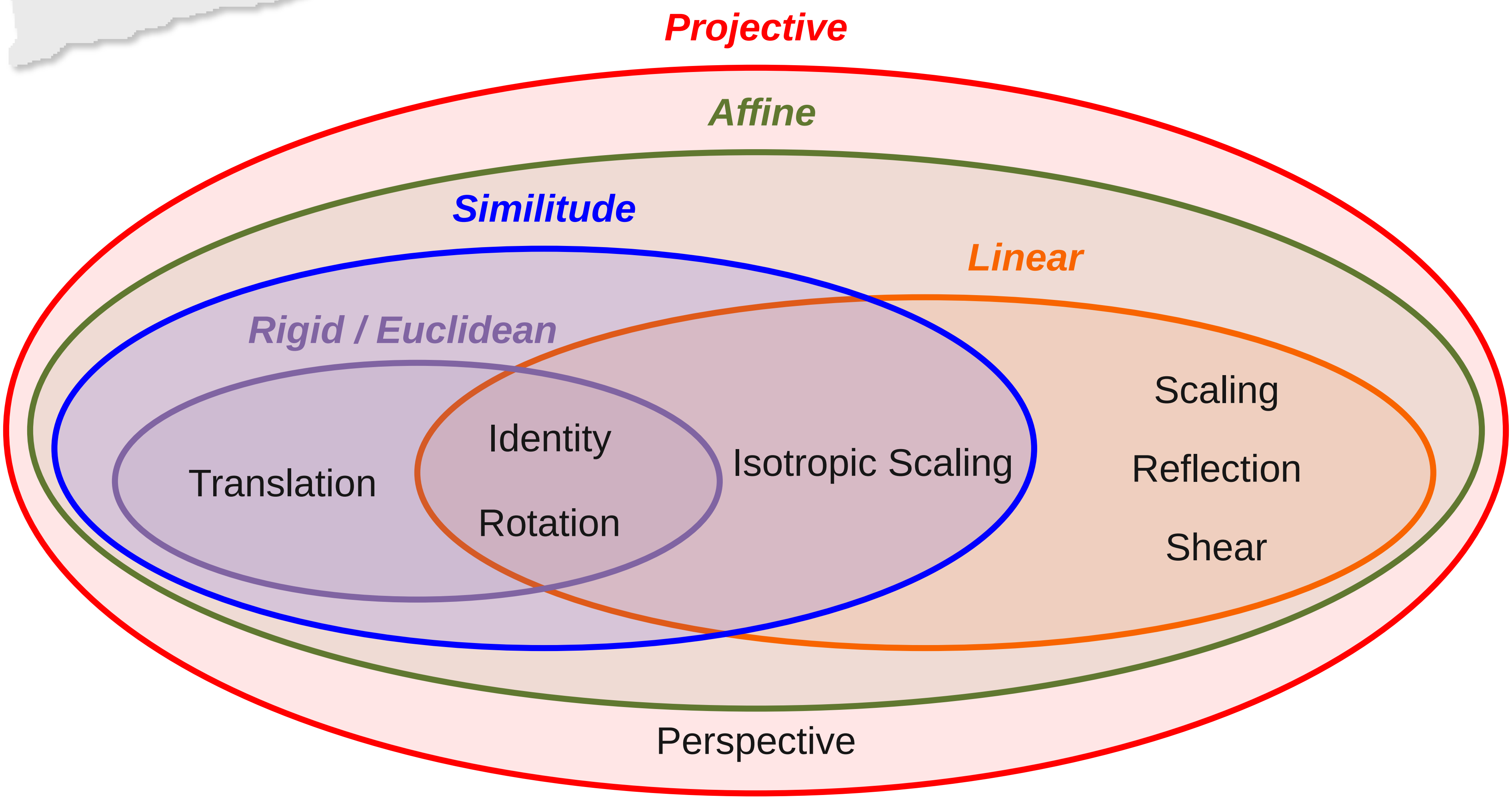
Linear Algebra Extension

Plücker Coordinates

- An alternative to represent flat geometric elements
- Points, lines and planes as elementary types
- Allow the development of more general solution
- Not fully compatible with transformation matrices

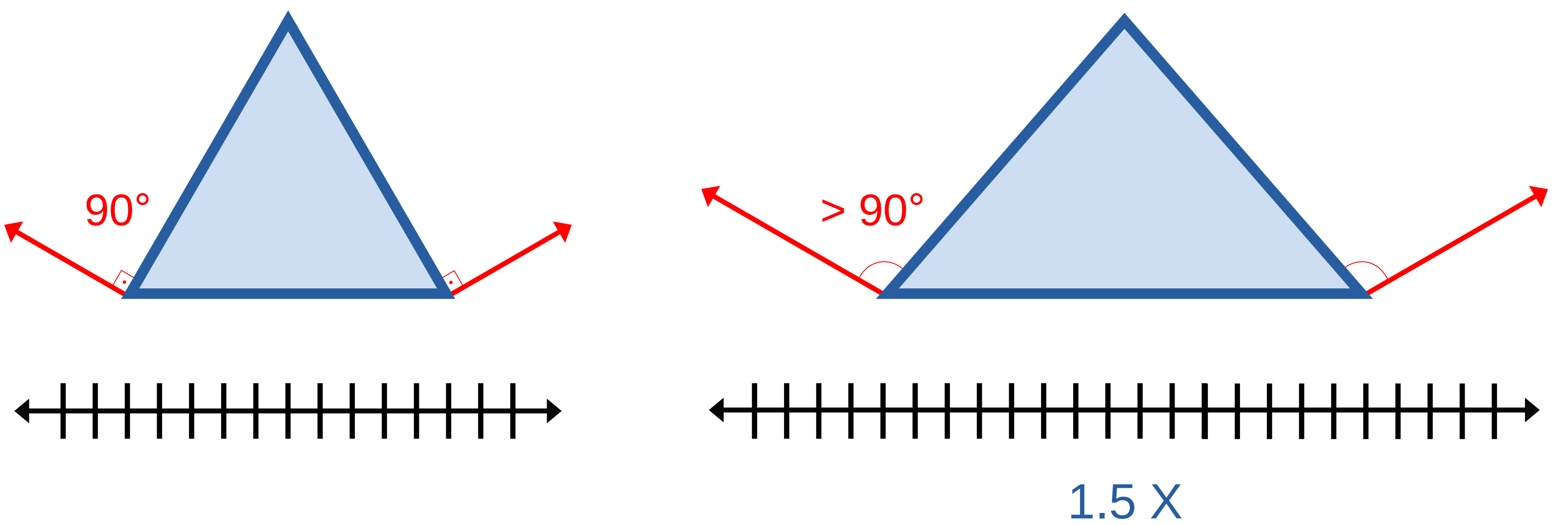
J. Stolfi (1991) Oriented projective geometry. Academic Press Professional, Inc.

