Introduction to Geometric Algebra

Lecture III

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Lecture III

Checkpoint

Checkpoint, Lecture I

- Multivector space $\bigwedge \mathbb{R}^n$

e.g., Basis for $\bigwedge \mathbb{R}^3$

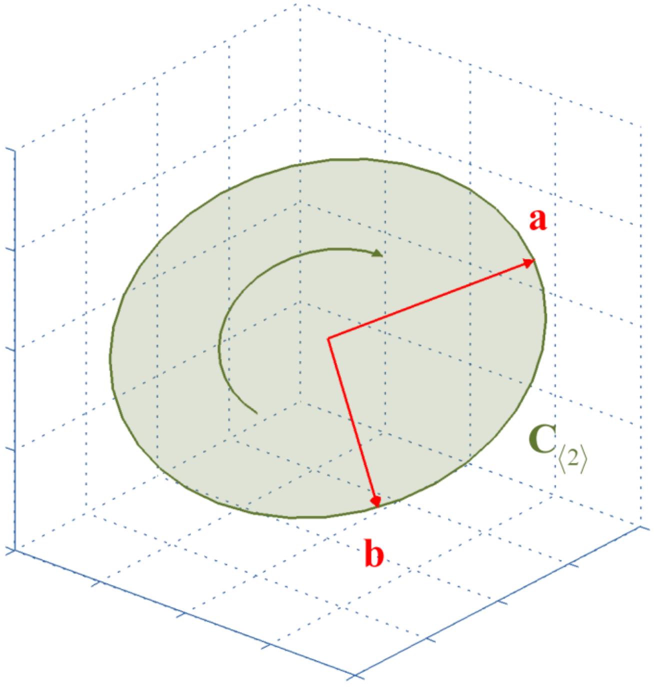
$$\{\ \underbrace{1,}_{\text{Scalars}}\ \underbrace{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,}_{\text{Vector Space}}\ \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2,\ \mathbf{e}_1 \wedge \mathbf{e}_3,\ \mathbf{e}_2 \wedge \mathbf{e}_3,}_{\text{Bivector Space}}\ \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3}_{\text{Trivector Space}}\ \}$$

Scalars	Vector Space	Bivector Space	Trivector Space
$\mathbb{R}$	$\mathbb{R}^3$	$\bigwedge^2 \mathbb{R}^3$	$\bigwedge^3 \mathbb{R}^3$

Checkpoint, Lecture I

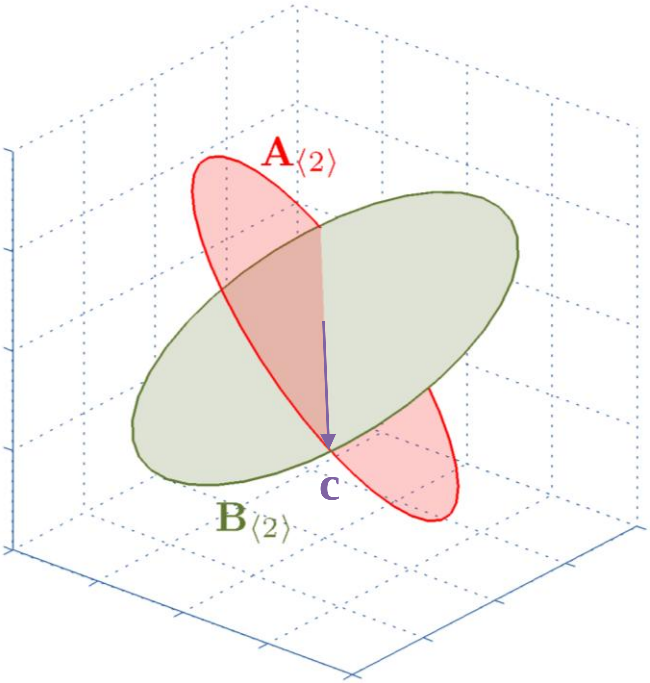
- k-D oriented subspaces (or k-blades) as primitives
- k-blades are built as
  - the outer product of k vectors

$$\mathbf{C}_{\langle 2 \rangle} = \mathbf{a} \wedge \mathbf{b}$$



Checkpoint, Lecture I

- k-D oriented subspaces (or k-blades) as primitives
- k-blades are built as
  - the outer product of k vectors
  - the regressive product of n-k pseudovectors



$$\mathbf{c} = \mathbf{A}_{\langle 2 \rangle} \vee \mathbf{B}_{\langle 2 \rangle}$$

Checkpoint, Lecture II

- Metric spaces
  - Bilinear form $\mathrm{Q}(\mathbf{a}, \mathbf{b})$ defines a metric on the vector space, e.g., Euclidean metric
  - Metric matrix

$$\mathrm{M} = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix}$$

$$m_{ij} = \mathrm{Q}(\mathbf{e}_i, \mathbf{e}_j)$$

Checkpoint, Lecture II

- Metric spaces
  - Bilinear form $Q(\mathbf{a},\mathbf{b})$ defines a metric on the vector space, e.g., Euclidean metric
  - Metric matrix

- Some inner products
  - Inner product of vectors
  - Scalar product
  - Left contraction
  - Right contraction

The scalar product is a particular case of the left and right contractions

$$\mathbf{A}_{\langle k \rangle} * \mathbf{B}_{\langle k \rangle} = \mathbf{A}_{\langle k \rangle} \rfloor \mathbf{B}_{\langle k \rangle} = \mathbf{A}_{\langle k \rangle} \lfloor \mathbf{B}_{\langle k \rangle}$$

These metric products are backward compatible for 1-blades

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} * \mathbf{b} = \mathbf{a} \rfloor \mathbf{b} = \mathbf{a} \lfloor \mathbf{b}$$

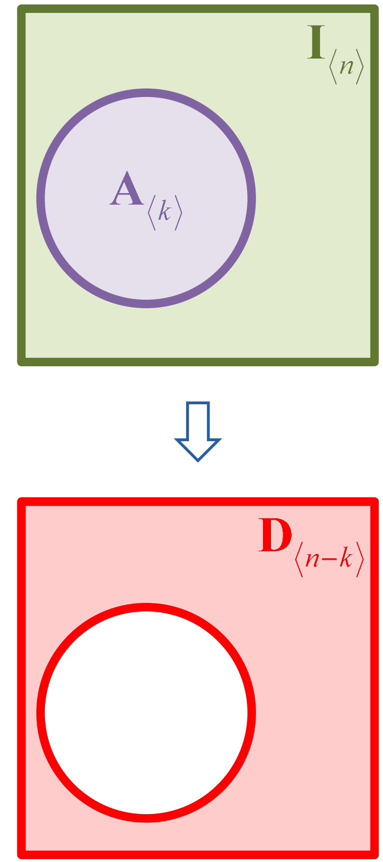
Checkpoint, Lecture II

- Dualization

$$\mathbf{A}^{*}_{\langle k \rangle} = \mathbf{D}_{\langle n-k \rangle} = \mathbf{A}_{\langle k \rangle} \rfloor \mathbf{I}^{-1}_{\langle n \rangle}$$

