Introduction to Geometric Algebra

Lecture II

Leandro A. F. Fernandes
laffernandes@inf.ufrgs.br

Manuel M. Oliveira
oliveira@inf.ufrgs.br

Visgraf - Summer School in Computer Graphics - 2010

CG UFRGS

.inf INSTITUTO DE INFORMÁTICA UFRGS

UFRGS UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL

CNPq Conselho Nacional de Desenvolvimento Científico e Tecnológico

Checkpoint

- Multivector space $\bigwedge \mathbb{R}^n$

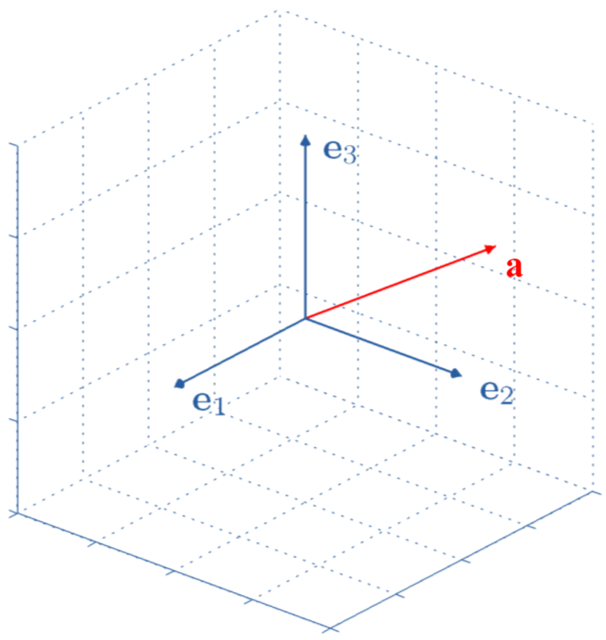
e.g., Basis for $\bigwedge \mathbb{R}^3$

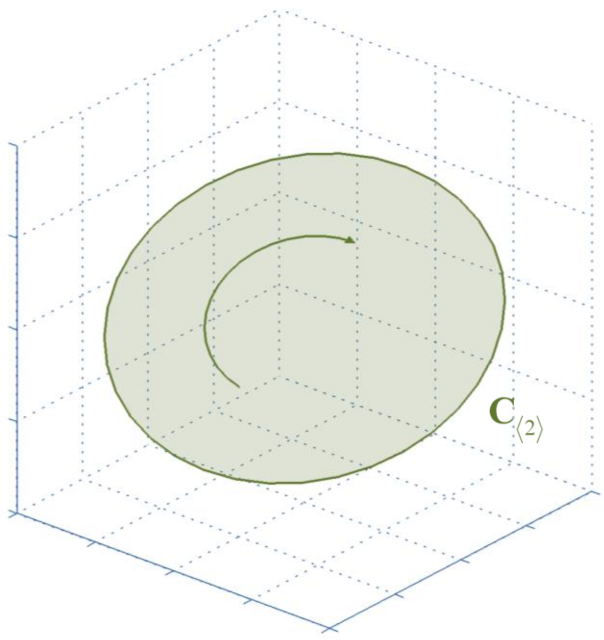
$$\{\; \underbrace{1,}_{\text{Scalars}} \quad \underbrace{\mathbf{e}_1, \quad \mathbf{e}_2, \quad \mathbf{e}_3,}_{\text{Vector Space}} \quad \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2, \quad \mathbf{e}_1 \wedge \mathbf{e}_3, \quad \mathbf{e}_2 \wedge \mathbf{e}_3,}_{\text{Bivector Space}} \quad \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3}_{\text{Trivector Space}} \;\}$$

Scalars	Vector Space	Bivector Space	Trivector Space
$\mathbb{R}$	$\mathbb{R}^3$	$\bigwedge^2 \mathbb{R}^3$	$\bigwedge^3 \mathbb{R}^3$