Using matrices to encode transformations



Drawbacks of transformation matrices

- Non-uniform scaling affects point vectors and normal vectors differently



Drawbacks of transformation matrices

- Non-uniform scaling affects point vectors and normal vectors differently
- Rotation matrices
  - Not suitable for interpolation
  - Encode the rotation axis and angle in a costly way

Drawbacks of transformation matrices

- Non-uniform scaling affects point vectors and normal vectors differently
- Rotation matrices
  - Not suitable for interp
  - Encode the rotation a

Linear Algebra Extension

Quaternion

- Represent and interpolate rotations consistently
- Can be combined with isotropic scaling
- Not well connected with other transformations
- Not compatible with Plücker coordinates
- Defined only in 3-D

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W. R. Hamilton (1844) On a new species of imaginary quantities connected with the theory of quaternions. In Proc. of the Royal Irish Acad., vol. 2, pp. 424-434.

Linear Algebra

- Standard language for geometric problems
- Well-known limitations
- Aggregates different formalisms to obtain complete solutions
  - Matrices
  - Plücker coordinates
  - Quaternions
- Jumping back and forth between formalisms requires custom and ad hoc conversions

Geometric Algebra

- High-level framework for geometric operations
- Geometric elements as primitives for computation
- Naturally generalizes and integrates
  - Plücker coordinates
  - Quaternions
  - Complex numbers
- Extends the same solution to
  - Higher dimensions
  - All kinds of geometric elements

Lecture I

Outline

Outline for this week

- Lecture I – Mon, January 11
  - Subspaces
  - Multivector space
  - Some non-metric products
- Lecture II – Tue, January 12
  - Metric spaces
  - Some inner products
  - Dualization and undualization
- Lecture III – Fri, January 15
  - Duality relationships between products
  - Blade factorization
  - Some non-linear products

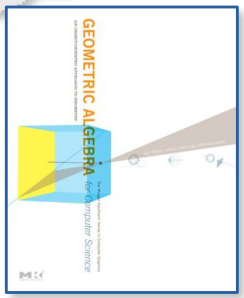
Outline for next week

- Lecture IV – Mon, January 18
  - Geometric product
  - Versors
  - Rotors
- Lecture V – Tue, January 19
  - Models of geometry
  - Euclidean vector space model
  - Homogeneous model
- Lecture VI – Fri, January 22
  - Conformal model
  - Concluding remarks

Reference material (on-line available)

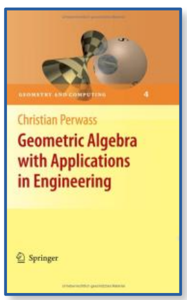
- Geometric Algebra: A powerful tool for solving geometric problems in visual computing
  L. A. F. Fernandes – M. M. Oliveira
  Tutorials of Sibgrapi (2009)
- Geometric Algebra: a Computational Framework for Geometrical Applications, Part 1
  L. Dorst – S. Mann
  IEEE Computer Graphics and Applications, 22(3):24-31 (2002)
- Geometric Algebra: a Computational Framework for Geometrical Applications, Part 2
  S. Mann – L. Dorst
  IEEE Computer Graphics and Applications, 22(4):58-67 (2002)

Reference material (books)



