Introduction to Geometric Algebra

Lecture V

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Lecture V

Checkpoint

Checkpoint

- The geometric product is the most fundamental product of geometric algebra
  - It is an invertible product

$$\mathbf{a}\,\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

Geometric product of vectors

$$\frac{D}{\mathbf{b}} = \frac{\mathbf{a}\,\cancel{\mathbf{b}}}{\cancel{\mathbf{b}}} = \mathbf{a}$$

Checkpoint

- The geometric product is the most fundamental product of geometric algebra
  - It is an invertible product
  - The subspace products can be derived from it

$$\mathbf{A}_{\langle r \rangle} \wedge \mathbf{B}_{\langle s \rangle} = \left\langle \mathbf{A}_{\langle r \rangle} \mathbf{B}_{\langle s \rangle} \right\rangle_{r+s}$$

Outer product

$$\mathbf{A}_{\langle r \rangle} * \mathbf{B}_{\langle s \rangle} = \left\langle \mathbf{A}_{\langle r \rangle} \mathbf{B}_{\langle s \rangle} \right\rangle_{0}$$

Scalar product

$$\mathbf{A}_{\langle r \rangle} \rfloor \mathbf{B}_{\langle s \rangle} = \left\langle \mathbf{A}_{\langle r \rangle} \mathbf{B}_{\langle s \rangle} \right\rangle_{s-r}$$

Left contraction

$$\mathbf{A}_{\langle r \rangle} \lfloor \mathbf{B}_{\langle s \rangle} = \left\langle \mathbf{A}_{\langle r \rangle} \mathbf{B}_{\langle s \rangle} \right\rangle_{r-s}$$

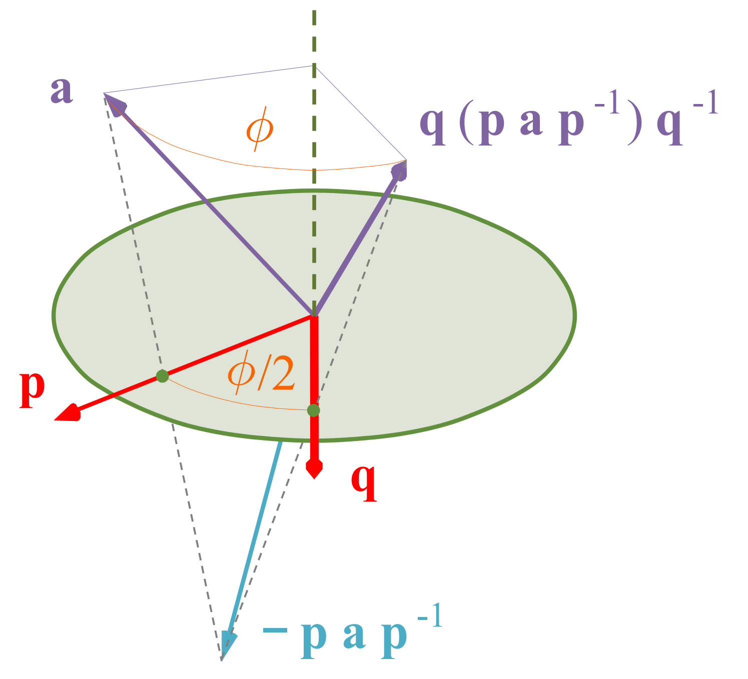
Right contraction

$$\mathbf{A}_{\langle r \rangle} \Delta \mathbf{B}_{\langle s \rangle} = \left\langle \mathbf{A}_{\langle r \rangle} \mathbf{B}_{\langle s \rangle} \right\rangle_{max}$$

Delta product

Checkpoint

- Versors encode linear transformations, e.g.,
  - Reflections
  - Rotations (rotors)



Today

- Lecture V – Tue, January 19
  - Models of geometry
  - Euclidean vector space model
  - Homogeneous model

Lecture V

Models of Geometry

What does a Model of Geometry do?

- Assumes a metric to the space

$$\mathbf{a} \cdot \mathbf{b} = \mathrm{Q}\left(\mathbf{a}, \mathbf{b}\right)$$

- Gives a geometrical interpretation to subspaces
  - Directions, points, straight lines, circles, etc.

- Makes versors behave like some transformation type
  - Scaling, rotation, translation, etc.