- Undualization

$$\mathbf{D}^{-*}_{\langle n-k \rangle} = \mathbf{A}_{\langle k \rangle} = \mathbf{D}_{\langle n-k \rangle} \rfloor \mathbf{I}_{\langle n \rangle}$$



Venn Diagrams

$$\left(\mathbf{A}^{*}_{\langle k \rangle}\right)^{-*} = \mathbf{A}_{\langle k \rangle}$$

By taking the undual, the dual representation of a blade can be correctly mapped back to its direct representation

Today

- Lecture III – Fri, January 15
  - Duality relationships between products
  - Blade factorization
  - Some non-linear products

Lecture III

Duality Relationships Between Products

The outer product and the left contraction

- Duality of subspaces

$$\mathbf{X}^{*}_{\langle k \rangle} = \mathbf{X}_{\langle k \rangle} \rfloor \mathbf{I}^{-1}_{\langle n \rangle}$$

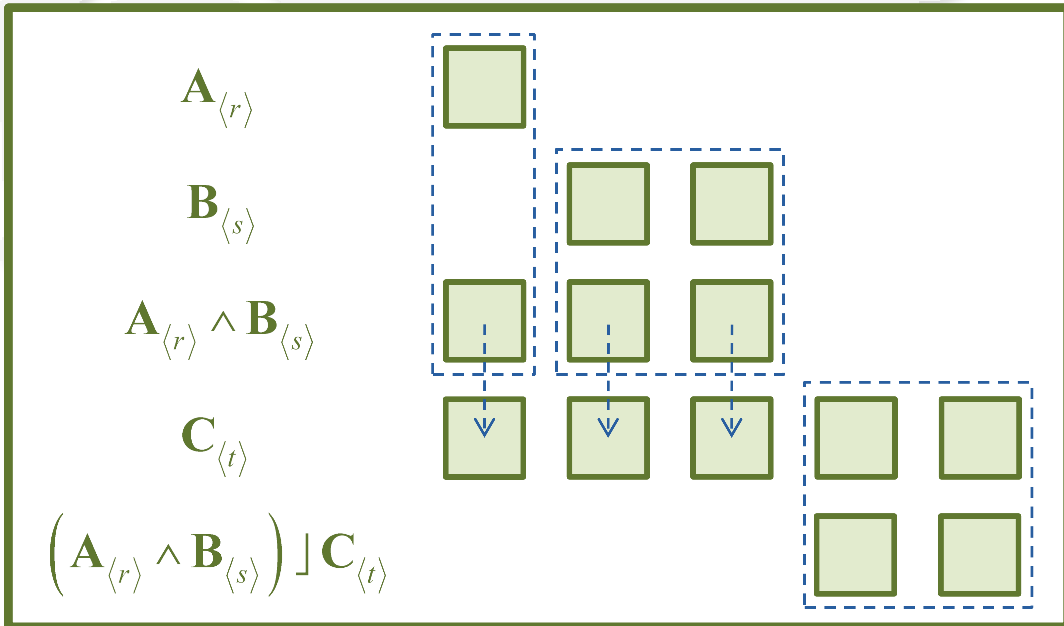
- Dual of the outer product

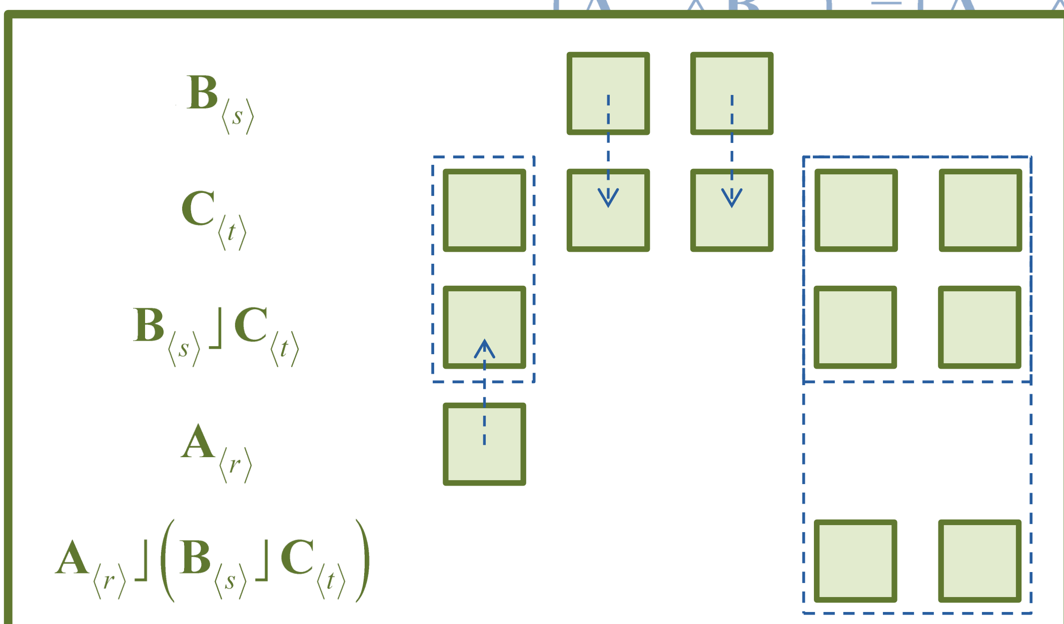
$$\left(\mathbf{A}_{\langle r \rangle} \wedge \mathbf{B}_{\langle s \rangle}\right)^{*} = \left(\mathbf{A}_{\langle r \rangle} \wedge \mathbf{B}_{\langle s \rangle}\right) \rfloor \mathbf{I}^{-1}_{\langle n \rangle}$$

Universally Valid

$$\left(\mathbf{A}_{\langle r \rangle} \wedge \mathbf{B}_{\langle s \rangle}\right) \rfloor \mathbf{C}_{\langle t \rangle} = \mathbf{A}_{\langle r \rangle} \rfloor \left(\mathbf{B}_{\langle s \rangle} \rfloor \mathbf{C}_{\langle t \rangle}\right)$$

e left contraction





Universally Valid

$$\left(\mathbf{A}_{\langle r\rangle} \wedge \mathbf{B}_{\langle s\rangle}\right) \rfloor \mathbf{C}_{\langle t\rangle} = \mathbf{A}_{\langle r\rangle} \rfloor \left(\mathbf{B}_{\langle s\rangle} \rfloor \mathbf{C}_{\langle t\rangle}\right)$$

The outer product and the left contraction

- Duality of subspaces

$$\mathbf{X}^{*}_{\langle k \rangle} = \mathbf{X}_{\langle k \rangle} \rfloor \mathbf{I}^{-1}_{\langle n \rangle}$$