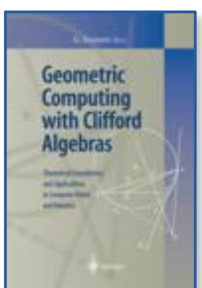
Geometric algebra for computer science

L. Dorst – D. Fontijne – S. Mann
Morgan Kaufmann Publishers (2007)



Geometric algebra with applications in engineering

C. Perwass
Springer Publishing Company (2009)



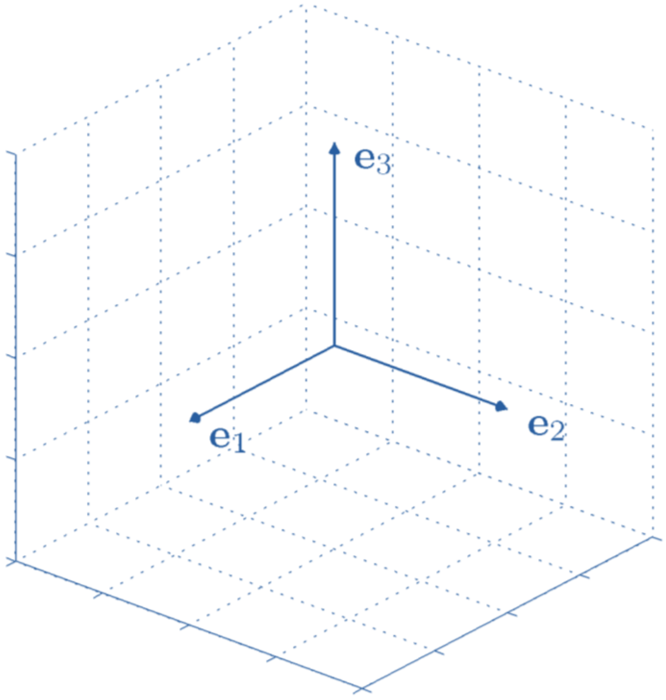
Geometric computing with Clifford algebras

G. Sommer
Springer Publishing Company (2001)

Lecture I

Subspaces

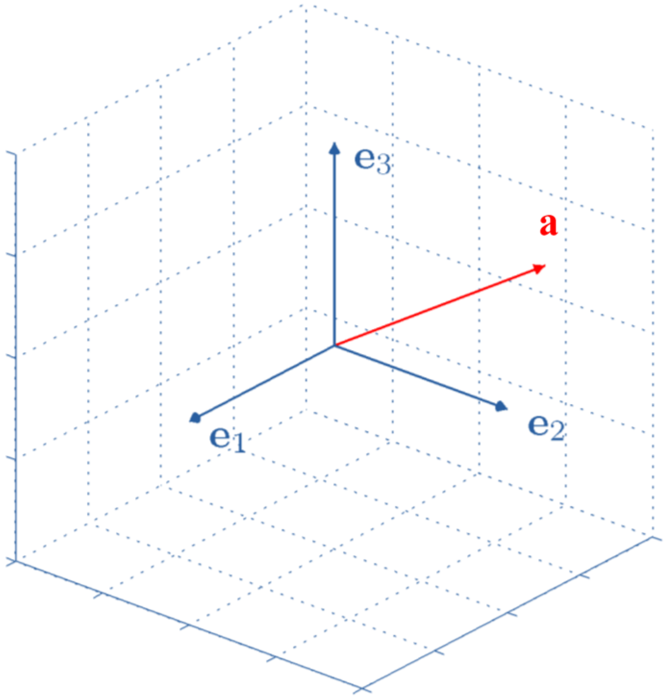
Vector space



A vector space consists, by definition, of elements called vectors

$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbb{R}^3$

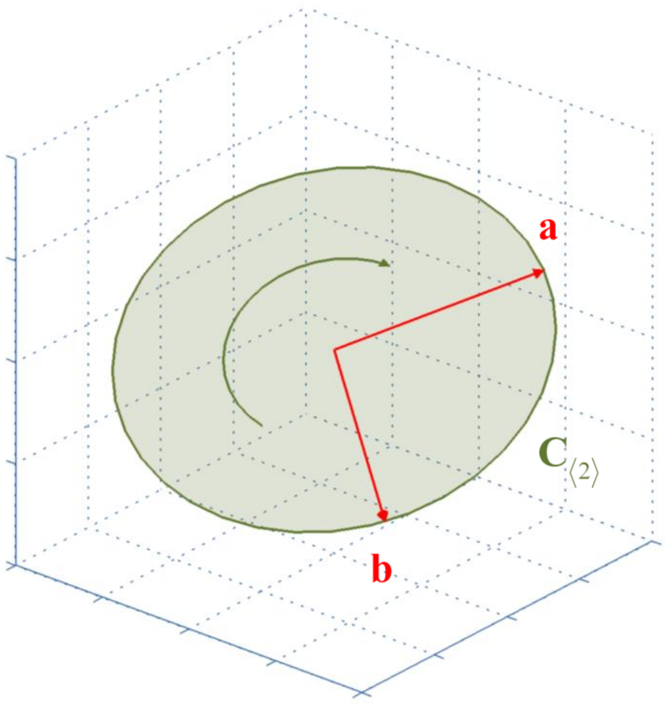
Vector in a vector space



$$\mathbf{a} = \alpha_1\ \mathbf{e}_1 + \alpha_2\ \mathbf{e}_2 + \alpha_3\ \mathbf{e}_3$$

$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbb{R}^3$

Spanning subspaces



$$C_{\langle 2 \rangle} = \mathbf{a} \wedge \mathbf{b}$$

The resulting subspace is
a primitive for computation!

Geometric Meaning

The subspace spanned
by vectors **a** and **b**

k-D oriented subspaces

- k-D oriented subspaces (or k-blades) are built as the outer product of k vectors spanning it, for $0 \leq k \leq n$

$\mathbf{B}_{\langle 0 \rangle} = \beta$	0-blade
$\mathbf{B}_{\langle 1 \rangle} = \mathbf{b}$	1-blade
$\mathbf{B}_{\langle 2 \rangle} = \mathbf{b}_1 \wedge \mathbf{b}_2$	2-blade
$\mathbf{B}_{\langle 3 \rangle} = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{b}_3$	3-blade
$\mathbf{B}_{\langle n \rangle} = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_n$	n-blade

Properties of subspaces

Attitude The equivalence class $\alpha\,\mathbf{B}_{\langle k \rangle}$, for any $\alpha \in \mathbb{R}$