Lecture V

Euclidean Vector Space Model of Geometry

Euclidean vector space model

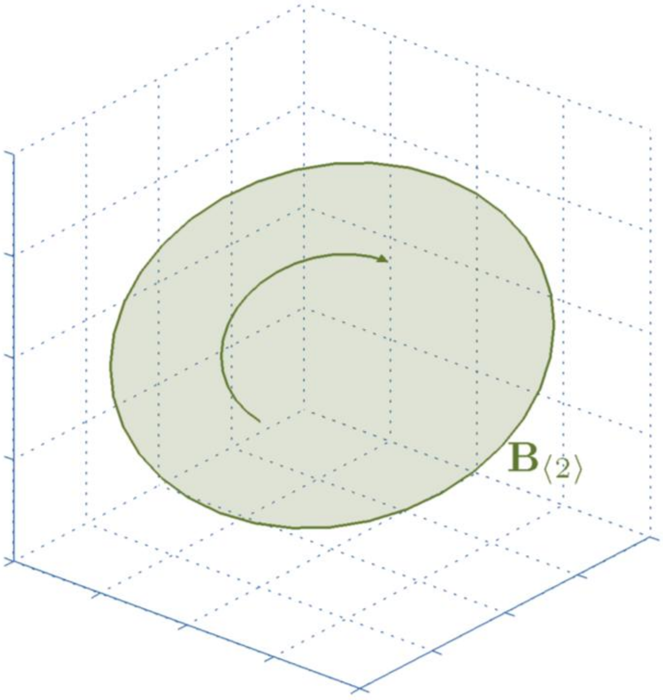
- Euclidean metric

$$
M = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array} \\
\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}
\end{array}
\begin{array}{c} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_n \end{array}
$$

Euclidean vector space model

- Euclidean metric
- Blades
  - Euclidean subspaces

Geometrically, an Euclidean subspace
is a flat in n-dimensional Euclidean space
that passes through the origin.

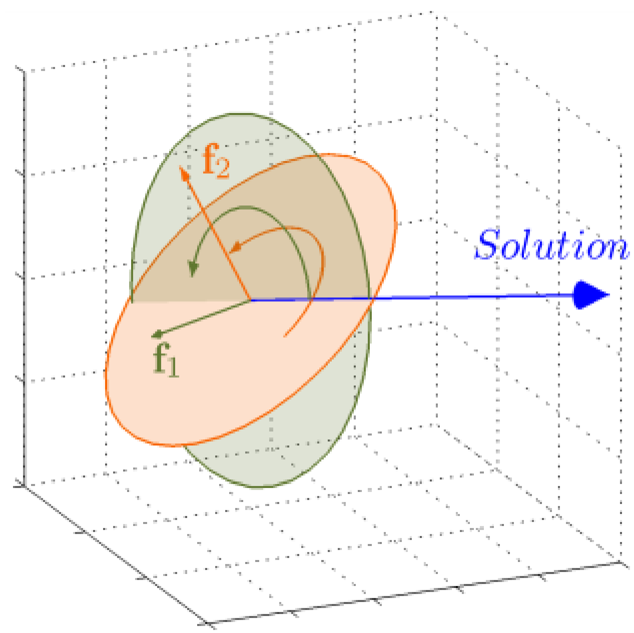


Solving homogeneous systems of linear equations with n variables

$$\begin{cases} 2\mathbf{e}_1 - 3\mathbf{e}_2 \qquad\quad = 0 \\ \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3 = 0 \end{cases}$$

$$\mathbf{f}_1 = 2\mathbf{e}_1 - 3\mathbf{e}_2$$
$$\mathbf{f}_2 = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3$$

$$\left(\mathbf{f}_1 \wedge \mathbf{f}_2\right)^{-*} = 9\mathbf{e}_1 + 6\mathbf{e}_2 + \mathbf{e}_3$$



Each equation of the system is the dual of and hyperplane that passes through the origin.

Solving homogeneous systems of linear equations with n variables

- The general approach is

$$\left(\mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \cdots \wedge \mathbf{f}_k\right)^{-*}$$

