Introduction to Geometric Algebra

Lecture IV

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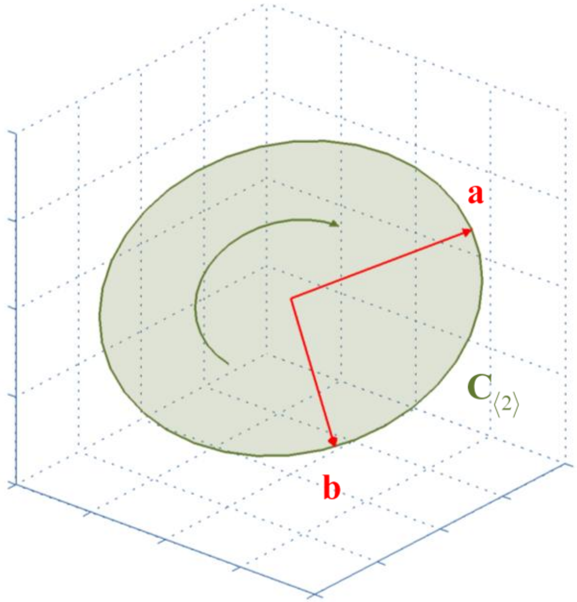
Lecture IV

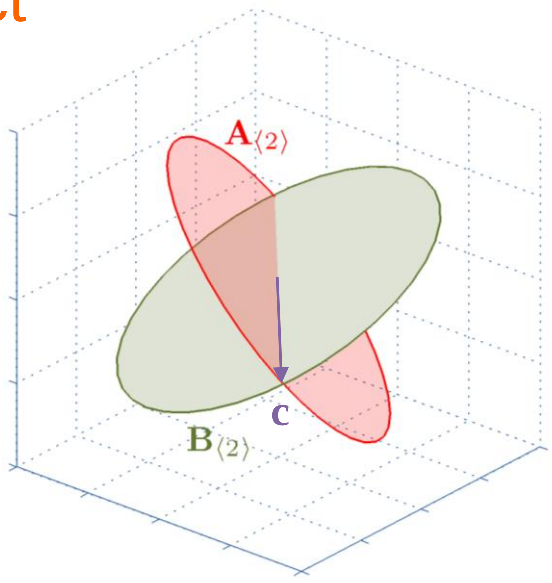
Checkpoint

Checkpoint, Lecture I

- Multivector space $\bigwedge \mathbb{R}^n$
- Non-metric products
  - The outer product
  - The regressive product



$$C_{\langle 2 \rangle} = a \wedge b$$



$$c = A_{\langle 2 \rangle} \vee B_{\langle 2 \rangle}$$

Checkpoint, Lecture II

- Metric spaces
  - Bilinear form $Q(\mathbf{a},\mathbf{b})$ defines a metric on the vector space, e.g., Euclidean metric
  - Metric matrix
- Some inner products
  - Inner product of vectors
  - Scalar product
  - Left contraction
  - Right contraction

The scalar product is a particular case of the left and right contractions

$$\mathbf{A}_{\langle k \rangle} * \mathbf{B}_{\langle k \rangle} = \mathbf{A}_{\langle k \rangle} \rfloor \mathbf{B}_{\langle k \rangle} = \mathbf{A}_{\langle k \rangle} \lfloor \mathbf{B}_{\langle k \rangle}$$

These metric products are backward compatible for 1-blades

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} * \mathbf{b} = \mathbf{a} \rfloor \mathbf{b} = \mathbf{a} \lfloor \mathbf{b}$$

Checkpoint, Lecture II

- Dualization

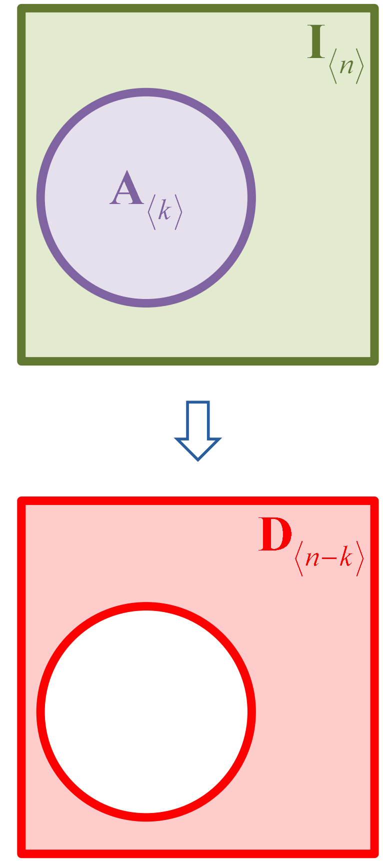
$$\mathbf{A}^{*}_{\langle k\rangle} = \mathbf{D}_{\langle n-k\rangle} = \mathbf{A}_{\langle k\rangle} \rfloor \mathbf{I}^{-1}_{\langle n\rangle}$$