Checkpoint

- k-D oriented subspaces (or k-blades) as primitives

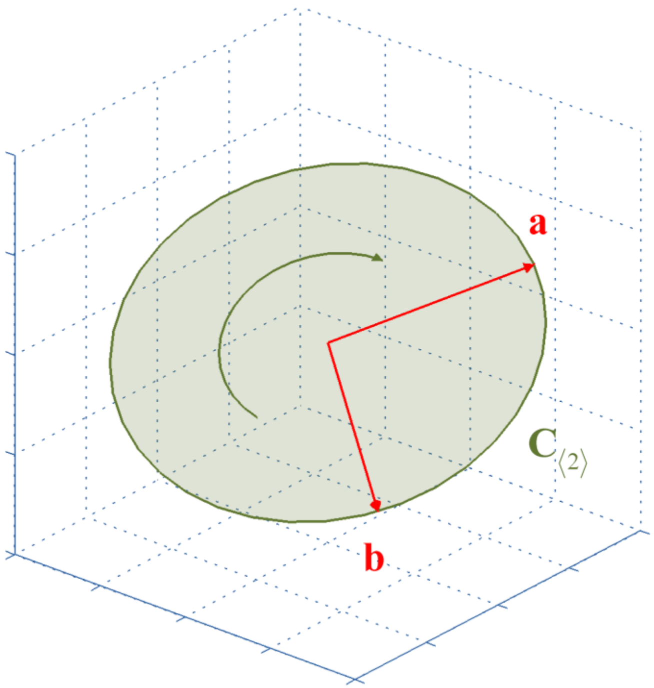




Checkpoint

- k-blades are built as
  - the outer product of k vectors

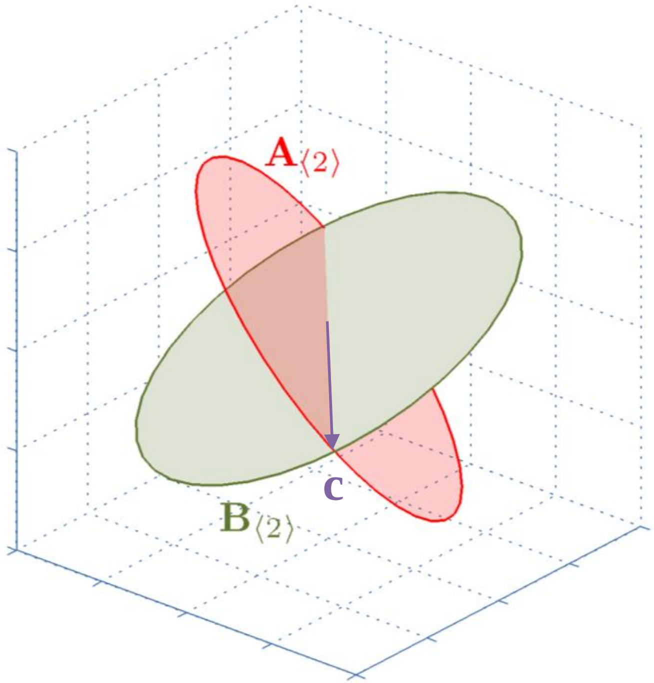
$$\mathbf{C}_{\langle 2 \rangle} = \mathbf{a} \wedge \mathbf{b}$$



Checkpoint

- k-blades are built as
  - the outer product of k vectors
  - the regressive product of n-k pseudovectors

$$\mathbf{c} = \mathbf{A}_{\langle 2 \rangle} \vee \mathbf{B}_{\langle 2 \rangle}$$



Checkpoint

- Properties of subspaces

Attitude	The equivalence class $\alpha \mathbf{B}_{\langle k \rangle}$, for any $\alpha \in \mathbb{R}$
Weight	The value of α in $\mathbf{B}_{\langle k \rangle} = \alpha \mathbf{J}_{\langle k \rangle}$, where $\mathbf{J}_{\langle k \rangle}$ is a reference blade with the same attitude as $\mathbf{B}_{\langle k \rangle}$
Orientation	The sign of the weight relative to $\mathbf{J}_{\langle k \rangle}$
Direction	The combination of attitude and orientation

Checkpoint

Multivector — The weighted combination of basis elements of $\bigwedge \mathbb{R}^n$

***k*-vector** — The weighted combination of basis elements of $\bigwedge^k \mathbb{R}^n$

Step

***k*-blade** — The outer product of k vector factors

Grade

Today

- Lecture II – Tue, January 12
  - Metric spaces
  - Some inner products
  - Dualization and undualization

Lecture II

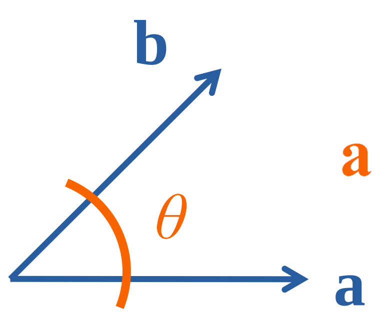
Metric and Some Inner Products

The inner product of vectors

$$\mathbf{a} \cdot \mathbf{b} = \boxed{\mathrm{Q}(\mathbf{a}, \mathbf{b})}$$

<u>Geometric Meaning</u>

Measures the relation between vectors $\mathbf{a}$ and $\mathbf{b}$



$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

<u>Bilinear Form</u>

A scalar-valued function of vectors, like in Linear Algebra

It defines a metric on the vector space

** Euclidean Metric

The inner product of vectors

$$\mathbf{a} \cdot \mathbf{b} = \boxed{Q(\mathbf{a}, \mathbf{b})}$$

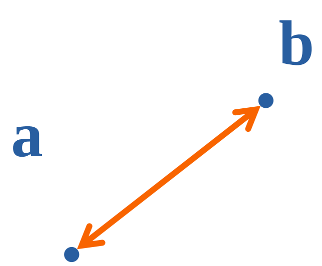
Geometric Meaning

Measures the relation between vectors $\mathbf{a}$ and $\mathbf{b}$

Bilinear Form

A scalar-valued function of vectors, like in Linear Algebra

It defines a metric on the vector space



$$\mathbf{a} \cdot \mathbf{b} = d_E^2(a, b)$$

Euclidean distance between points

** Non-Euclidean Metric!

Properties of the inner product of vectors

$$\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

Symmetry $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

Scalars commute $\mathbf{a} \cdot (\beta\, \mathbf{b}) = \beta (\mathbf{a} \cdot \mathbf{b})$