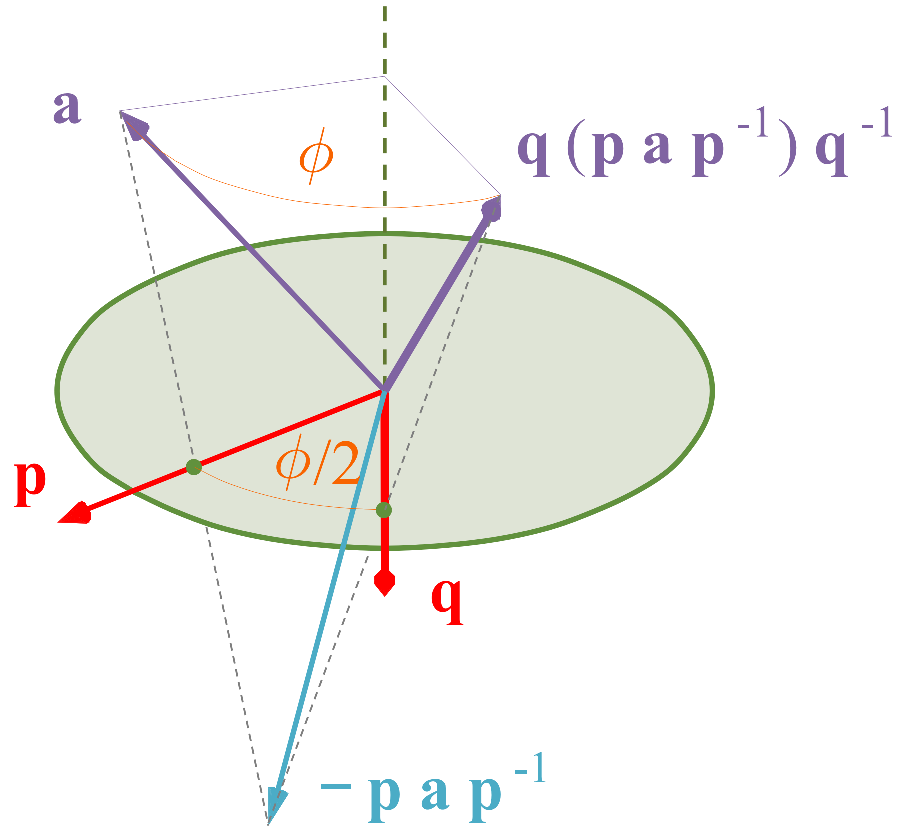
$$1 \leq k \leq n$$

- When the system has no solution
  - The resulting subspace will be zero
- When the system is underdeterminated
  - The resulting subspace has more than one dimension

Euclidean vector space model

- Euclidean metric
- Blades
  - Euclidean subspaces
- Versors
  - Reflections
  - Rotations (rotors)

Rotations as double reflections



Rotations as the exponential of 2-blades

Rotation angle (in radians)

$$R = \exp\left(-\frac{\phi}{2}\boxed{\mathbf{B}_{\langle 2 \rangle}}\right)$$

Unit rotation plane

Rotations as the exponential of 2-blades

$$\boldsymbol{R} = \exp\left(\right.$$

The exponential of k-blades for arbitrary metric spaces

$$\exp\left(\mathbf{A}_{\langle k\rangle}\right) = 1 + \frac{\mathbf{A}_{\langle k\rangle}}{1!} + \frac{\mathbf{A}_{\langle k\rangle}^2}{2!} + \frac{\mathbf{A}_{\langle k\rangle}^3}{3!} + \cdots$$

$$= \begin{cases} \cos\alpha + \frac{\sin\alpha}{\alpha}\mathbf{A}_{\langle k\rangle} & \text{for } \mathbf{A}_{\langle k\rangle}^2 < 0 \\ 1 + \mathbf{A}_{\langle k\rangle} & \text{for } \mathbf{A}_{\langle k\rangle}^2 = 0 \\ \cosh\alpha + \frac{\sinh\alpha}{\alpha}\mathbf{A}_{\langle k\rangle} & \text{for } \mathbf{A}_{\langle k\rangle}^2 > 0 \end{cases},$$

$$\text{where } \alpha = \sqrt{\mathrm{abs}(\mathbf{A}_{\langle k\rangle}^2)}.$$

Rotations as the exponential of 2-blades

$$\boldsymbol{R} = \exp\left(-\frac{\phi}{2}\mathbf{B}_{\langle 2\rangle}\right)$$

$$= \boxed{\cos\left(\frac{\phi}{2}\right)} \boxed{-\sin\left(\frac{\phi}{2}\right)\mathbf{B}_{\langle 2\rangle}}$$

Quaternions

$$H = \boxed{\alpha} + \boxed{\beta_1\, i + \beta_2\, j + \beta_3\, k}$$

$$\alpha, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$$

$$i^2 = j^2 = k^2 = i\, j\, k = -1$$

Logarithm of a rotation in a plane

$$\log(\boldsymbol{R}) = \frac{\langle \boldsymbol{R} \rangle_2}{\left\| \langle \boldsymbol{R} \rangle_2 \right\|} \tan^{-1}\left(\frac{\left\| \langle \boldsymbol{R} \rangle_2 \right\|}{\left\| \langle \boldsymbol{R} \rangle_0 \right\|} \right)$$

This equation is defined only for rotors
3-D Euclidean vector spaces.