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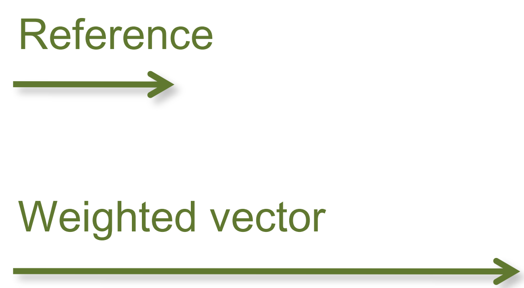
Weight The value of α in $\mathbf{B}_{\langle k \rangle} = \alpha \mathbf{J}_{\langle k \rangle}$, where $\mathbf{J}_{\langle k \rangle}$ is a reference blade with the same attitude as $\mathbf{B}_{\langle k \rangle}$



Properties of subspaces

Attitude — The equivalence class $\alpha \mathbf{B}_{\langle k \rangle}$, for any $\alpha \in \mathbb{R}$

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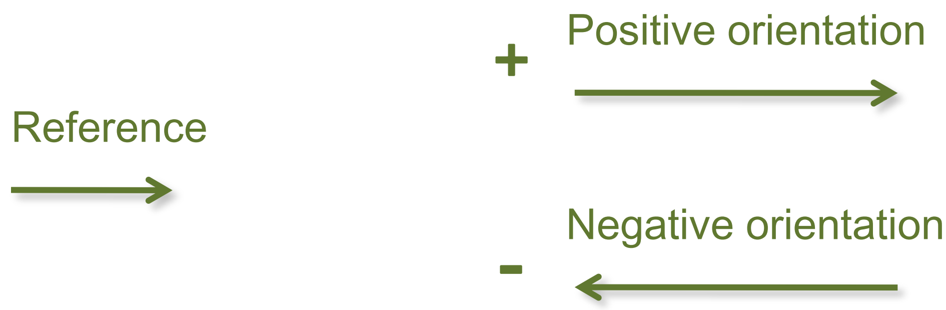
Orientation — The sign of the weight relative to $\mathbf{J}_{\langle k \rangle}$



Properties of subspaces

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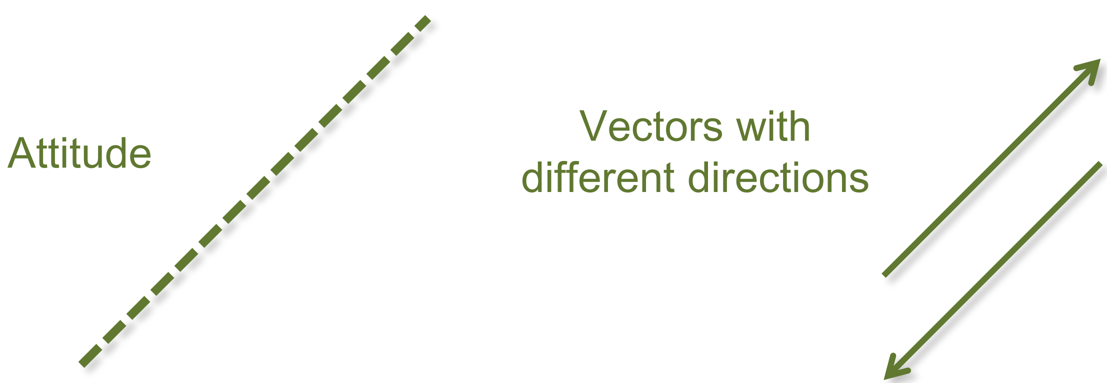
Orientation The sign of the weight relative to $\mathbf{J}_{\langle k \rangle}$

Direction The combination of attitude and orientation



We need a basis for k-D subspaces

n-D Vector Space

$$\mathbb{R}^n$$

consists of 1-D elements
called vectors, in the basis

$$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{n-1}, \mathbf{e}_n\}$$

It is not enough!

2^n-D Multivector Space

$$\bigwedge \mathbb{R}^n$$

can handle k-D elements,
for $0 \leq k \leq n$

Basis for multivector space $\bigwedge \mathbb{R}^3$

$$\left\{ 1, \quad \mathbf{e}_1, \quad \mathbf{e}_2, \quad \mathbf{e}_3, \quad \mathbf{e}_1 \wedge \mathbf{e}_2, \quad \mathbf{e}_1 \wedge \mathbf{e}_3, \quad \mathbf{e}_2 \wedge \mathbf{e}_3, \quad \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \right\}$$

Scalars	Vector Space	Bivector Space	Trivector Space
$\mathbb{R} = \bigwedge^0 \mathbb{R}^3$	$\mathbb{R}^3 = \bigwedge^1 \mathbb{R}^3$	$\bigwedge^2 \mathbb{R}^3$	$\bigwedge^3 \mathbb{R}^3$
$C(3,0) = 1$	$C(3,1) = 3$	$C(3,2) = 3$	$C(3,3) = 1$

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Multivectors

- Basis for multivector space $\bigwedge \mathbb{R}^3$

$$\underbrace{\{\ 1,}_{\text{Scalars}}\ \underbrace{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,}_{\text{Vector Space}}\ \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2,\ \mathbf{e}_1 \wedge \mathbf{e}_3,\ \mathbf{e}_2 \wedge \mathbf{e}_3,}_{\text{Bivector Space}}\ \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\ \}}_{\text{Trivector Space}}$$