- Undualization

$$\mathbf{D}^{-*}_{\langle n-k\rangle} = \mathbf{A}_{\langle k\rangle} = \mathbf{D}_{\langle n-k\rangle} \rfloor \mathbf{I}_{\langle n\rangle}$$

$$\left(\mathbf{A}^{*}_{\langle k\rangle}\right)^{-*} = \mathbf{A}_{\langle k\rangle}$$

By taking the undual, the dual representation of a blade can be correctly mapped back to its direct representation



Venn Diagrams

Checkpoint, Lecture III

- Duality relationships between products
  - Dual of the outer product

$$\left(\mathbf{A}_{\langle r\rangle} \wedge \mathbf{B}_{\langle s\rangle}\right)^{*} = \mathbf{A}_{\langle r\rangle} \rfloor \mathbf{B}^{*}_{\langle s\rangle}$$

  - Dual of the left contraction

$$\left(\mathbf{A}_{\langle r\rangle} \rfloor \mathbf{B}_{\langle s\rangle}\right)^{*} = \mathbf{A}_{\langle r\rangle} \wedge \mathbf{B}^{*}_{\langle s\rangle}$$

Checkpoint, Lecture III

- Some non-linear products
  - Meet of blades

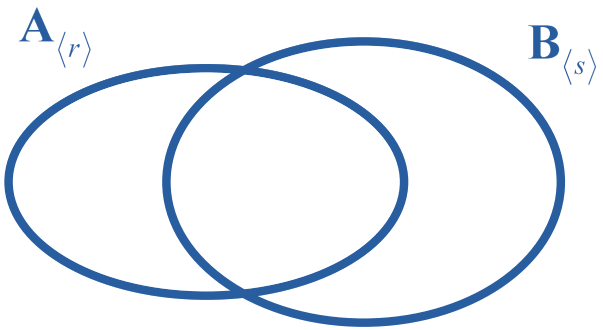
$$\mathbf{A}_{\langle r \rangle} \cap \mathbf{B}_{\langle s \rangle}$$

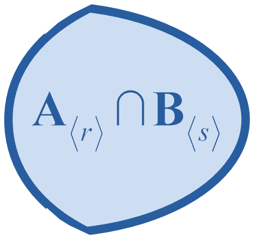
  - Join of blades

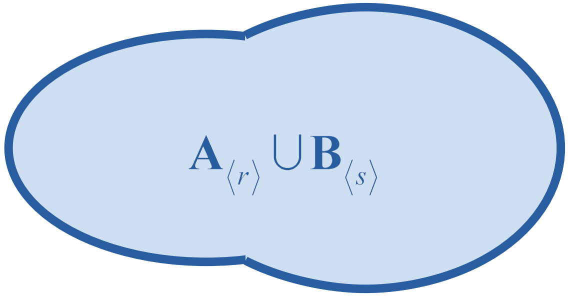
$$\mathbf{A}_{\langle r \rangle} \cup \mathbf{B}_{\langle s \rangle}$$