The logarithm of a general rotor
is an open problem.

When $\boldsymbol{R} = 1$ the logarithm is virtually zero.
But the ambiguity at $\boldsymbol{R} = -1$ cannot be resolved without
making an arbitrary choice for the rotation plane.

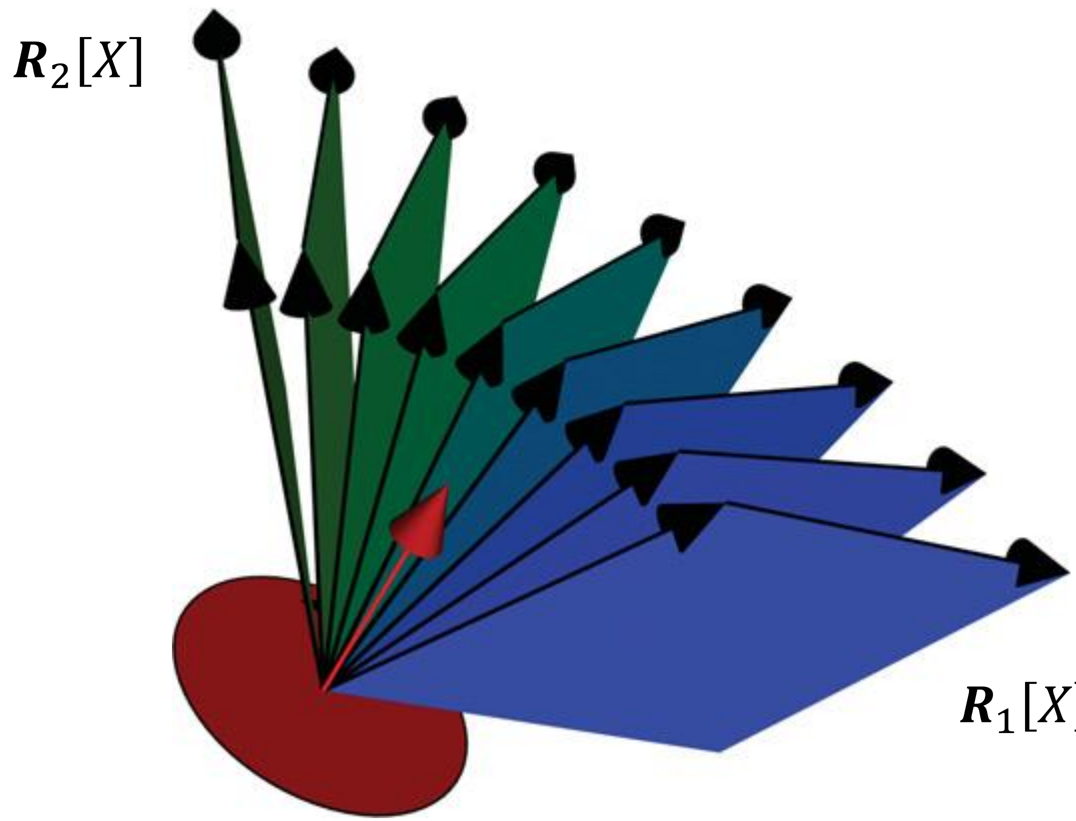
Rotation interpolation

$$\boldsymbol{R} = \frac{\boldsymbol{R}_2}{\boldsymbol{R}_1}$$

Rotor to be interpolated

$$\boldsymbol{S} = \exp\left(\frac{\log(\boldsymbol{R})}{n}\right)$$

Rotation step (it is applied n times)



Adapted from L. Dorst, D. Fontijine, S. Mann. *Geometric algebra for computer science*. Morgan Kaufmann Publishers, 2007.