Distributivity $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

The metric matrix

- A practical way to implement the bilinear form Q
- Defines the inner product of pairs of basis vectors

$$\mathrm{M} = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix}$$

$$m_{ij} = \mathrm{Q}\left(\mathbf{e}_i, \mathbf{e}_j\right)$$

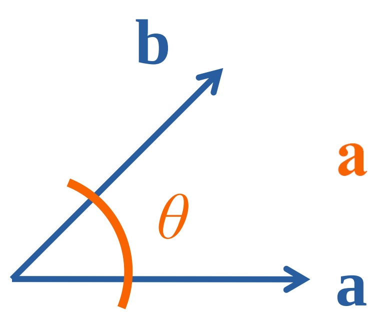
The metric matrix

- A practical way to implement the bilinear form Q
- Defines the inner pr

$$M =$$

Orthogonal 3-D Euclidean Metric

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



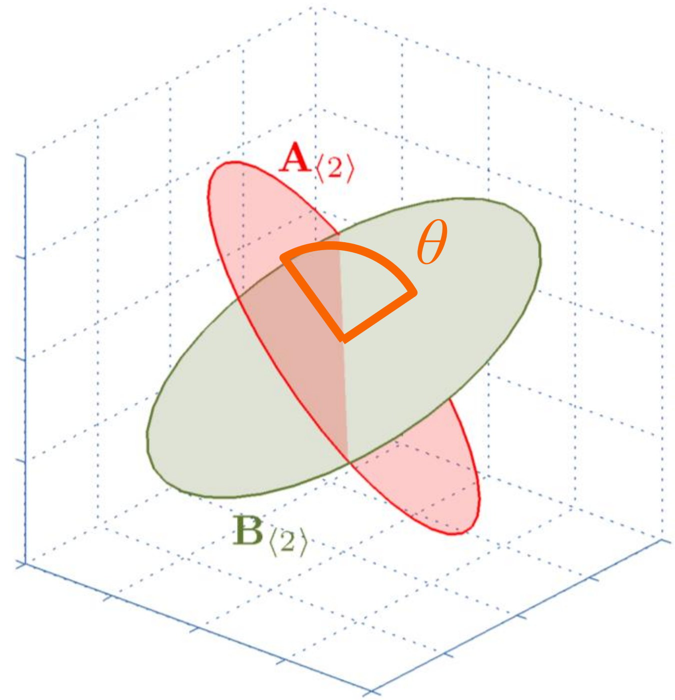
$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

UFRGS

The scalar product

All piecewise functions presented here are an abuse of notation. In practice, they are implicitly defined by the computation of the products.

$$\mathbf{A}_{\langle r \rangle} * \mathbf{B}_{\langle s \rangle} = \begin{cases} \chi & r = s \\ 0 & r \neq s \end{cases}$$



Geometric Meaning

Extends the inner product of vectors to subspaces having the same dimensionality

$$\mathbf{A}_{\langle 2 \rangle} * \mathbf{B}_{\langle 2 \rangle} = \left\| \mathbf{A}_{\langle 2 \rangle} \right\| \left\| \mathbf{B}_{\langle 2 \rangle} \right\| \cos \theta$$

UFRGS

** Euclidean Metric

Properties of the scalar product

$$*: \bigwedge^k \mathbb{R}^n \times \bigwedge^k \mathbb{R}^n \rightarrow \mathbb{R}$$

Symmetry $\mathbf{A}_{\langle r \rangle} * \mathbf{B}_{\langle s \rangle} = \mathbf{B}_{\langle s \rangle} * \mathbf{A}_{\langle r \rangle}$

Scalars commute $\mathbf{A}_{\langle r \rangle} * \left(\beta\, \mathbf{B}_{\langle s \rangle} \right) = \beta \left(\mathbf{A}_{\langle r \rangle} * \mathbf{B}_{\langle s \rangle} \right)$

Distributivity $\mathbf{A}_{\langle r \rangle} * \left(\mathbf{B}_{\langle s \rangle} + \mathbf{C}_{\langle t \rangle} \right) = \mathbf{A}_{\langle r \rangle} * \mathbf{B}_{\langle s \rangle} + \mathbf{A}_{\langle r \rangle} * \mathbf{C}_{\langle t \rangle}$

Backward compatible for 1-blades

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} * \mathbf{b}$$

Squared (reverse) norm of subspaces

$$\left\| \mathbf{A}_{\langle k \rangle} \right\|^2 = \mathbf{A}_{\langle k \rangle} * \tilde{\mathbf{A}}_{\langle k \rangle}$$

Important

The norm is a metric property of subspaces, and it is conceptually different from weight

$$\tilde{\mathbf{A}}_{\langle k \rangle} = (-1)^{k(k-1)/2} \, \mathbf{A}_{\langle k \rangle}$$

Reverse ($+ + - -\ \ + + - - \ldots$ pattern over k)

** Euclidean Metric

UFRGS

Inverse of blades

- The inverse of a blade is another blade with the same attitude, the inverse weight and, eventually, a different orientation

$$\mathbf{A}_{\langle k \rangle}^{-1} = \frac{\tilde{\mathbf{A}}_{\langle k \rangle}}{\left\| \mathbf{A}_{\langle k \rangle} \right\|^2}$$

Inverse

$$\left\| \mathbf{A}_{\langle k \rangle} \right\|^2 = \mathbf{A}_{\langle k \rangle} * \tilde{\mathbf{A}}_{\langle k \rangle}$$