- Dual of the outer product

$$\begin{aligned}\left(\mathbf{A}_{\langle r \rangle} \wedge \mathbf{B}_{\langle s \rangle}\right)^{*} &= \left(\mathbf{A}_{\langle r \rangle} \wedge \mathbf{B}_{\langle s \rangle}\right) \rfloor \mathbf{I}^{-1}_{\langle n \rangle} \\ &= \mathbf{A}_{\langle r \rangle} \rfloor \left(\mathbf{B}_{\langle s \rangle} \rfloor \mathbf{I}^{-1}_{\langle n \rangle}\right) \\ &= \mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}^{*}_{\langle s \rangle}\end{aligned}$$

Universally Valid

$$\left(\mathbf{A}_{\langle r \rangle} \wedge \mathbf{B}_{\langle s \rangle}\right) \rfloor \mathbf{C}_{\langle t \rangle} = \mathbf{A}_{\langle r \rangle} \rfloor \left(\mathbf{B}_{\langle s \rangle} \rfloor \mathbf{C}_{\langle t \rangle}\right)$$

The left contraction and the outer product

- Duality of subspaces

$$\mathbf{X}^*_{\langle k \rangle} = \mathbf{X}_{\langle k \rangle} \rfloor \mathbf{I}^{-1}_{\langle n \rangle}$$

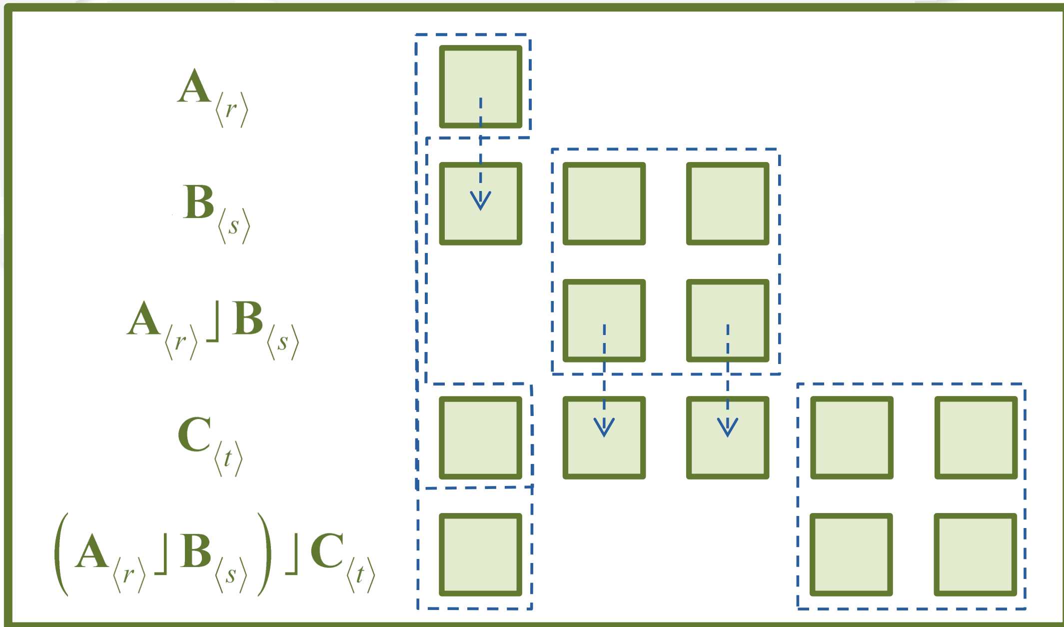
- Dual of the left contraction

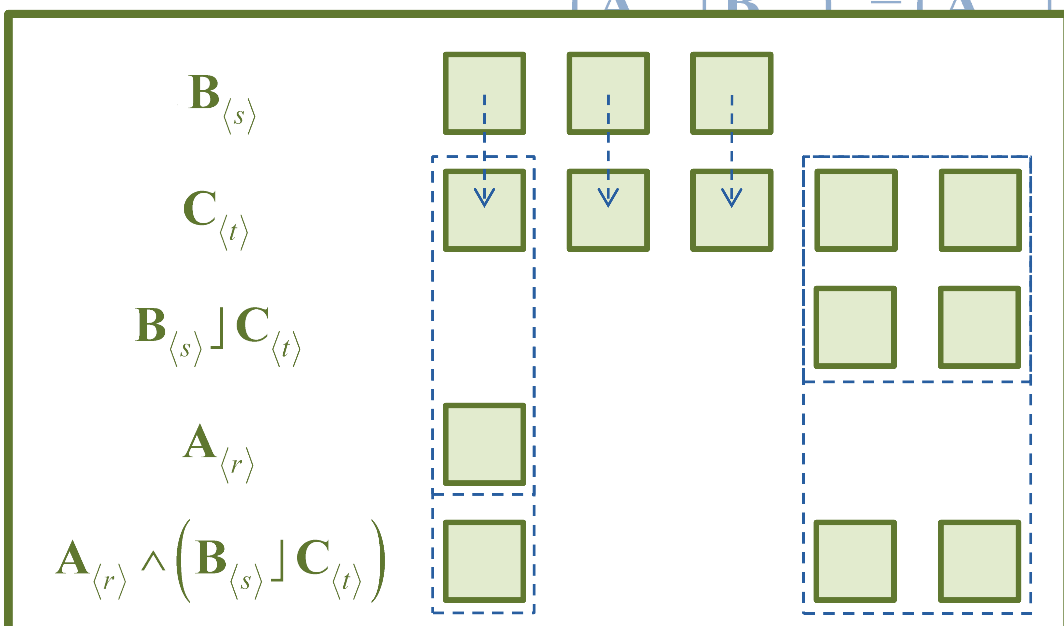
$$\left(\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle}\right)^* = \left(\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle}\right) \rfloor \mathbf{I}^{-1}_{\langle n \rangle}$$

Valid when $\mathbf{A}_{\langle r \rangle} \subseteq \mathbf{C}_{\langle t \rangle}$

$$\left(\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle}\right) \rfloor \mathbf{C}_{\langle t \rangle} = \mathbf{A}_{\langle r \rangle} \wedge \left(\mathbf{B}_{\langle s \rangle} \rfloor \mathbf{C}_{\langle t \rangle}\right)$$

...the outer product





Valid when $\mathbf{A}_{\langle r \rangle} \subseteq \mathbf{C}_{\langle t \rangle}$

$$\left(\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle}\right) \rfloor \mathbf{C}_{\langle t \rangle} = \mathbf{A}_{\langle r \rangle} \wedge \left(\mathbf{B}_{\langle s \rangle} \rfloor \mathbf{C}_{\langle t \rangle}\right)$$

The left contraction and the outer product

- Duality of subspaces

$$\mathbf{X}^{*}_{\langle k \rangle} = \mathbf{X}_{\langle k \rangle} \rfloor \mathbf{I}^{-1}_{\langle n \rangle}$$

- Dual of the left contraction

$$\begin{aligned}\left(\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle}\right)^{*} &= \left(\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle}\right) \rfloor \mathbf{I}^{-1}_{\langle n \rangle} \\ &= \mathbf{A}_{\langle r \rangle} \wedge \left(\mathbf{B}_{\langle s \rangle} \rfloor \mathbf{I}^{-1}_{\langle n \rangle}\right) \\ &= \mathbf{A}_{\langle r \rangle} \wedge \mathbf{B}^{*}_{\langle s \rangle}\end{aligned}$$