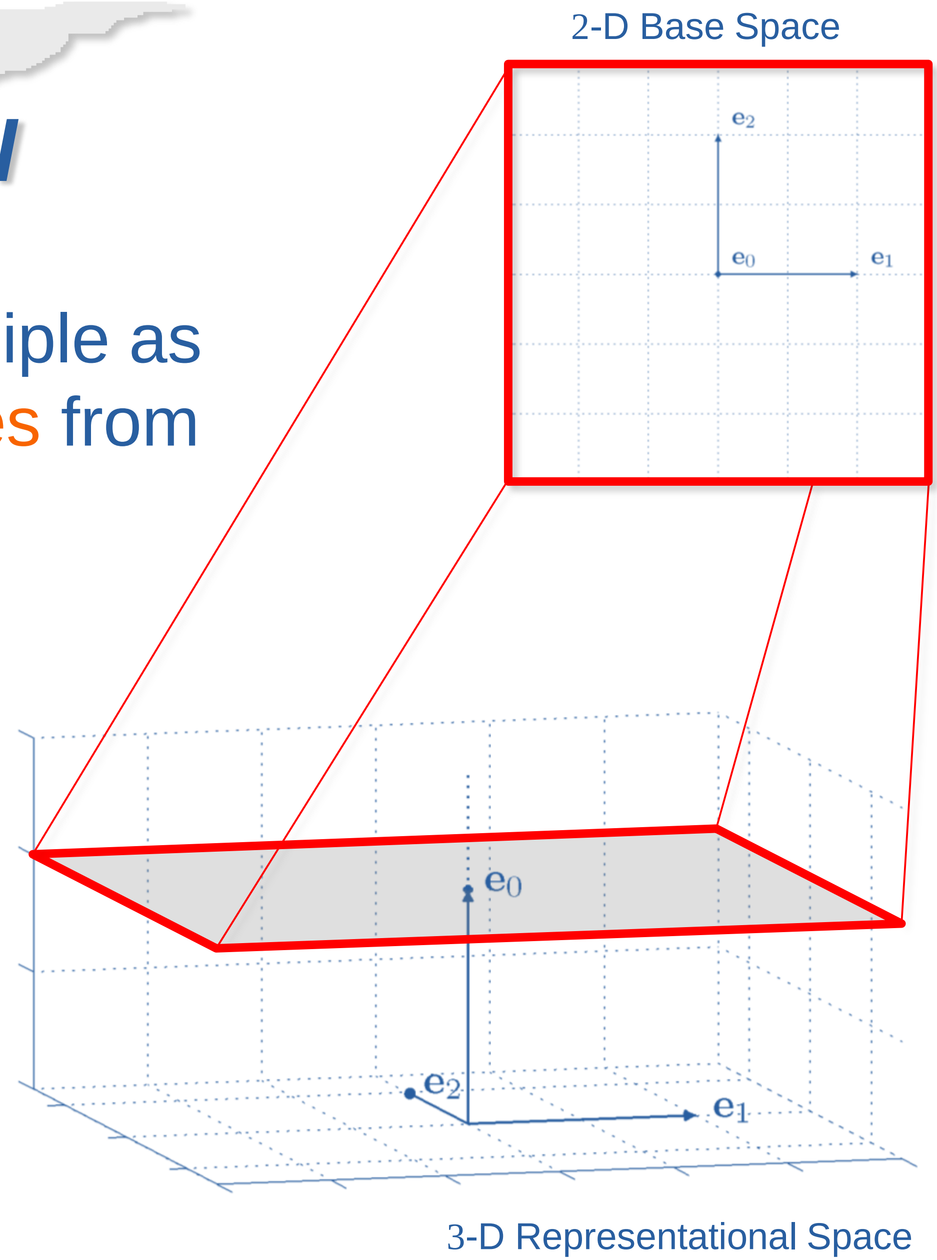
Lecture V

Homogeneous Model of Geometry

Homogeneous model

- The same modeling principle as homogeneous coordinates from linear algebra

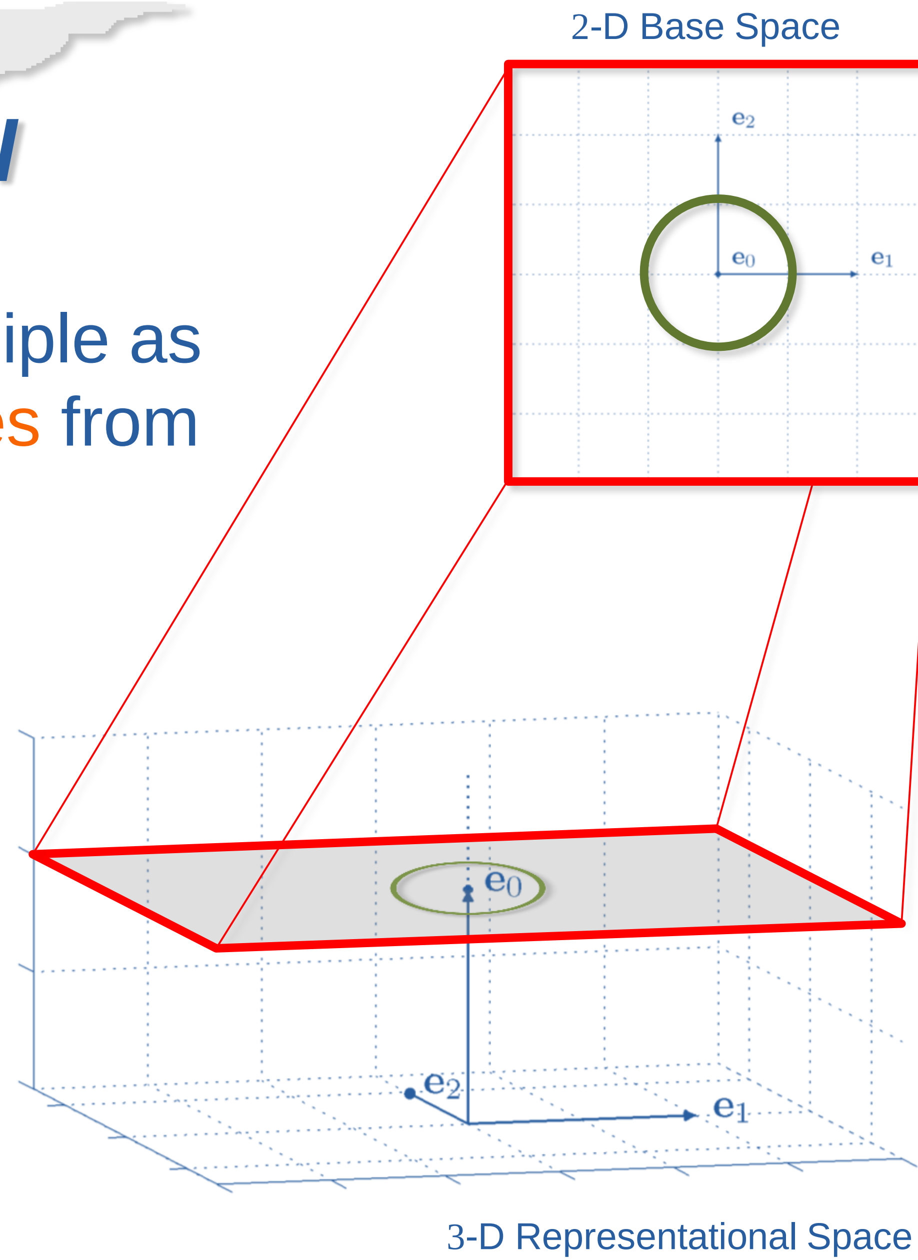
The d-dimensional base space is embedded in a $(d+1)$-dimensional representational vector space.



Homogeneous model

- The same modeling principle as homogeneous coordinates from linear algebra

The extra basis vector is interpreted as point at origin.



Homogeneous model

- The same modeling principle as homogeneous coordinates from linear algebra

- Euclidean metric

$$
M = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}
\begin{array}{cccccc}
\mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_d & \\
\end{array}
$$

$$
M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{array}{c} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_d \end{array}
$$

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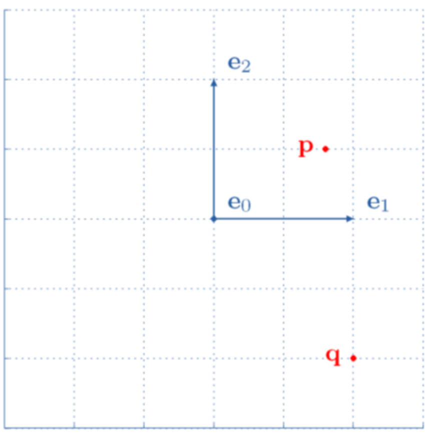
Homogeneous model

- The same modeling principle as homogeneous coordinates from linear algebra
- Euclidean metric
- Well suited to compute with oriented k-flats
  - 0-flat $\longrightarrow$ point
  - 1-flat $\longrightarrow$ straigh line
  - 2-flat $\longrightarrow$ plane
  - etc.