Squared (reverse) norm

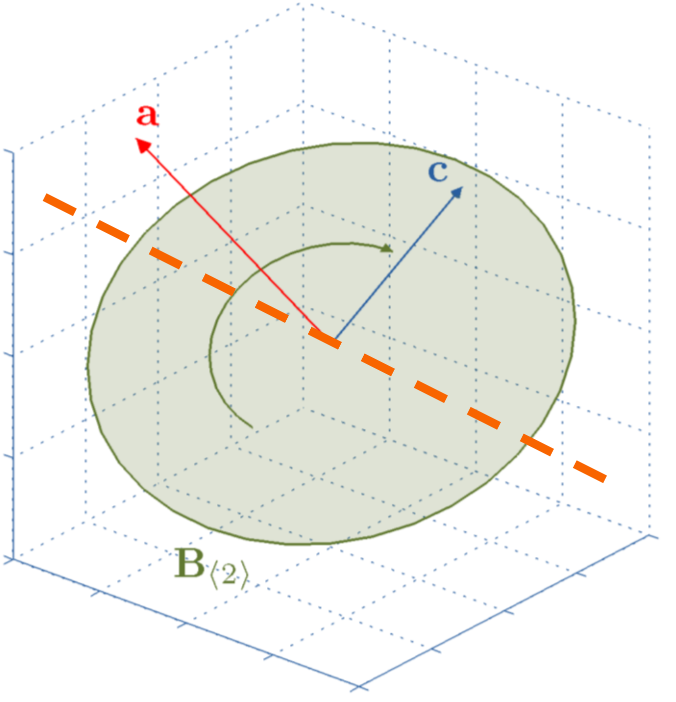
$$\mathbf{A}_{\langle k \rangle} \mathbf{A}_{\langle k \rangle}^{-1} = \mathbf{A}_{\langle k \rangle}^{-1} \mathbf{A}_{\langle k \rangle} = 1$$

$$\tilde{\mathbf{A}}_{\langle k \rangle} = (-1)^{k(k-1)/2} \mathbf{A}_{\langle k \rangle}$$

Reverse ($+ + - - \; + + - - \ldots$ pattern over k)

The left contraction

$$\mathbf{A}_{\langle r\rangle} \rfloor \mathbf{B}_{\langle s\rangle} = \begin{cases} \mathbf{C}_{\langle s-r\rangle} & s \geq r \\ 0 & s < r \end{cases}$$



Geometric Meaning

Remove from $\mathbf{B}_{\langle s\rangle}$ the part that is like $\mathbf{A}_{\langle r\rangle}$

Some observations

- The outcome of $\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle}$ represents a subspace that is contained in $\mathbf{B}_{\langle s \rangle}$

$$\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle} = \left(\mathbf{A}'_{\langle r-1 \rangle} \wedge \mathbf{a} \right) \rfloor \mathbf{B}_{\langle s \rangle} = \mathbf{A}'_{\langle r-1 \rangle} \rfloor \left(\mathbf{a} \rfloor \mathbf{B}_{\langle s \rangle} \right)$$

- The grade of $\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle}$ is $r - s$

Some observations

- The outcome of $\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle}$ is perpendicular to the subspace $\mathbf{A}_{\langle r \rangle}$
- For a vector $\mathbf{x}$, having $\mathbf{x} \rfloor \mathbf{C}_{\langle t \rangle} = 0$ mean that $\mathbf{x}$ is perpendicular to all vectors in $\mathbf{C}_{\langle t \rangle}$

Properties of the left contraction

$$\rfloor : \bigwedge^r \mathbb{R}^n \times \bigwedge^s \mathbb{R}^n \rightarrow \bigwedge^{s-r} \mathbb{R}^n$$

Symmetry $\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle} = \mathbf{B}_{\langle s \rangle} \rfloor \mathbf{A}_{\langle r \rangle}, \text{ iif } r = s$

Scalars commute $\mathbf{A}_{\langle r \rangle} \rfloor \left(\beta \, \mathbf{B}_{\langle s \rangle} \right) = \beta \left(\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle} \right)$

Distributivity $\mathbf{A}_{\langle r \rangle} \rfloor \left(\mathbf{B}_{\langle s \rangle} + \mathbf{C}_{\langle t \rangle} \right) = \mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle} + \mathbf{A}_{\langle r \rangle} \rfloor \mathbf{C}_{\langle t \rangle}$

The scalar product is a particular case of the left contraction

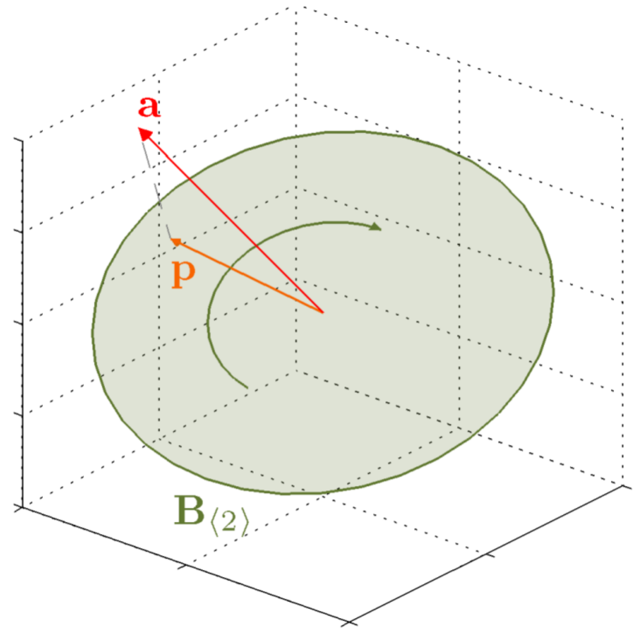
$$\mathbf{A}_{\langle k \rangle} * \mathbf{B}_{\langle k \rangle} = \mathbf{A}_{\langle k \rangle} \rfloor \mathbf{B}_{\langle k \rangle}$$