Scalars	Vector Space	Bivector Space	Trivector Space
$\mathbb{R}$	$\mathbb{R}^3$	$\bigwedge^2 \mathbb{R}^3$	$\bigwedge^3 \mathbb{R}^3$

- Multivector

$$\begin{aligned} M = {} & \alpha_1 \\ & + \alpha_2\, \mathbf{e}_1 + \alpha_3\, \mathbf{e}_2 + \alpha_4\, \mathbf{e}_3 \\ & + \alpha_5\, \mathbf{e}_1 \wedge \mathbf{e}_2 + \alpha_6\, \mathbf{e}_1 \wedge \mathbf{e}_3 + \alpha_7\, \mathbf{e}_2 \wedge \mathbf{e}_3 \\ & + \alpha_8\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \end{aligned}$$

Definition issues

Multivector	The weighted combination of basis elements of $\bigwedge \mathbb{R}^n$
k-vector Step	The weighted combination of basis elements of $\bigwedge^k \mathbb{R}^n$
k-blade Grade	The outer product of k vector factors

Notable *k*-vectors

- Only these k-vectors are always also blades in n-D

k-vector	Linear Space	Special Name
0-vector	$\bigwedge^0 \mathbb{R}^n = \mathbb{R}$	scalar
1-vector	$\bigwedge^1 \mathbb{R}^n = \mathbb{R}^n$	vector
$(n$-$1)$-vector	$\bigwedge^{n-1} \mathbb{R}^n$	pseudovector
n-vector	$\bigwedge^n \mathbb{R}^n$	pseudoscalar

Lecture I

Outer product

Properties of the outer product

$$\wedge : \bigwedge^r \mathbb{R}^n \times \bigwedge^s \mathbb{R}^n \rightarrow \bigwedge^{r+s} \mathbb{R}^n$$

Antisymmetry	$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}, \quad \text{thus } \mathbf{c} \wedge \mathbf{c} = 0$
Scalars commute	$\mathbf{a} \wedge (\beta\, \mathbf{b}) = \beta (\mathbf{a} \wedge \mathbf{b})$
Distributivity	$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$
Associativity	$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$

Computing with the outer product

- Basis for multivector space $\bigwedge \mathbb{R}^3$

$$\{\ \underbrace{1,}_{\text{Scalars}}\ \ \underbrace{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,}_{\text{Vector Space}}\ \ \underbrace{\mathbf{e}_1\wedge\mathbf{e}_2,\ \mathbf{e}_1\wedge\mathbf{e}_3,\ \mathbf{e}_2\wedge\mathbf{e}_3,}_{\text{Bivector Space}}\ \ \underbrace{\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3}_{\text{Trivector Space}}\ \}$$

Scalars	Vector Space	Bivector Space	Trivector Space
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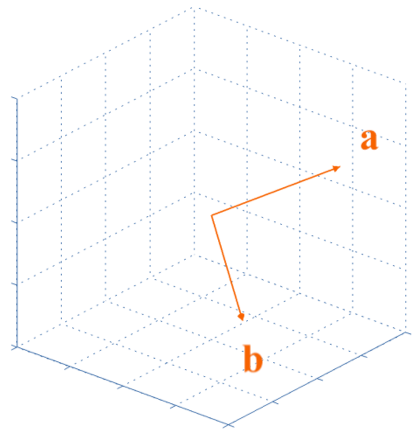
$$\mathbf{b} = \beta_1\,\mathbf{e}_1 + \beta_2\,\mathbf{e}_2 + \beta_3\,\mathbf{e}_3$$



Computing with the outer product

- Basis for multivector space $\bigwedge \mathbb{R}^3$

$$\{ \underbrace{1,}_{\text{Scalars}} \quad \underbrace{\mathbf{e}_1, \quad \mathbf{e}_2, \quad \mathbf{e}_3,}_{\text{Vector Space}} \quad \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2, \quad \mathbf{e}_1 \wedge \mathbf{e}_3, \quad \mathbf{e}_2 \wedge \mathbf{e}_3,}_{\text{Bivector Space}} \quad \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3}_{\text{Trivector Space}} \}$$

Scalars	Vector Space	Bivector Space	Trivector Space
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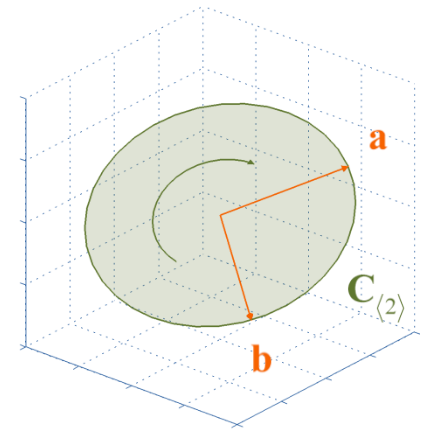
$$\mathbf{b} = \beta_1\,\mathbf{e}_1 + \beta_2\,\mathbf{e}_2 + \beta_3\,\mathbf{e}_3$$

$$\mathbf{C}_{\langle 2 \rangle} = \mathbf{a} \wedge \mathbf{b}$$



Computing with th…

- Basis for multivector s…

$$\{\ 1,\quad \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\quad \mathbf{e}_1 \ldots$$

Scalars — $\mathbb{R}$

Vector Space — $\mathbb{R}^3$

$$\mathbf{a} = \alpha_1\,\mathbf{e}_1 + \alpha_2\,\mathbf{e}_2 + \alpha_3\,\mathbf{e}_3$$