Vectors interpreted as points



General proper point

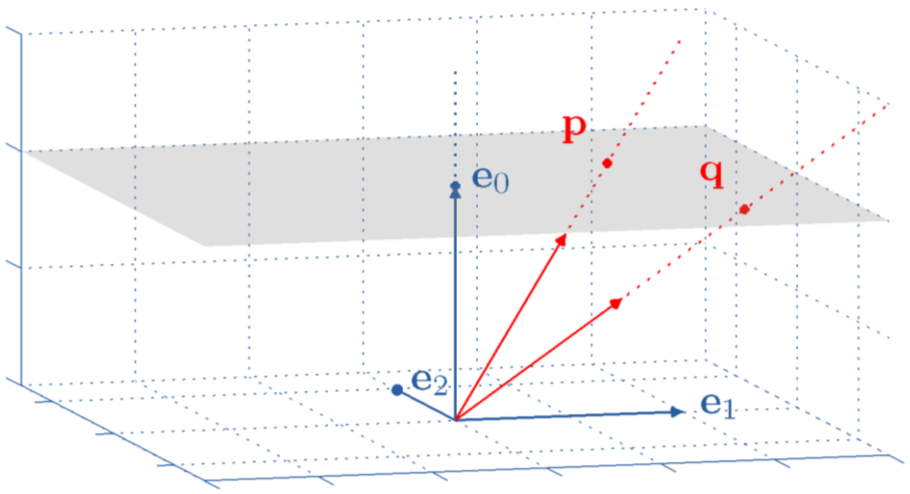
$$\mathbf{p} = \gamma\left(\mathbf{e}_0 + \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \cdots + \alpha_d\mathbf{e}_d\right)$$

$$\text{for } \gamma \neq 0$$

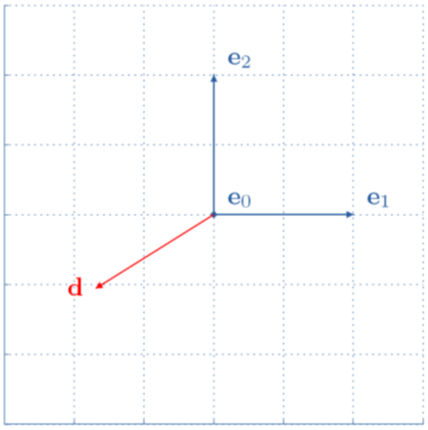
Proper points have finite location.

The scalar factor γ does not affect the location of a proper point.

Unit proper points have γ equal to one.



Vectors interpreted as directions



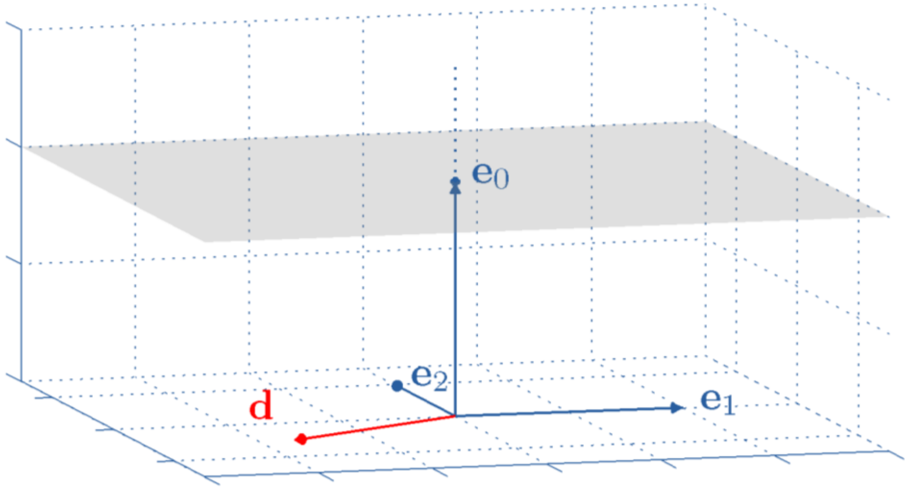
Improper point

$$\mathbf{d} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \cdots + \alpha_d \mathbf{e}_d$$

Improper points are points at infinity.

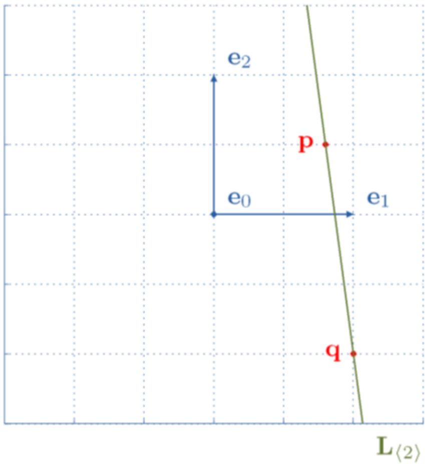
Improper points can be seen as directions, because they are in purely directional space of the total space.

Unlike proper points, directions have the coefficient of $\mathbf{e}_0$ equal to zero.