Backward compatible for 1-blades

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} * \mathbf{b} = \mathbf{a} \rfloor \mathbf{b}$$

Orthogonal projection of blades

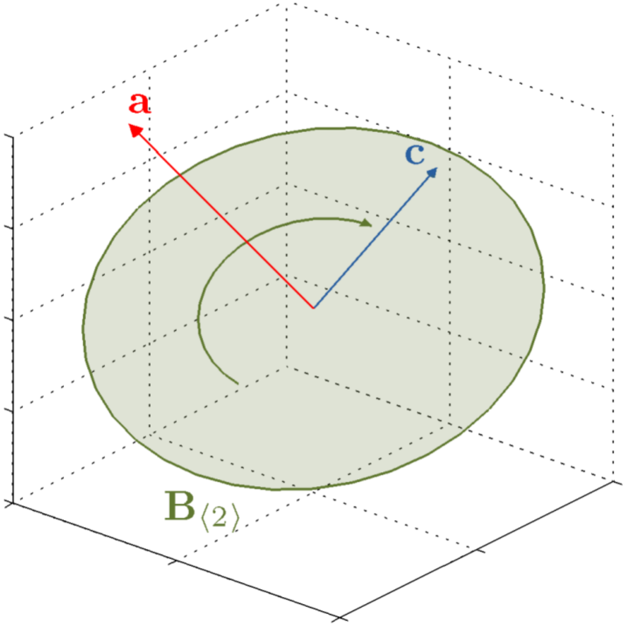
Computing the orthogonal projection of a vector onto a 2-blade



Orthogonal projection of blades

- Step 1

$$\mathbf{c} = \mathbf{a} \rfloor \mathbf{B}_{\langle 2 \rangle}$$



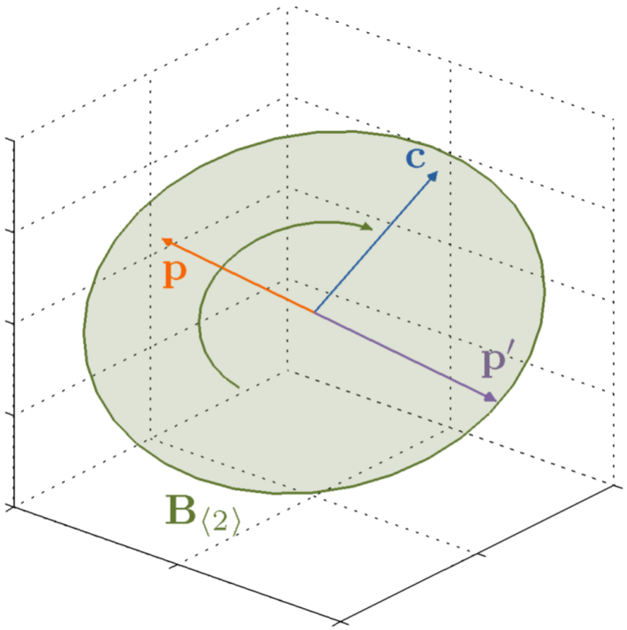
$$\mathbf{c} \subset \mathbf{B}_{\langle 2 \rangle}$$

Orthogonal projection of blades

- Step 2

$$\mathbf{c} = \mathbf{a} \rfloor \mathbf{B}_{\langle 2 \rangle}$$

$$\mathbf{p}' = \mathbf{c} \rfloor \mathbf{B}_{\langle 2 \rangle}$$



Warning

The resulting vector has the same attitude of the intended result, but a different weight and orientation

$$\mathbf{c} \subset \mathbf{B}_{\langle 2 \rangle}$$

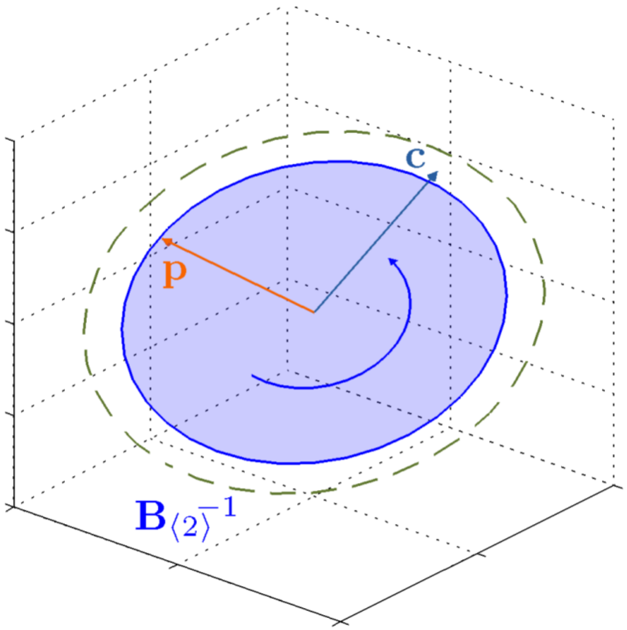
$$\mathbf{p}' \subset \mathbf{B}_{\langle 2 \rangle}$$