Valid when $\mathbf{A}_{\langle r \rangle} \subseteq \mathbf{C}_{\langle t \rangle}$

$$\left(\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle}\right) \rfloor \mathbf{C}_{\langle t \rangle} = \mathbf{A}_{\langle r \rangle} \wedge \left(\mathbf{B}_{\langle s \rangle} \rfloor \mathbf{C}_{\langle t \rangle}\right)$$

Duality relationships between products

- Dual of the outer product

$$\left(\mathbf{A}_{\langle r\rangle} \wedge \mathbf{B}_{\langle s\rangle}\right)^* = \mathbf{A}_{\langle r\rangle} \rfloor \mathbf{B}^*_{\langle s\rangle}$$

- Dual of the left contraction

$$\left(\mathbf{A}_{\langle r\rangle} \rfloor \mathbf{B}_{\langle s\rangle}\right)^* = \mathbf{A}_{\langle r\rangle} \wedge \mathbf{B}^*_{\langle s\rangle}$$

The cross product incorporated

- The cross product (in 3-D space only) as the dual of the outer product (universally applicable)

$$\mathbf{a} \times \mathbf{b} \equiv (\mathbf{a} \wedge \mathbf{b})^{*}$$

Lecture I

$$\begin{aligned}\mathbf{a} \wedge \mathbf{b} = &(\alpha_1\beta_2 - \alpha_2\beta_1)\,\mathbf{e}_1 \wedge \mathbf{e}_2 \\ &+ (\alpha_1\beta_3 - \alpha_3\beta_1)\,\mathbf{e}_1 \wedge \mathbf{e}_3 \\ &+ (\alpha_2\beta_3 - \alpha_3\beta_2)\,\mathbf{e}_2 \wedge \mathbf{e}_3\end{aligned}$$

Lecture II

$$\begin{aligned}\mathbf{B}_{\langle 2\rangle} = \mathbf{a} \wedge \mathbf{b} = &\gamma_1\,\mathbf{e}_1 \wedge \mathbf{e}_2 \\ &+ \gamma_2\,\mathbf{e}_1 \wedge \mathbf{e}_3 \\ &+ \gamma_3\,\mathbf{e}_2 \wedge \mathbf{e}_3\end{aligned}$$

$$\mathbf{B}^{*}_{\langle 2\rangle} = \gamma_3\,\mathbf{e}_1 - \gamma_2\mathbf{e}_2 + \gamma_1\mathbf{e}_3$$

By replacing…

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (\mathbf{a} \wedge \mathbf{b})^{*} \\ &= \gamma_3\,\mathbf{e}_1 - \gamma_2\mathbf{e}_2 + \gamma_1\mathbf{e}_3 \\ &= \ (\alpha_2\beta_3 - \alpha_3\beta_2)\,\mathbf{e}_1 \\ &\quad -(\alpha_1\beta_3 - \alpha_3\beta_1)\,\mathbf{e}_2 \\ &\quad +(\alpha_1\beta_2 - \alpha_2\beta_1)\,\mathbf{e}_3\end{aligned}$$

Lecture III

Blade Factorization

The blade factorization problem

- Find, for a given blade $\mathbf{B}_{\langle k \rangle}$, a set of k vectors $\mathbf{b}_i$ such that

$$\mathbf{B}_{\langle k \rangle} = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k$$

Why one may want to factorize a blade

- To use the factors as input to libraries that cannot handle blades
- To implement another low-level algorithm

Good News!

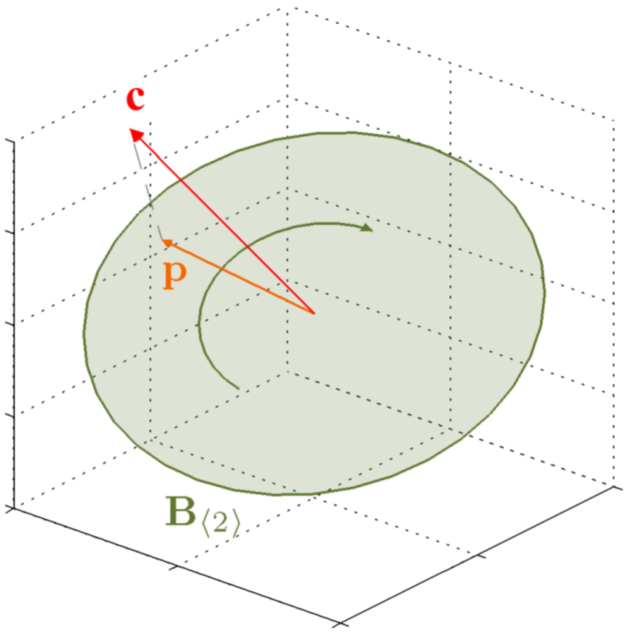
We are only concerned with the outer product and consequently are allowed to choose any convenient metric e.g., Euclidean metric

How to find the factors?

- One may project candidate vectors $\mathbf{c}$ onto $\mathbf{B}_{\langle k \rangle}$

$$\mathbf{p} = \left(\mathbf{c} \rfloor \mathbf{B}_{\langle k \rangle}^{-1}\right) \rfloor \mathbf{B}_{\langle k \rangle}$$

$$\mathbf{p} \subseteq \mathbf{B}_{\langle k \rangle}, \text{ iif } \mathbf{p} \neq 0$$



How to find the factors?

- One may project candidate vectors $\mathbf{c}$ onto $\mathbf{B}_{\langle k \rangle}$

$$\mathbf{p} = \left(\mathbf{c} \rfloor \mathbf{B}_{\langle k \rangle}^{-1}\right) \rfloor \mathbf{B}_{\langle k \rangle}$$

All nonzero blades are invertible under Euclidean metric

$$\mathbf{p} \subseteq \mathbf{B}_{\langle k \rangle}, \text{ iif } \mathbf{p} \neq 0$$

- By find k linearly independent vectors $\mathbf{p}_i$ a factorization of $\mathbf{B}_{\langle k \rangle}$ is found (up to a scale)

Blade factorization

Input nonzero blade and $k > 0$

procedure blade_factorization $\left(\mathbf{B}_{\langle k \rangle}\right)$

- $\beta \leftarrow \left\| \mathbf{B}_{\langle k \rangle} \right\|$
- $\mathbf{C} \leftarrow \mathbf{B}_{\langle k \rangle} / \beta$
- $\mathbf{E}_{\langle k \rangle} \leftarrow$ **basis blade with the largest coefficient**
- **for all but one of the vectors** $\mathbf{e}_i$ **that span** $\mathbf{E}_{\langle k \rangle}$ **do**
  - $\mathbf{p}_i \leftarrow \left(\mathbf{e}_i \rfloor \mathbf{C}^{-1}\right) \rfloor \mathbf{C}$ $\quad // \textit{Orthogonal projection.}$
  - $\mathbf{f}_i \leftarrow \mathbf{p}_i / \left\| \mathbf{p}_i \right\|$ $\quad // \textit{Blade normalization.}$
  - $\mathbf{C} \leftarrow \mathbf{f}_i^{-1} \rfloor \mathbf{C}$

  end for
- $\mathbf{f}_k \leftarrow \mathbf{C} / \left\| \mathbf{C} \right\|$
- **return** $\left\{ \beta, \mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_k \right\}$

end procedure

By assuming Euclidean metric…

- The algorithm also works for null blade in the actual metric
- The output is a scalar value and a set of orthonormal factors in Euclidean metric