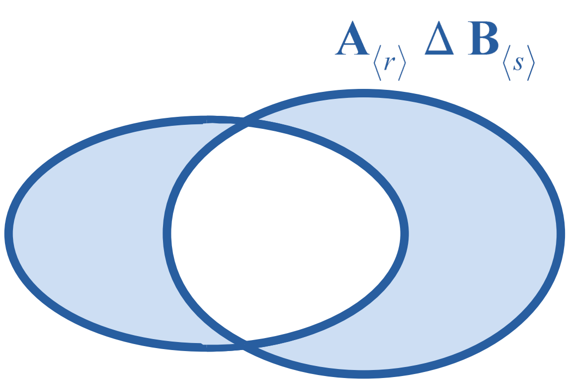
  - Delta product of blades

$$\mathbf{A}_{\langle r \rangle} \,\Delta\, \mathbf{B}_{\langle s \rangle}$$









Today

- Lecture IV – Mon, January 18
  - Geometric product
  - Versors
  - Rotors

Lecture IV

Geometric Product

Geometric product of vectors

$$\boxed{\mathbf{a}\,\mathbf{b}} = \boxed{\mathbf{a}\cdot\mathbf{b}} + \boxed{\mathbf{a}\wedge\mathbf{b}}$$

Denoted by a white space, like standard multiplication

Inner Product

Outer Product

<u>Unique Feature</u>

An invertible product for vectors!

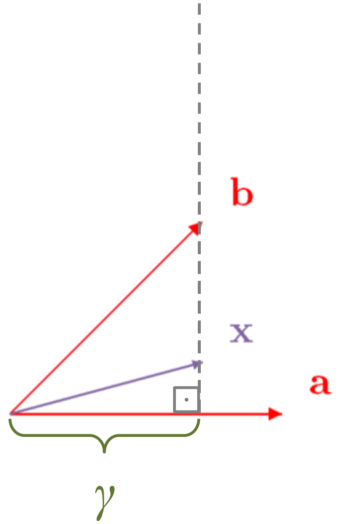
Geometric product of vectors

$$\mathbf{a}\,\mathbf{b} =$$

Unique Feature

An invertible product for vectors!

The Inner Product is not Invertible



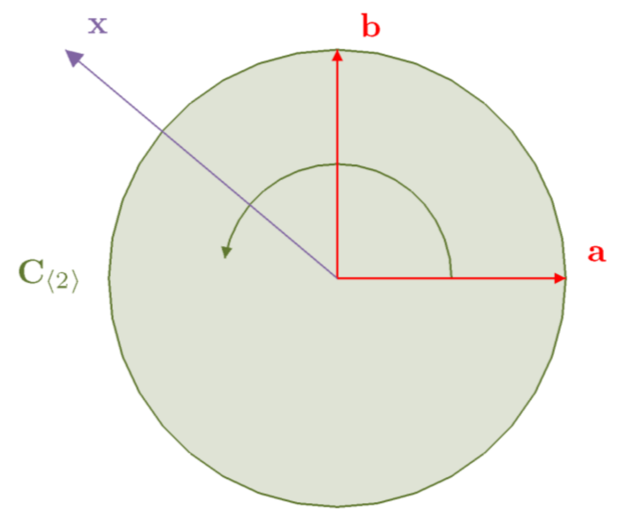
$$\gamma = \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{x}$$

** Euclidean Metric

Geometric product of vectors

$$\mathbf{a}\,\mathbf{b} =$$

The Outer Product is not Invertible



$$C_{\langle 2 \rangle} = \mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{x}$$

<u>Unique Feature</u>

An invertible product for vectors!

Geometric product of vectors

$$\mathbf{a}\,\mathbf{b} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}$$

$$D = \mathbf{a}\,\mathbf{b}$$

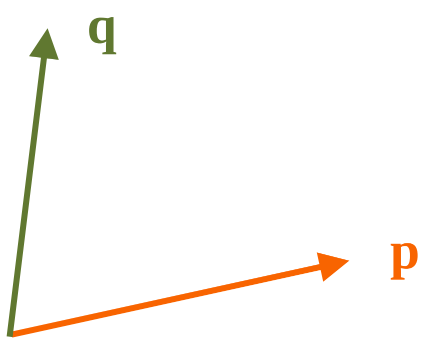
Unique Feature

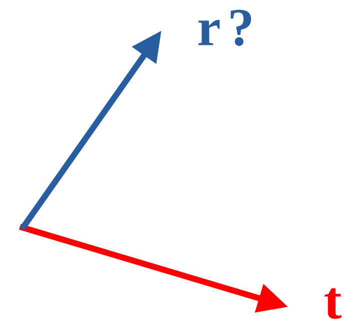
An invertible product for vectors!

$$\boxed{\frac{D}{\mathbf{b}}} = \frac{\mathbf{a}\,\cancel{\mathbf{b}}}{\cancel{\mathbf{b}}} = \mathbf{a}$$