Orthogonal projection of blades

- Step 2

$$\mathbf{c} = \mathbf{a} \rfloor \mathbf{B}_{\langle 2 \rangle}$$

$$\mathbf{p} = \mathbf{c} \rfloor \mathbf{B}_{\langle 2 \rangle}^{-1}$$



By using the inverse of $\mathbf{B}_{\langle 2 \rangle}$, one fixes the weight and orientation of the resulting vector

$$\mathbf{c} \subset \mathbf{B}_{\langle 2 \rangle}$$

$$\mathbf{p} \subset \mathbf{B}_{\langle 2 \rangle}$$

Orthogonal projection of blades

- The general case, i.e., in an arbitrary metric space

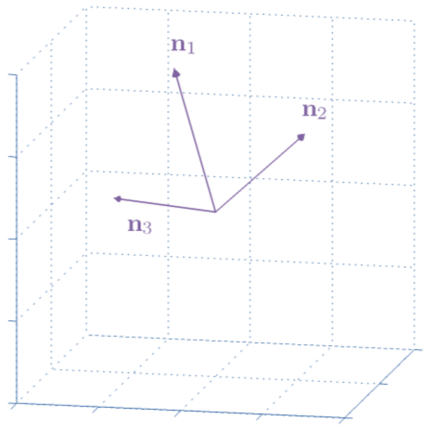
$$\mathbf{P}_{\langle r \rangle} = \left(\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle}^{-1} \right) \rfloor \mathbf{B}_{\langle s \rangle}$$

<u>Important</u>

The resulting blade has the same grade of the projected blade, so the input blades can not have orthogonal factors

$$\mathbf{P}_{\langle r \rangle} \subseteq \mathbf{B}_{\langle s \rangle}$$

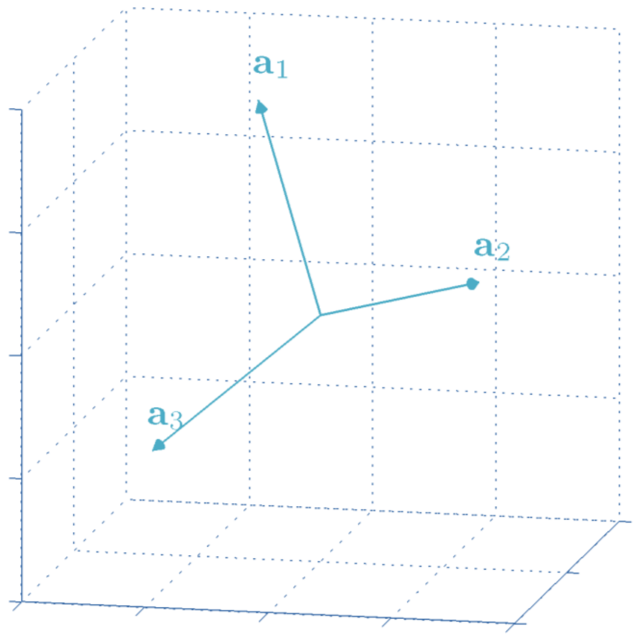
Basis orthogonalization



procedure orthogonal_basis $(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k)$

- $\mathbf{S} \leftarrow 1$
- **for all** $i \in \{1, 2, \cdots, k\}$ **do**
  - $\mathbf{T} \leftarrow \mathbf{S} \wedge \mathbf{a}_i$
  - **if** $\|\mathbf{T}\|^2 = 0$
    - **abort** // *Input vectors are dependent.*
  - **end if**
  - $\mathbf{n}_i \leftarrow \mathbf{S}^{-1} \rfloor \mathbf{T}$
  - $\mathbf{S} \leftarrow \mathbf{T}$
- **end for**
- **return** $\{\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}_k\}$

end procedure



** Euclidean Metric

The right contraction

$$\mathbf{B}_{\langle s\rangle} \lfloor \mathbf{A}_{\langle r\rangle} = \begin{cases} \mathbf{C}_{\langle s-r\rangle} & s \geq r \\ 0 & s < r \end{cases}$$

<u>Geometric Meaning</u>

Remove from $\mathbf{B}_{\langle s\rangle}$ the part

that is like $\mathbf{A}_{\langle r\rangle}$

$$\mathbf{B}_{\langle s\rangle} \lfloor \mathbf{A}_{\langle r\rangle} = \left(\tilde{\mathbf{A}}_{\langle r\rangle} \rfloor \tilde{\mathbf{B}}_{\langle s\rangle}\right)^{\sim} = (-1)^{r(s+1)} \mathbf{A}_{\langle r\rangle} \rfloor \mathbf{B}_{\langle s\rangle}$$