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$$\mathbf{C}_{\langle 2 \rangle} = \mathbf{a} \wedge \mathbf{b}$$

$$\mathbf{C}_{\langle 2 \rangle} = \left(\alpha_1\,\mathbf{e}_1 + \alpha_2\,\mathbf{e}_2 + \alpha_3\,\mathbf{e}_3\right) \wedge \left(\beta_1\,\mathbf{e}_1 + \beta_2\,\mathbf{e}_2 + \beta_3\,\mathbf{e}_3\right)$$

$$\begin{aligned}\mathbf{C}_{\langle 2 \rangle} &= \alpha_1\,\mathbf{e}_1 \wedge \left(\beta_1\,\mathbf{e}_1 + \beta_2\,\mathbf{e}_2 + \beta_3\,\mathbf{e}_3\right) \\ &+ \alpha_2\,\mathbf{e}_2 \wedge \left(\beta_1\,\mathbf{e}_1 + \beta_2\,\mathbf{e}_2 + \beta_3\,\mathbf{e}_3\right) \\ &+ \alpha_3\,\mathbf{e}_3 \wedge \left(\beta_1\,\mathbf{e}_1 + \beta_2\,\mathbf{e}_2 + \beta_3\,\mathbf{e}_3\right)\end{aligned}$$

$$\begin{aligned}\mathbf{C}_{\langle 2 \rangle} &= \alpha_1\beta_1\,\mathbf{e}_1 \wedge \mathbf{e}_1 + \alpha_1\beta_2\,\mathbf{e}_1 \wedge \mathbf{e}_2 + \alpha_1\beta_3\,\mathbf{e}_1 \wedge \mathbf{e}_3 \\ &+ \alpha_2\beta_1\,\mathbf{e}_2 \wedge \mathbf{e}_1 + \alpha_2\beta_2\,\mathbf{e}_2 \wedge \mathbf{e}_2 + \alpha_2\beta_3\,\mathbf{e}_2 \wedge \mathbf{e}_3 \\ &+ \alpha_3\beta_1\,\mathbf{e}_3 \wedge \mathbf{e}_1 + \alpha_3\beta_2\,\mathbf{e}_3 \wedge \mathbf{e}_2 + \alpha_3\beta_3\,\mathbf{e}_3 \wedge \mathbf{e}_3\end{aligned}$$

Antisymmetry	$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$, thus $\mathbf{c} \wedge \mathbf{c} = 0$
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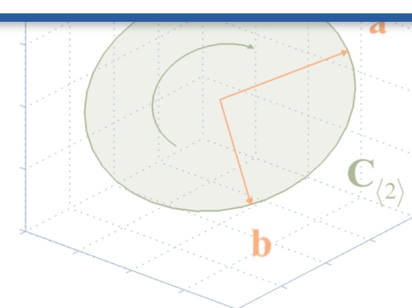


Computing with th

- Basis for multivector s

$$\{ \; 1, \quad \mathbf{e}_1, \; \mathbf{e}_2, \; \mathbf{e}_3, \quad \mathbf{e}_1 \ldots$$

Scalars $\mathbb{R}$ Vector Space $\mathbb{R}^3$

$$\mathbf{a} = \alpha_1 \, \mathbf{e}_1 + \alpha_2 \, \mathbf{e}_2 + \alpha_3 \, \mathbf{e}_3$$

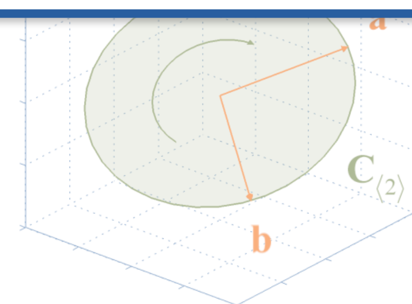
$$\mathbf{b} = \beta_1 \, \mathbf{e}_1 + \beta_2 \, \mathbf{e}_2 + \beta_3 \, \mathbf{e}_3$$

$$\mathbf{C}_{\langle 2 \rangle} = \alpha_1\beta_1 \, \mathbf{e}_1 \wedge \mathbf{e}_1 + \alpha_1\beta_2 \, \mathbf{e}_1 \wedge \mathbf{e}_2 + \alpha_1\beta_3 \, \mathbf{e}_1 \wedge \mathbf{e}_3 + \alpha_2\beta_1 \, \mathbf{e}_2 \wedge \mathbf{e}_1 + \alpha_2\beta_2 \, \mathbf{e}_2 \wedge \mathbf{e}_2 + \alpha_2\beta_3 \, \mathbf{e}_2 \wedge \mathbf{e}_3 + \alpha_3\beta_1 \, \mathbf{e}_3 \wedge \mathbf{e}_1 + \alpha_3\beta_2 \, \mathbf{e}_3 \wedge \mathbf{e}_2 + \alpha_3\beta_3 \, \mathbf{e}_3 \wedge \mathbf{e}_3$$

$$\mathbf{C}_{\langle 2 \rangle} = \alpha_1\beta_2 \, \mathbf{e}_1 \wedge \mathbf{e}_2 + \alpha_1\beta_3 \, \mathbf{e}_1 \wedge \mathbf{e}_3 - \alpha_2\beta_1 \, \mathbf{e}_1 \wedge \mathbf{e}_2 + \alpha_2\beta_3 \, \mathbf{e}_2 \wedge \mathbf{e}_3 - \alpha_3\beta_1 \, \mathbf{e}_1 \wedge \mathbf{e}_3 - \alpha_3\beta_2 \, \mathbf{e}_2 \wedge \mathbf{e}_3$$

$$\mathbf{C}_{\langle 2 \rangle} = (\alpha_1\beta_2 - \alpha_2\beta_1) \, \mathbf{e}_1 \wedge \mathbf{e}_2 + (\alpha_1\beta_3 - \alpha_3\beta_1) \, \mathbf{e}_1 \wedge \mathbf{e}_3 + (\alpha_2\beta_3 - \alpha_3\beta_2) \, \mathbf{e}_2 \wedge \mathbf{e}_3$$