Inverse geometric product,
denoted by a slash,
like standard division

Intuitive solutions for simple problems





$$\mathbf{r} = \left(\frac{\mathbf{q}}{\mathbf{p}}\right)\mathbf{t}$$

** Euclidean Metric

Geometric product and multivector space

- The geometric product of two vectors is an element of mixed dimensionality

$$\{\ \underbrace{1,}_{\text{Scalars}}\quad \underbrace{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,}_{\text{Vector Space}}\quad \underbrace{\mathbf{e}_1\wedge\mathbf{e}_2,\ \mathbf{e}_1\wedge\mathbf{e}_3,\ \mathbf{e}_2\wedge\mathbf{e}_3,}_{\text{Bivector Space}}\quad \underbrace{\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3}_{\text{Trivector Space}}\ \}$$

$$\mathbb{R} \qquad \mathbb{R}^3 \qquad \textstyle\bigwedge^2 \mathbb{R}^3 \qquad \bigwedge^3 \mathbb{R}^3$$

$$\mathbf{a} = \alpha_1\,\mathbf{e}_1 + \alpha_2\,\mathbf{e}_2 + \alpha_3\,\mathbf{e}_3$$

$$\mathbf{b} = \beta_1\,\mathbf{e}_1 + \beta_2\,\mathbf{e}_2 + \beta_3\,\mathbf{e}_3$$

Geometric product and multivector space

- The geometric product of two vectors is an element of mixed dimensionality

$$\{ \underbrace{1,}_{\text{Scalars}} \quad \underbrace{\mathbf{e}_1, \quad \mathbf{e}_2, \quad \mathbf{e}_3,}_{\text{Vector Space}} \quad \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2, \quad \mathbf{e}_1 \wedge \mathbf{e}_3, \quad \mathbf{e}_2 \wedge \mathbf{e}_3,}_{\text{Bivector Space}} \quad \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3}_{\text{Trivector Space}} \}$$

$$\mathbb{R} \qquad \mathbb{R}^3 \qquad \bigwedge^2 \mathbb{R}^3 \qquad \bigwedge^3 \mathbb{R}^3$$

$$\mathbf{a}\,\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

<u>Geometric Meaning</u>

The interpretation of the resulting element depends on the operands.

Properties of the geometric product

$$\bigwedge \mathbb{R}^n \times \bigwedge \mathbb{R}^n \to \bigwedge \mathbb{R}^n$$

Scalars commute $A(\beta\, B) = \beta(A\, B)$

Distributivity $A(B + C) = A\, B + A\, C$

Associativity $A(B\, C) = (A\, B)\, C$

Neither fully symmetric nor fully antisymmetric $\exists\, A, B \in \bigwedge \mathbb{R}^n \quad : \quad A\, B \neq B\, A$

Lecture IV

Geometric product of basis blades

Geometric product of basis blades

- Let's assume an orthogonal metric, i.e.,

$$\mathbf{e}_i \cdot \mathbf{e}_j = m_i \, \delta^i_j$$

With an orthogonal metric, there are two cases to be handled

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Kronecker delta function

$$m_i \in \mathbb{R}$$

Metric factor

Geometric product of basis blades

- Let's assume an orthogonal metric, i.e.,

$$\mathbf{e}_i \cdot \mathbf{e}_j = m_i \, \delta^i_j$$

  - Case 1: blades consisting of different orthogonal factors

$$\mathbf{e}_1 \, \mathbf{e}_2 \cdots \mathbf{e}_k = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_k$$

The geometric product is equivalent to the outer product

  - Case 2: blades with some common factors

$$(\mathbf{e}_1 \wedge \mathbf{e}_2)(\mathbf{e}_2 \wedge \mathbf{e}_3) = \mathbf{e}_1 \, \mathbf{e}_2 \, \mathbf{e}_2 \, \mathbf{e}_3 = \mathbf{e}_1 \, m_2 \, \mathbf{e}_3 = m_2 \, \mathbf{e}_1 \wedge \mathbf{e}_3$$

The dependent-basis factors are replaced by metric factors

Geometric product of basis blades

- Let's assume a non-orthogonal metric, e.g.,

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \mathbf{o} & \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_d & \boldsymbol{\infty} \end{array} \\
\mathrm{M} = & \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{array}{c} \mathbf{o} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_d \\ \boldsymbol{\infty} \end{array}
\end{array}
$$