Lecture III

The Meet and Join of Blades

The meet and join of blades

$$\mathbf{A}_{\langle r \rangle} \cap \mathbf{B}_{\langle s \rangle}$$

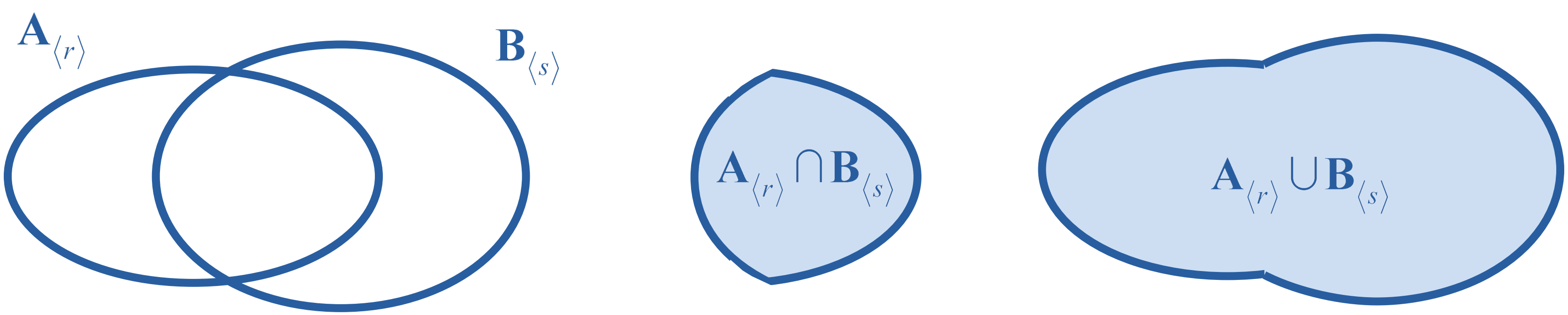
Meet of Blades

$$\mathbf{A}_{\langle r \rangle} \cup \mathbf{B}_{\langle s \rangle}$$

Join of Blades

<u>Geometric Meaning</u>

The geometric version of intersection and union from set theory.



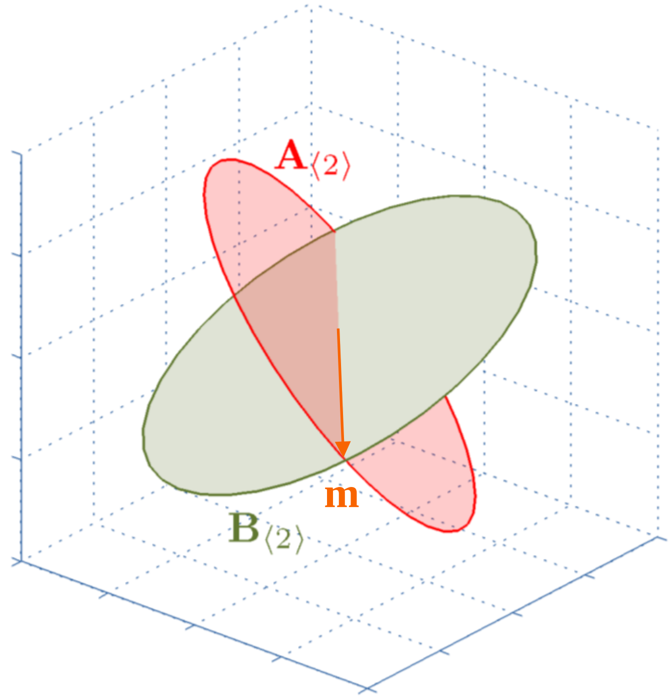
Venn Diagrams

The meet and join of blades

Common Subspace

$$\mathbf{A}_{\langle r \rangle} = \mathbf{A}'_{\langle r-t \rangle} \wedge \mathbf{M}_{\langle t \rangle}$$

$$\mathbf{B}_{\langle s \rangle} = \mathbf{M}_{\langle t \rangle} \wedge \mathbf{B}'_{\langle s-t \rangle}$$



$$\mathbf{A}_{\langle r \rangle} \cap \mathbf{B}_{\langle s \rangle} = \mathbf{M}_{\langle t \rangle}$$

$$\mathbf{A}_{\langle r \rangle} \cup \mathbf{B}_{\langle s \rangle} = \mathbf{A}'_{\langle r-t \rangle} \wedge \mathbf{M}_{\langle t \rangle} \wedge \mathbf{B}'_{\langle s-t \rangle}$$

$$\mathbf{A}_{\langle 2 \rangle} = \mathbf{a} \wedge \mathbf{m}$$

$$\mathbf{B}_{\langle 2 \rangle} = \mathbf{m} \wedge \mathbf{b}$$

The meet and join of blades

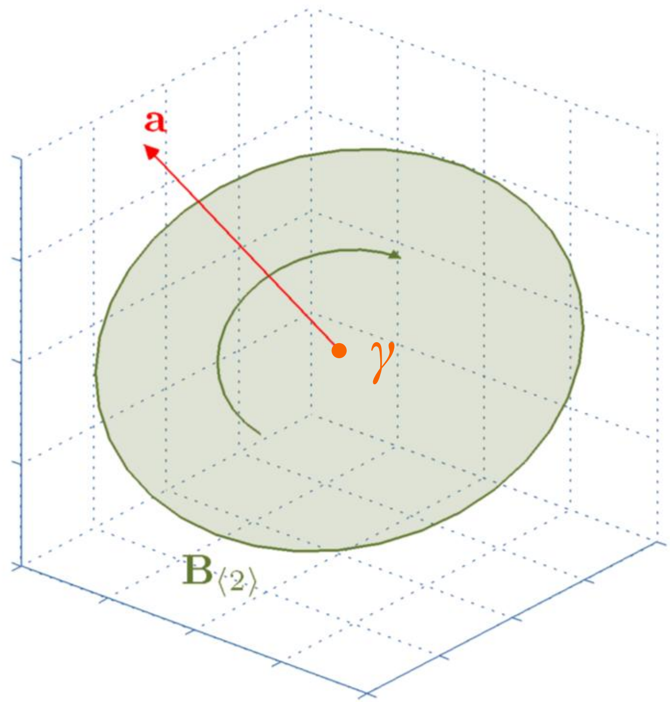
Common Subspace

$$\mathbf{A}_{\langle r\rangle} = \mathbf{A}'_{\langle r-t\rangle} \wedge \mathbf{M}_{\langle t\rangle}$$