$\mathbf{C} \ldots \mathbf{b}$



Antisymmetry	$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$, thus $\mathbf{c} \wedge \mathbf{c} = 0$
Scalars commute	$\mathbf{a} \wedge (\beta \, \mathbf{b}) = \beta (\mathbf{a} \wedge \mathbf{b})$
Distributivity	$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$

Computing with the outer product

- Basis for multivector space $\bigwedge \mathbb{R}^3$

$$\{\underbrace{1,}_{\text{Scalars}} \quad \underbrace{\mathbf{e}_1, \quad \mathbf{e}_2, \quad \mathbf{e}_3,}_{\text{Vector Space}} \quad \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2, \quad \mathbf{e}_1 \wedge \mathbf{e}_3, \quad \mathbf{e}_2 \wedge \mathbf{e}_3,}_{\text{Bivector Space}} \quad \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3}_{\text{Trivector Space}}\}$$

Scalars	Vector Space	Bivector Space	Trivector Space
$\mathbb{R}$	$\mathbb{R}^3$	$\bigwedge^2 \mathbb{R}^3$	$\bigwedge^3 \mathbb{R}^3$

$$\mathbf{a} = \alpha_1\, \mathbf{e}_1 + \alpha_2\, \mathbf{e}_2 + \alpha_3\, \mathbf{e}_3$$

$$\mathbf{b} = \beta_1\, \mathbf{e}_1 + \beta_2\, \mathbf{e}_2 + \beta_3\, \mathbf{e}_3$$

$$\begin{aligned} \mathbf{C}_{\langle 2 \rangle} = \mathbf{a} \wedge \mathbf{b} = &\left(\alpha_1\beta_2 - \alpha_2\beta_1\right) \mathbf{e}_1 \wedge \mathbf{e}_2 \\ &+ \left(\alpha_1\beta_3 - \alpha_3\beta_1\right) \mathbf{e}_1 \wedge \mathbf{e}_3 \\ &+ \left(\alpha_2\beta_3 - \alpha_3\beta_2\right) \mathbf{e}_2 \wedge \mathbf{e}_3 \end{aligned}$$