Apply the spectral theorem from linear algebra and reduce the problem to the orthogonal metric case

** The spectral theorem states that a matrix is orthogonally diagonalizable if and only if it is symmetric.

Geometric product of basis blades

- For non-orthogonal metrics
  1. Compute the eigenvectors and eigenvalues of the metric matrix
  2. Represent the input with respect to the eigenbasis
     - Apply a change of basis using the inverse of the eigenvector matrix
  3. Compute the geometric product on this new orthogonal basis
     - The eigenvalues specify the new orthogonal metric
  4. Get back to the original basis
     - Apply a change of basis using the original eigenvector matrix

Geometric product of basis blades

- For non-orthogonal metrics
  1. Compute the eigenvectors and eigenvalues of the metric matrix
  2. Represent the input with respect to the eigenbasis
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     orthogonal bas
     - The eigenvalu
  4. Get back to the
     - Apply a chang

See the Supplementary Material A of the Tutorial at Sibgrapi 2009 for a purest treatment of the geometric product

Lecture IV

Subspace Products from Geometric Product

The most fundamental product of GA

- The subspace products can be derived from the ge[...]

The "grade extraction" operation
extracts grade parts from multivector

$$M = \alpha_1 + \alpha_2\,\mathbf{e}_1 + \alpha_3\,\mathbf{e}_2 + \alpha_4\,\mathbf{e}_3 + \alpha_5\,\mathbf{e}_1 \wedge \mathbf{e}_2 + \alpha_6\,\mathbf{e}_1 \wedge \mathbf{e}_3 + \alpha_7\,\mathbf{e}_2 \wedge \mathbf{e}_3 + \alpha_8\,\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$$

A general multivector variable in $\bigwedge \mathbb{R}^3$

$$\langle M \rangle_2 = \alpha_5\,\mathbf{e}_1 \wedge \mathbf{e}_2 + \alpha_6\,\mathbf{e}_1 \wedge \mathbf{e}_3 + \alpha_7\,\mathbf{e}_2 \wedge \mathbf{e}_3$$

The most fundamental product of GA

- The subspace products can be derived from the geometric product

$$\mathbf{A}_{\langle r\rangle} \wedge \mathbf{B}_{\langle s\rangle} = \left\langle \mathbf{A}_{\langle r\rangle} \mathbf{B}_{\langle s\rangle} \right\rangle_{r+s}$$

Outer product

$$\mathbf{A}_{\langle r\rangle} * \mathbf{B}_{\langle s\rangle} = \left\langle \mathbf{A}_{\langle r\rangle} \mathbf{B}_{\langle s\rangle} \right\rangle_{0}$$

Scalar product

$$\mathbf{A}_{\langle r\rangle} \rfloor \mathbf{B}_{\langle s\rangle} = \left\langle \mathbf{A}_{\langle r\rangle} \mathbf{B}_{\langle s\rangle} \right\rangle_{s-r}$$

Left contraction

$$\mathbf{A}_{\langle r\rangle} \lfloor \mathbf{B}_{\langle s\rangle} = \left\langle \mathbf{A}_{\langle r\rangle} \mathbf{B}_{\langle s\rangle} \right\rangle_{r-s}$$

Right contraction

$$\mathbf{A}_{\langle r\rangle} \Delta \mathbf{B}_{\langle s\rangle} = \left\langle \mathbf{A}_{\langle r\rangle} \mathbf{B}_{\langle s\rangle} \right\rangle_{max}$$