$$\mathbf{B}_{\langle s\rangle} = \mathbf{M}_{\langle t\rangle} \wedge \mathbf{B}'_{\langle s-t\rangle}$$

$$\mathbf{A}_{\langle r\rangle} \cap \mathbf{B}_{\langle s\rangle} = \mathbf{M}_{\langle t\rangle}$$

$$\mathbf{A}_{\langle r\rangle} \cup \mathbf{B}_{\langle s\rangle} = \mathbf{A}'_{\langle r-t\rangle} \wedge \mathbf{M}_{\langle t\rangle} \wedge \mathbf{B}'_{\langle s-t\rangle}$$



$$\mathbf{a} \cup \mathbf{B}_{\langle 2\rangle} = \mathbf{I}_{\langle 3\rangle} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$$

$$\mathbf{a} \cap \mathbf{B}_{\langle 2\rangle} = \gamma = \mathbf{a} \rfloor \mathbf{B}^*_{\langle 2\rangle}$$

Relationships between meet and join

$$\mathbf{A}_{\langle r \rangle} = \mathbf{A}'_{\langle r-t \rangle} \wedge \mathbf{M}_{\langle t \rangle}$$
$$\mathbf{B}_{\langle s \rangle} = \mathbf{M}_{\langle t \rangle} \wedge \mathbf{B}'_{\langle s-t \rangle}$$

$\Rightarrow$

$$\mathbf{A}'_{\langle r-t \rangle} = \mathbf{A}_{\langle r \rangle} \lfloor \mathbf{M}^{-1}_{\langle t \rangle}$$
$$\mathbf{B}'_{\langle s-t \rangle} = \mathbf{M}^{-1}_{\langle t \rangle} \rfloor \mathbf{B}_{\langle s \rangle}$$

Don’t worry about the inverse, because
meet and join are independent
of the particular metric

Relationships between meet and join

$$\mathbf{A}_{\langle r\rangle} = \mathbf{A}'_{\langle r-t\rangle} \wedge \mathbf{M}_{\langle t\rangle} \qquad \mathbf{A}'_{\langle r-t\rangle} = \mathbf{A}_{\langle r\rangle} \lfloor \mathbf{M}^{-1}_{\langle t\rangle}$$

$$\mathbf{B}_{\langle s\rangle} = \mathbf{M}_{\langle t\rangle} \wedge \mathbf{B}'_{\langle s-t\rangle} \quad \Rightarrow \quad \mathbf{B}'_{\langle s-t\rangle} = \mathbf{M}^{-1}_{\langle t\rangle} \rfloor \mathbf{B}_{\langle s\rangle}$$

$$\mathbf{J}_{\langle r+s-t\rangle} = \mathbf{A}_{\langle r\rangle} \cup \mathbf{B}_{\langle s\rangle} = \mathbf{A}_{\langle r\rangle} \wedge \left(\mathbf{M}^{-1}_{\langle t\rangle} \rfloor \mathbf{B}_{\langle s\rangle}\right)$$

$$= \left(\mathbf{A}_{\langle r\rangle} \lfloor \mathbf{M}^{-1}_{\langle t\rangle}\right) \wedge \mathbf{B}_{\langle s\rangle}$$

$$\mathbf{M}_{\langle t\rangle} = \mathbf{A}_{\langle r\rangle} \cap \mathbf{B}_{\langle s\rangle} = \left(\mathbf{B}_{\langle s\rangle} \rfloor \mathbf{J}^{-1}_{\langle r+s-t\rangle}\right) \rfloor \mathbf{A}_{\langle r\rangle}$$

$$= \mathbf{B}^{*}_{\langle s\rangle} \rfloor \mathbf{A}_{\langle r\rangle}$$

$$= \left(\mathbf{B}^{*}_{\langle s\rangle} \wedge \mathbf{A}^{*}_{\langle r\rangle}\right)^{-*}$$

This is not the dual relative to the pseudoscalar $\mathbf{I}_{\langle n\rangle}$ of the total space, but of the pseudoscalar $\mathbf{J}_{\langle r+s-t\rangle}$ within which the problem resides.

Lecture III

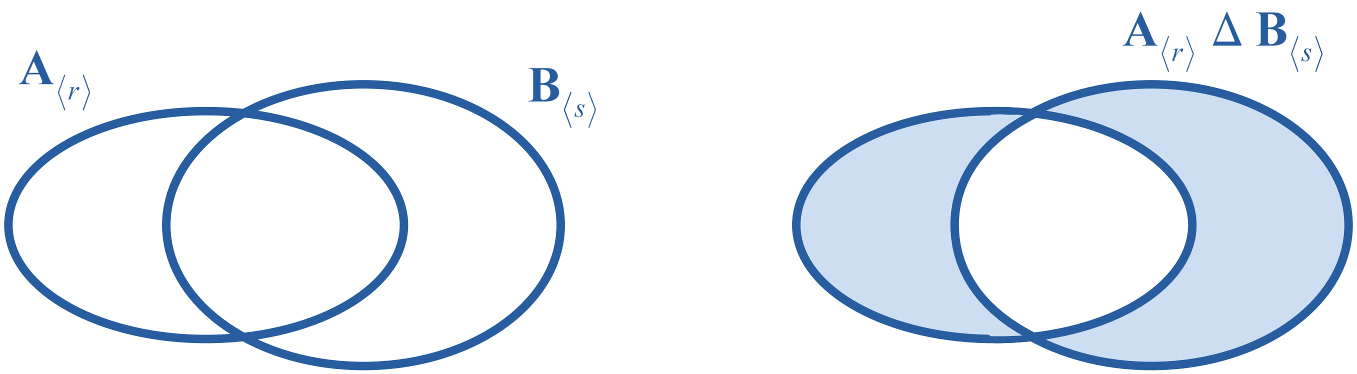
The Delta Product of Blades

The delta product of blades

$$\mathbf{A}_{\langle r \rangle} \, \Delta \, \mathbf{B}_{\langle s \rangle}$$

Geometric Meaning

The symmetric difference of the factors in $\mathbf{A}_{\langle r \rangle}$ and $\mathbf{B}_{\langle s \rangle}$.