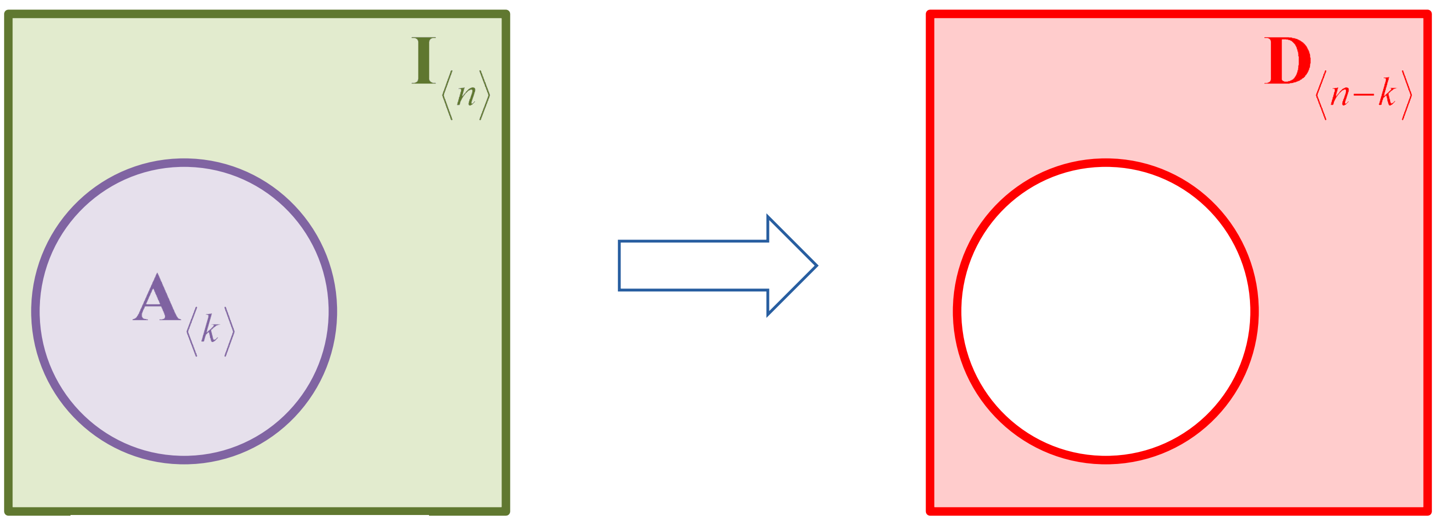
Lecture I

The regressive product

The notion of duality

- The complementary grade of a grade k is n-k

$$\bigwedge^{k} \mathbb{R}^{n} \leftrightarrow \bigwedge^{n-k} \mathbb{R}^{n}$$

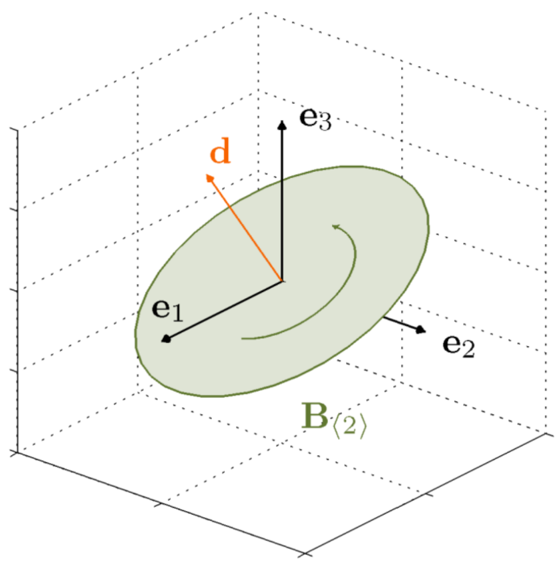


Venn Diagrams

The notion of duality

- The complementary grade of a grade k is n-k

$$\bigwedge^k \mathbb{R}^n \leftrightarrow \bigwedge^{n-k} \mathbb{R}^n$$



The notion of duality

- The complementary grade of a grade k is n-k

$$\bigwedge^{k} \mathbb{R}^{n} \leftrightarrow \bigwedge^{n-k} \mathbb{R}^{n}$$

- The regressive product is correctly dual to the outer product

$$\wedge \leftrightarrow \vee$$

- k-blade are also built as the regressive product of n-k pseudovectors

Regressive Product

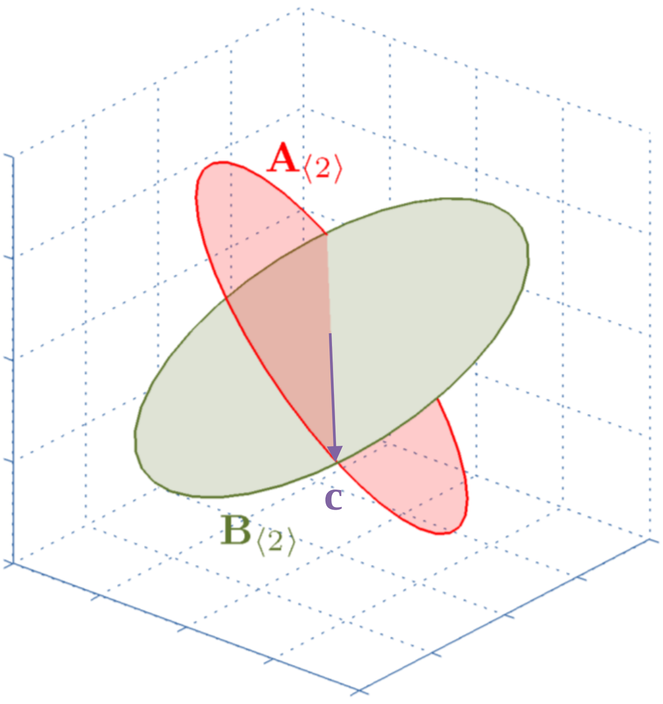
- Returns the subspace shared by two blades

$$A_{\langle 2 \rangle} = \mathbf{a} \wedge \mathbf{c}$$

$$B_{\langle 2 \rangle} = \mathbf{c} \wedge \mathbf{b}$$

$$\text{for } \mathbf{I}_{\langle 3 \rangle} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$$

$$\mathbf{c} = \mathbf{A}_{\langle 2 \rangle} \vee \mathbf{B}_{\langle 2 \rangle}$$



Properties of the regressive product

$$\vee : \bigwedge^{n-r} \mathbb{R}^n \times \bigwedge^{n-s} \mathbb{R}^n \to \bigwedge^{n-(r+s)} \mathbb{R}^n$$

Antisymmetry

$$\mathbf{A}_{\langle n-1\rangle} \vee \mathbf{B}_{\langle n-1\rangle} = -\mathbf{B}_{\langle n-1\rangle} \vee \mathbf{A}_{\langle n-1\rangle},$$

$$\text{thus } \mathbf{C}_{\langle n-1\rangle} \vee \mathbf{C}_{\langle n-1\rangle} = 0$$

Scalars commute

$$\mathbf{A}_{\langle n-1\rangle} \vee \left(\beta\, \mathbf{B}_{\langle n-1\rangle}\right) = \beta \left(\mathbf{A}_{\langle n-1\rangle} \vee \mathbf{B}_{\langle n-1\rangle}\right)$$

Distributivity

$$\mathbf{A}_{\langle n-1\rangle} \vee \left(\mathbf{B}_{\langle n-1\rangle} + \mathbf{C}_{\langle n-1\rangle}\right) = \mathbf{A}_{\langle n-1\rangle} \vee \mathbf{B}_{\langle n-1\rangle} + \mathbf{A}_{\langle n-1\rangle} \vee \mathbf{C}_{\langle n-1\rangle}$$

Associativity

$$\mathbf{A}_{\langle n-1\rangle} \vee \left(\mathbf{B}_{\langle n-1\rangle} \vee \mathbf{C}_{\langle n-1\rangle}\right) = \left(\mathbf{A}_{\langle n-1\rangle} \vee \mathbf{B}_{\langle n-1\rangle}\right) \vee \mathbf{C}_{\langle n-1\rangle}$$

Credits

Hermann G. Grassmann
(1809-1877)

Grassmann, H. G. (1877) *Verwendung der Ausdehnungslehre fur die allgemeine Theorie der Polaren und den Zusammenhang algebraischer Gebilde*. **J. Reine Angew. Math. (Crelle's J.)**, Walter de Gruyter Und Co., 84, 273-283