Delta product

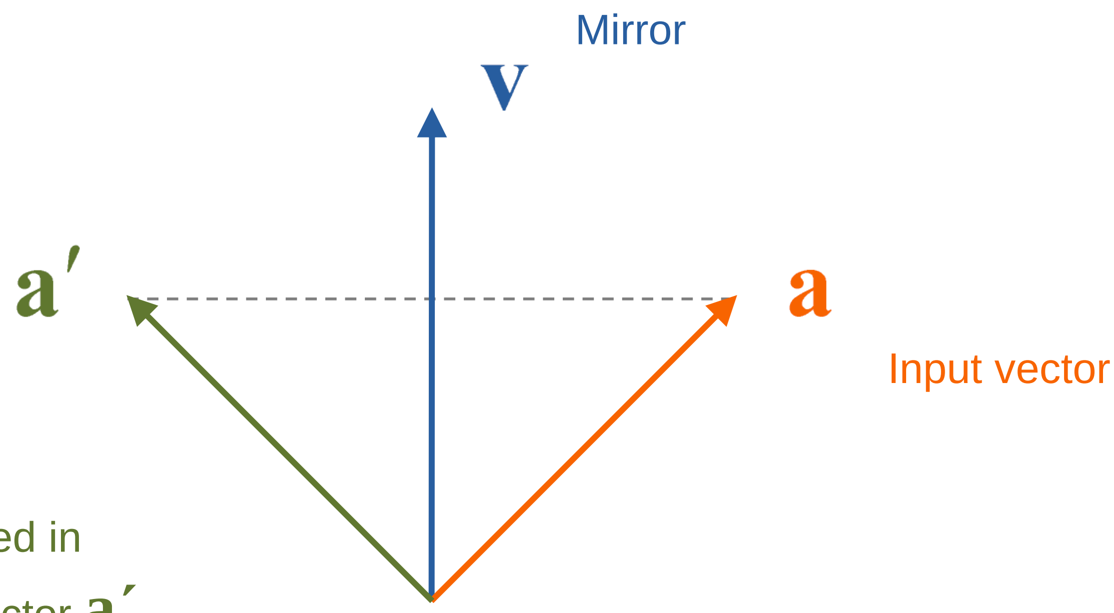
The largest grade such that the result is not zero

Lecture IV

Orthogonal Transformations as Versors

Reflection of vectors

$$\mathbf{a}' = -\mathbf{v}\,\mathbf{a}\,\mathbf{v}^{-1}$$



Vector $\mathbf{a}$ was reflected in vector $\mathbf{v}$, resulting in vector $\mathbf{a}'$

k-Versor

$$\mathbf{a}' = -\mathbf{v}_1\ \mathbf{a}\ \mathbf{v}_1^{-1}$$

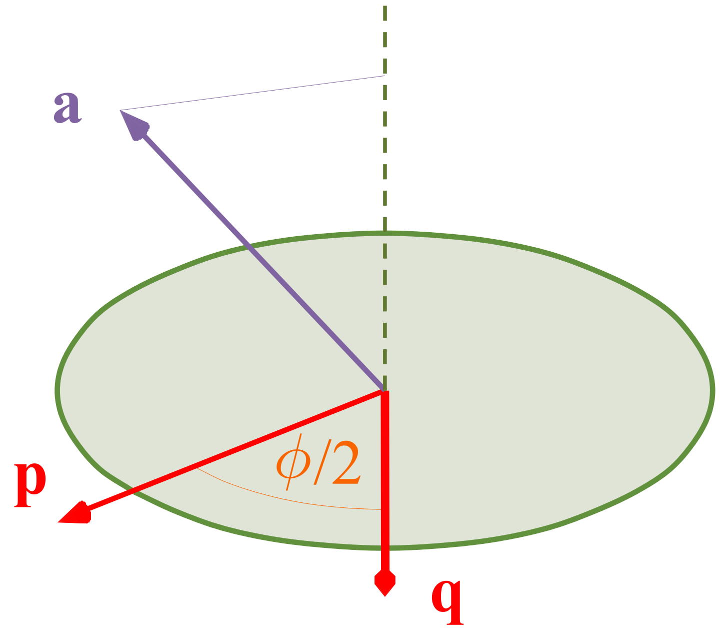
$$\mathbf{a}'' = +\mathbf{v}_2\ \mathbf{v}_1\ \mathbf{a}\ \mathbf{v}_1^{-1}\ \mathbf{v}_2^{-1}$$

$$\mathbf{a}''' = \left(-1\right)^k \left(\mathbf{v}_k \cdots \mathbf{v}_2\ \mathbf{v}_1\right) \mathbf{a} \left(\mathbf{v}_1^{-1}\ \mathbf{v}_2^{-1} \cdots \mathbf{v}_k^{-1}\right)$$