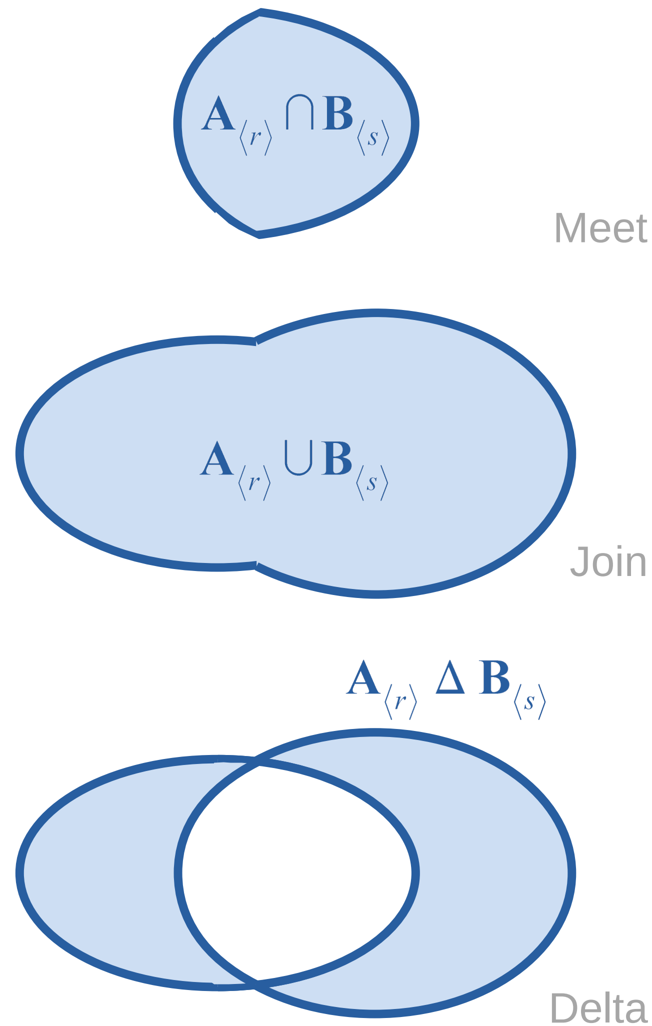


Venn Diagrams

Computing the grade of the meet and join

$$grade\left(\mathbf{A}_{\langle r\rangle} \cap \mathbf{B}_{\langle s\rangle}\right) = \frac{r + s - grade\left(\mathbf{A}_{\langle r\rangle}\ \Delta\ \mathbf{B}_{\langle s\rangle}\right)}{2}$$

$$grade\left(\mathbf{A}_{\langle r\rangle} \cup \mathbf{B}_{\langle s\rangle}\right) = \frac{r + s + grade\left(\mathbf{A}_{\langle r\rangle}\ \Delta\ \mathbf{B}_{\langle s\rangle}\right)}{2}$$



Venn Diagrams

Tests for containment

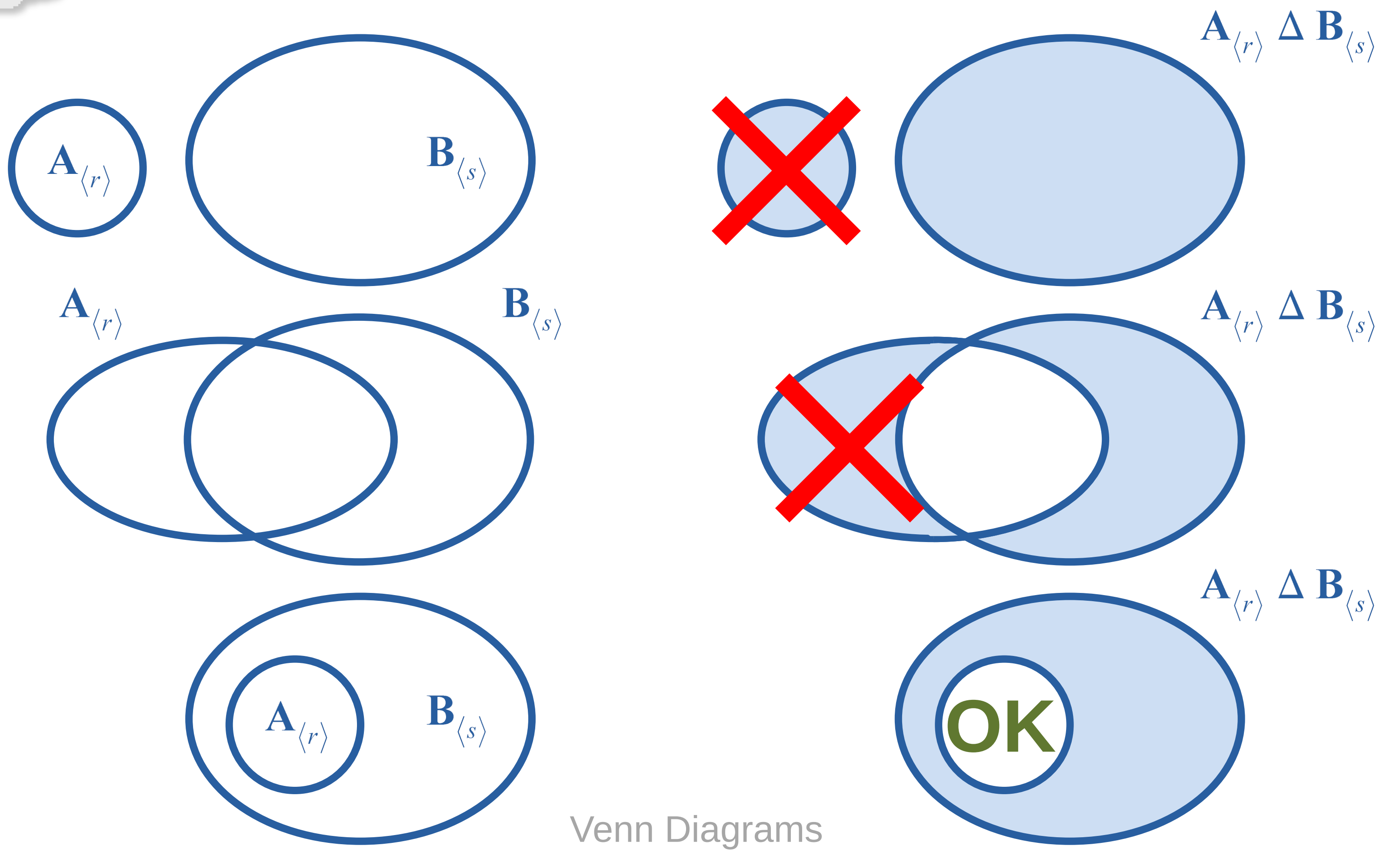
$$\mathbf{a} \wedge \mathbf{B}_{\langle s \rangle} = 0$$

This test returns true if and only if the vector $\mathbf{a} \subseteq \mathbf{B}_{\langle s \rangle}$.

$$\mathbf{A}_{\langle r \rangle} \wedge \left(\mathbf{A}_{\langle r \rangle} \, \Delta \, \mathbf{B}_{\langle s \rangle} \right) \neq 0$$

This test returns true if and only if $\mathbf{A}_{\langle r \rangle} \subseteq \mathbf{B}_{\langle s \rangle}$.