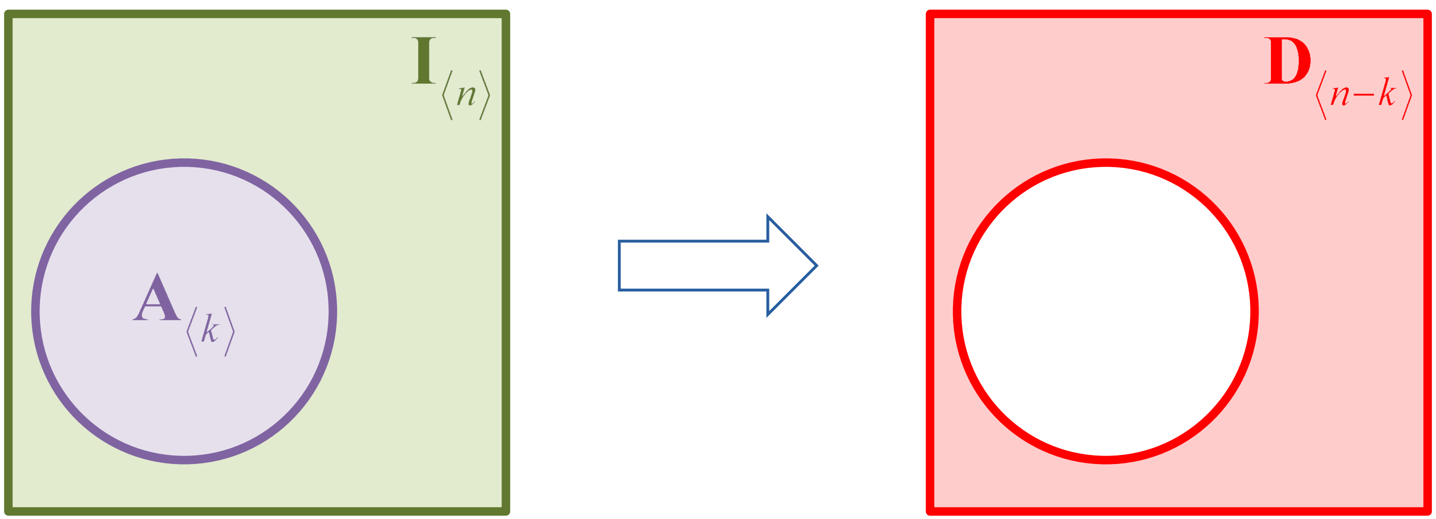
Lecture II

Dualization and Undualization

The notion of duality

- The complementary grade of a grade k is n-k

$$\bigwedge^k \mathbb{R}^n \leftrightarrow \bigwedge^{n-k} \mathbb{R}^n$$



Venn Diagrams

"Taking the dual" operation

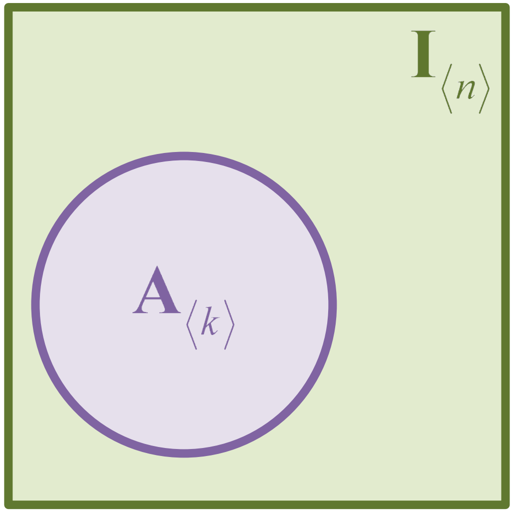
$$\mathbf{A}^{*}_{\langle k \rangle} = \mathbf{D}_{\langle n-k \rangle} = \mathbf{A}_{\langle k \rangle} \rfloor \mathbf{I}^{-1}_{\langle n \rangle}$$

$$\mathbf{I}_{\langle n \rangle} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n$$

Unit Pseudoscalar

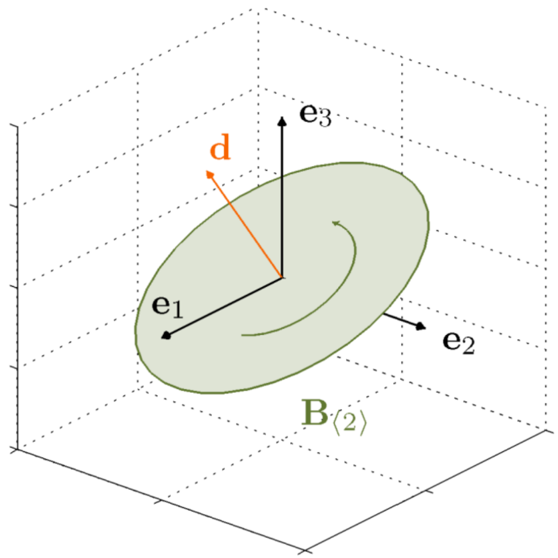
$$\mathbf{A}_{\langle k \rangle} \subseteq \mathbf{I}_{\langle n \rangle}$$

Condition



Venn Diagram

Dualization, step-by-step



$$\mathbf{B}_{\langle 2 \rangle} = \gamma_1 \, \mathbf{e}_1 \wedge \mathbf{e}_2 + \gamma_6 \, \mathbf{e}_1 \wedge \mathbf{e}_3 + \gamma_7 \, \mathbf{e}_2 \wedge \mathbf{e}_3 \qquad \mathbf{B}_{\langle 2 \rangle} \in \bigwedge^2 \mathbb{R}^3$$

$$\mathbf{d} = \mathbf{B}^{*}_{\langle 2 \rangle} \qquad \mathbf{d} \in \mathbb{R}^3$$