$$\mathbf{a}''' = \left(-1\right)^k V\ \mathbf{a}\ V^{-1}$$

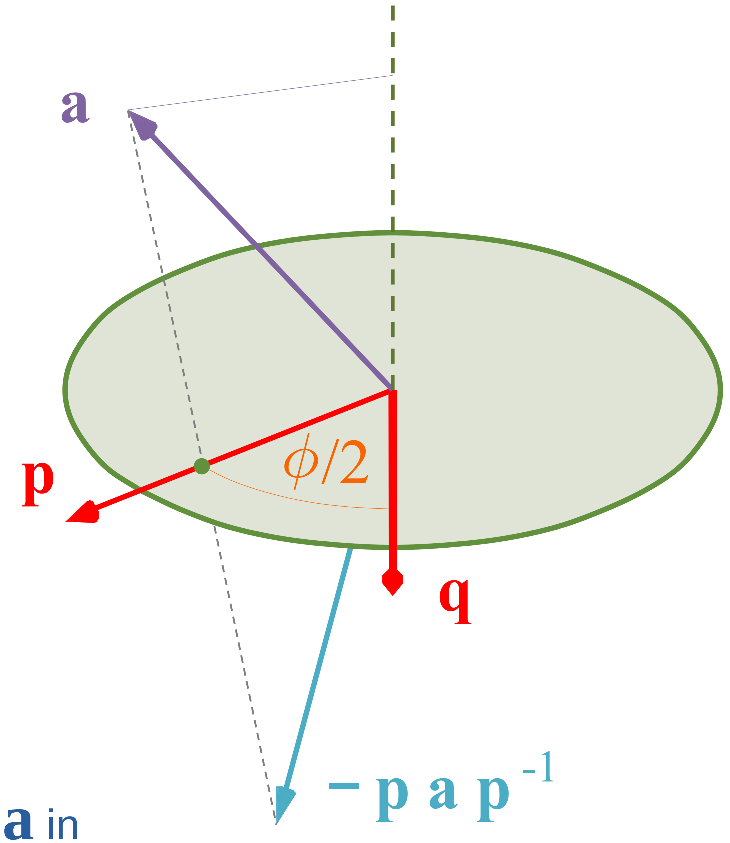
V is a *k*-versor. It is computed as the geometric product of *k* invertible vectors.

Rotation of subspaces



How to rotate vector $\mathbf{a}$ in the $\mathbf{p} \wedge \mathbf{q}$ plane by ϕ radians.

Rotation of subspaces



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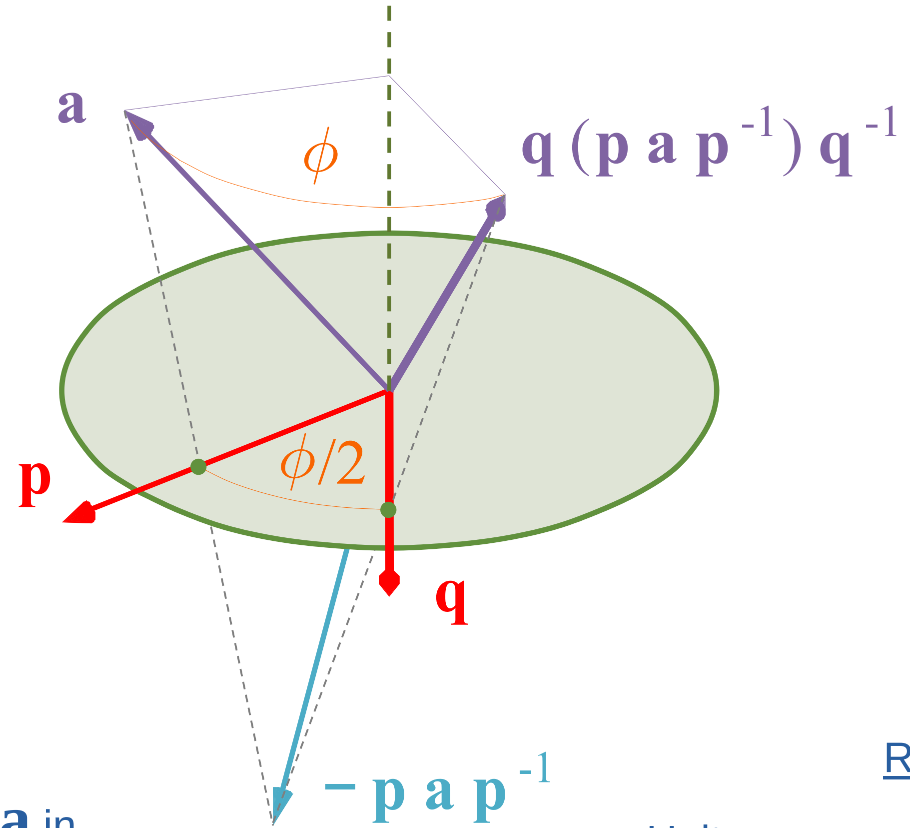