Tests for containment $\mathbf{A}_{\langle r \rangle} \wedge \left(\mathbf{A}_{\langle r \rangle} \,\Delta\, \mathbf{B}_{\langle s \rangle} \right) \neq 0$



Venn Diagrams

Lecture III

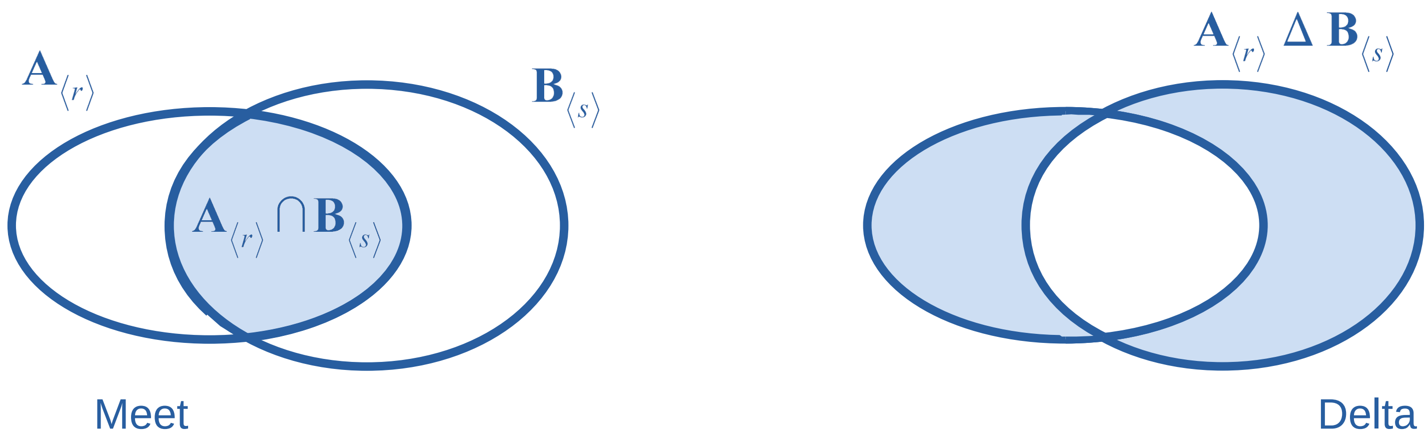
Computing the Meet and Join of Blades

Some observations

How the algorithm works

It starts with a scalar, and build the common subspace by the outer product of potential factors until it arrives at the true meet.

- Potential factors of the meet
  - They are factors of both input blades
  - They are not factors of the delta product



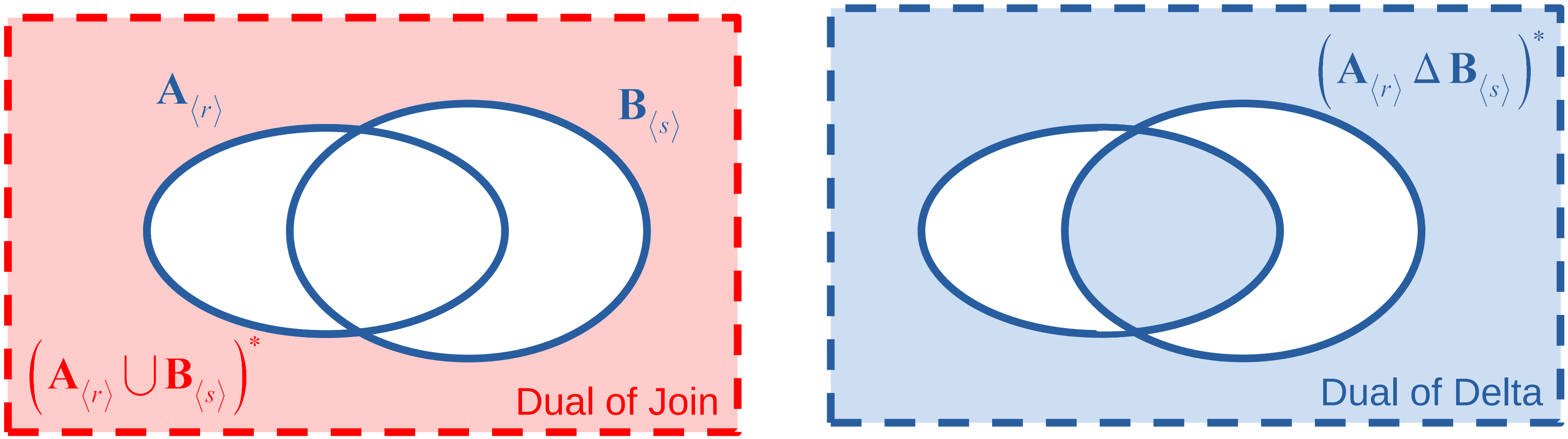
Venn Diagrams

How the algorithm works

It starts with a pseudoscalar, and remove factors from it until the true join is obtained.

Some observations

- Factors that should not be in the join
  - They are not factors of the input blades
  - They are factors of the dual of the delta product



Venn Diagrams

The algorithm

Swap input blades when it is necessary.

This may engender an extra sign: $(-1)^{(j-r)(j-s)}$

1. Input: blades $\mathbf{A}_{\langle r \rangle}$ and $\mathbf{B}_{\langle s \rangle}$, where $r \leq s$
2. Compute the dual of the delta product $\mathbf{S} = \left(\mathbf{A}_{\langle r \rangle} \Delta \mathbf{B}_{\langle s \rangle}\right)^*$ and factorize it in factors $\mathbf{s}_i$
3. Set $\mathbf{M} \leftarrow 1$ and $\mathbf{J} \leftarrow \mathbf{I}_{\langle n \rangle}$
4. For each of the factors $\mathbf{s}_i$:
   a. Compute the projection $\mathbf{p}_i = \left(\mathbf{s}_i \rfloor \mathbf{A}_{\langle r \rangle}^{-1}\right) \rfloor \mathbf{A}_{\langle r \rangle}$ and the rejection $\mathbf{r}_i = \left(\mathbf{s}_i \wedge \mathbf{A}_{\langle r \rangle}\right) \lfloor \mathbf{A}_{\langle r \rangle}^{-1}$
   b. For $\mathbf{p}_i \neq 0$, $\mathbf{M} \leftarrow \mathbf{M} \wedge \mathbf{p}_i$. If the grade of $\mathbf{M}$ is the required grade of the meet, then compute the join and break the loop. Otherwise continue with $\mathbf{s}_{i+1}$
   c. For $\mathbf{r}_i \neq 0$, $\mathbf{J} \leftarrow \mathbf{r}_i \rfloor \mathbf{J}$. If the grade of $\mathbf{J}$ is the required grade of the join, then computer the meet from the join and break the loop. Otherwise continue with $\mathbf{s}_{i+1}$
5. Output: blades $\mathbf{M}$ and $\mathbf{J}$

The rejection is a vector that is perpendicular to $\mathbf{A}_{\langle r \rangle}$.

Efficient factorization and join of blades

Fontijne, D. (2008) *Efficient algorithms for factorization and join of blades*. **In 3rd International Conference on Applied Geometric Algebras in Computer Science and Engineering**, Grimma, Germany

- 5 to 10 times faster than earlier algorithms
- The factors are linearly independents, but they are not orthogonal in general
- Remeber: the meet can be computed from the join