UFRGS

** Euclidean Metric

Dualization, s

$$\mathbf{d} = \mathbf{B}^{*}_{\langle 2 \rangle}$$

$$\mathbf{d} = \mathbf{B}_{\langle 2 \rangle} \rfloor \mathbf{I}^{-1}_{\langle 3 \rangle}$$

$$\mathbf{d} = \left(\gamma_1 \, \mathbf{e}_1 \wedge \mathbf{e}_2 + \gamma_2 \, \mathbf{e}_1 \wedge \mathbf{e}_3 + \gamma_3 \, \mathbf{e}_2 \wedge \mathbf{e}_3 \right) \rfloor \mathbf{I}^{-1}_{\langle 3 \rangle}$$

$$\mathbf{d} = + \gamma_1 \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \right) \rfloor \mathbf{I}^{-1}_{\langle 3 \rangle} + \gamma_2 \left(\mathbf{e}_1 \wedge \mathbf{e}_3 \right) \rfloor \mathbf{I}^{-1}_{\langle 3 \rangle} + \gamma_3 \left(\mathbf{e}_2 \wedge \mathbf{e}_3 \right) \rfloor \mathbf{I}^{-1}_{\langle 3 \rangle}$$

$$\mathbf{I}^{-1}_{\langle 3 \rangle} = \frac{\tilde{\mathbf{I}}_{\langle 3 \rangle}}{\mathbf{I}_{\langle 3 \rangle} * \tilde{\mathbf{I}}_{\langle 3 \rangle}} = \tilde{\mathbf{I}}_{\langle 3 \rangle}$$

$$\mathbf{I}^{-1}_{\langle 3 \rangle} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$$

$$\mathbf{I}^{-1}_{\langle 3 \rangle} = -\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$$

$$\mathbf{d} = - \gamma_1 \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \right) \rfloor \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \right) - \gamma_2 \left(\mathbf{e}_1 \wedge \mathbf{e}_3 \right) \rfloor \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \right) - \gamma_3 \left(\mathbf{e}_2 \wedge \mathbf{e}_3 \right) \rfloor \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \right)$$

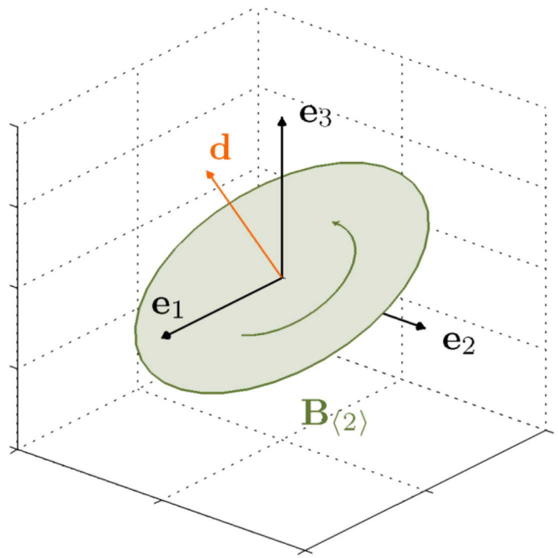
$$\mathbf{d} = + \gamma_1 \, \mathbf{e}_3 - \gamma_2 \, \mathbf{e}_2 + \gamma_3 \, \mathbf{e}_1$$

$$\mathbf{B}_{\langle 2 \rangle} = \gamma_1 \, \mathbf{e}_1 \wedge \mathbf{e}_2 + \gamma$$

$$\mathbf{d} = \mathbf{B}^{*}_{\langle 2 \rangle}$$

** Euclidean Metric

Dualization, step-by-step



$$\mathbf{B}_{\langle 2 \rangle} = \gamma_1\, \mathbf{e}_1 \wedge \mathbf{e}_2 + \gamma_6\, \mathbf{e}_1 \wedge \mathbf{e}_3 + \gamma_7\, \mathbf{e}_2 \wedge \mathbf{e}_3 \qquad \mathbf{B}_{\langle 2 \rangle} \in \bigwedge^2 \mathbb{R}^3$$

$$\mathbf{d} = \mathbf{B}^{*}_{\langle 2 \rangle} = \gamma_3\, \mathbf{e}_1 - \gamma_2\, \mathbf{e}_2 + \gamma_1\, \mathbf{e}_3 \qquad \mathbf{d} \in \mathbb{R}^3$$

** Euclidean Metric

Warning!

- The dual of the dual representation of a blade may not result in the direct representation of the blade

$$\left(\mathbf{A}^{*}_{\langle k\rangle}\right)^{*} = \mathbf{A}_{\langle k\rangle}$$

does not hold in the general case

$$\left(\mathbf{A}^{*}_{\langle k\rangle}\right)^{*} = (-1)^{n(n-1)/2}\,\mathbf{A}_{\langle k\rangle}$$

The successive application of two dualization operations may change the orientation of the resulting blade

"Taking the undual" operation

$$\mathbf{D}^{-*}_{\langle n-k \rangle} = \mathbf{A}_{\langle k \rangle} = \mathbf{D}_{\langle n-k \rangle} \rfloor \mathbf{I}_{\langle n \rangle}$$

$$\left(\mathbf{A}^{*}_{\langle k \rangle} \right)^{-*} = \mathbf{A}_{\langle k \rangle}$$

By taking the undual, the dual representation of a blade can be correctly mapped back to its direct representation