How to rotate vector $\mathbf{a}$ in the $\mathbf{p} \wedge \mathbf{q}$ plane by ϕ radians.

Rotors

Unit versors encoding rotations.
They are build as the geometric product of an even number of unit invertible vectors.

Versor product for general multivectors

$$M' = \begin{cases} V\ M\ V^{-1} & \text{for even versors} \\ V\ \widehat{M}\ V^{-1} & \text{for odd versors} \end{cases}$$

$$\widehat{\mathbf{B}}_{\langle t \rangle} = (-1)^t\,\mathbf{B}_{\langle t \rangle}$$

<u>Grade Involution</u>

The sign change under the grade involution exhibits a + - + - + - … pattern over the value of *t*.

The structure preservation property

$$V\left(A\circ B\right)V^{-1}=\left(V\,A\,V^{-1}\right)\circ\left(V\,B\,V^{-1}\right)$$

The structure preservation of
versors holds for the geometric product,
and hence to all other products
in geometric algebra.

The ○ symbol represents <u>any product</u>
of geometric algebra, and,
as a consequence, <u>any operation</u>
defined from the products

Inverse of versors

The inverse of versors is
computed as for the inverse of invertible blades

$$V^{-1} = \frac{\tilde{V}}{\left\| V \right\|^2}$$

Inverse

$$\left\| V \right\|^2 = \tilde{V} * V$$

Squared (reverse) norm

$$\tilde{\mathbf{A}}_{\langle k \rangle} = \left(-1\right)^{k(k-1)/2} \mathbf{A}_{\langle k \rangle}$$

Reverse ($+ + - -\ \ + + - - \ldots$ pattern over k)

The norm of rotors is equal to one,
so the inverse of a rotor is its reverse

$$R^{-1} = \tilde{R}$$

Multivector classification

- It can be used for blades and versors
  - Use Euclidean metric for blades
  - Use the actual metric for versors
- Test if $M/\left(M\,\widetilde{M}\right)$ is truly the inverse of the multivector

$$\mathrm{grade}\left(\widehat{M}\,M^{-1}\right) = 0 \qquad\qquad \widehat{M}\,M^{-1} = M^{-1}\widehat{M}$$

- Test the grade preservation property

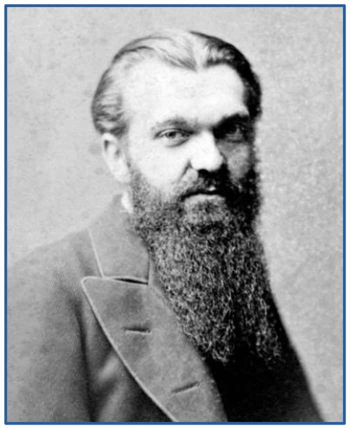
$$\mathrm{grade}\left(\widehat{M}\,\mathbf{e}_i\,\widetilde{M}\right) = 1$$

- If the multivector is of a single grade then it is a blade; otherwise it is a versor

Credits

William R. Hamilton
(1805-1865)

Hermann G. Grassmann
(1809-1877)



William K. Clifford
(1845-1879)

Clifford, W. K. (1878) *Applications of Grassmann's extensive algebra*. **Am. J. Math.**, Walter de Gruyter Und Co., vol. 1, n. 4, 350-358

Differences between algebras

- Clifford algebra
  - Developed in nongeometric directions
  - Permits us to construct elements by a universal addition
  - Arbitrary multivectors may be important

- Geometric algebra
  - The geometrically significant part of Clifford algebra
  - Only permits exclusively multiplicative constructions
    - The only elements that can be added are scalars, vectors, pseudovectors, and pseudoscalars
  - Only blades and versors are important