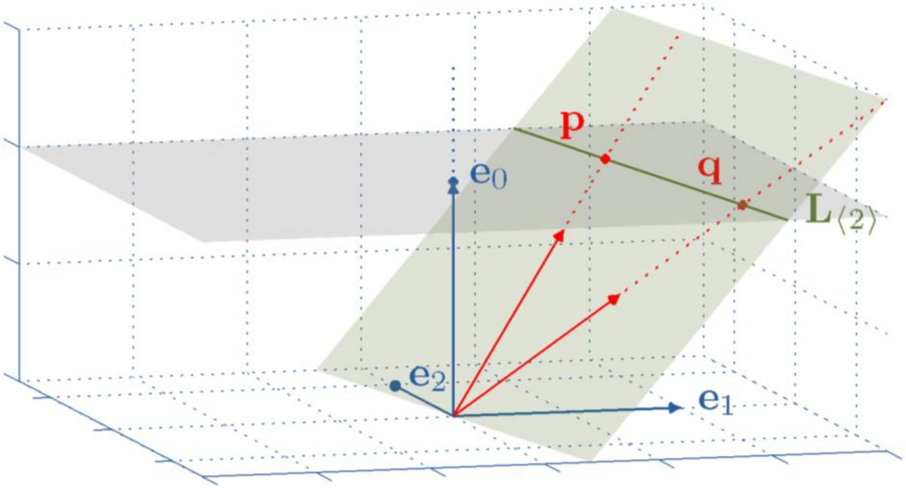


2-Blades interpreted as lines

2-D Base Space



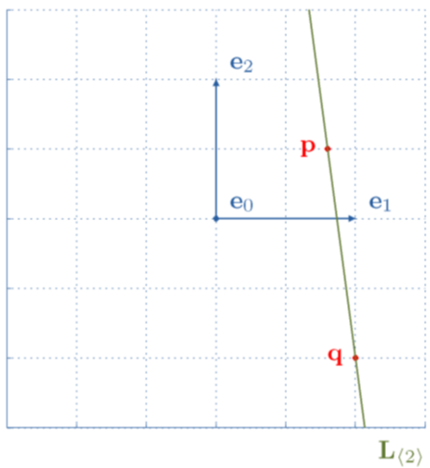
$$\mathbf{L}_{\langle 2\rangle} = \mathbf{p} \wedge \mathbf{q}$$



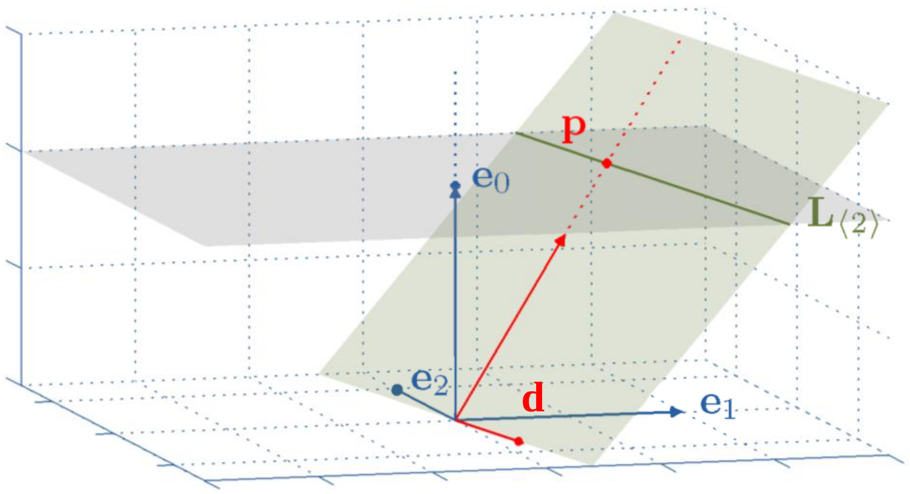
3-D Representational Space

2-Blades interpreted as lines

2-D Base Space



$$\mathbf{L}_{\langle 2 \rangle} = \mathbf{p} \wedge \mathbf{d}$$



3-D Representational Space

Building oriented *k*-flats

The coordinate representation of the homogeneous elements naturally embeds Plücker coordinates for line computation.

k-Flat from *k*+1 points

$$\mathbf{F}_{\langle k+1\rangle} = \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \cdots \wedge \mathbf{p}_{k+1}$$

k-Flat from support point and *k*-D direction

$$\mathbf{F}_{\langle k+1\rangle} = \mathbf{p} \wedge \mathbf{A}_{\langle k\rangle}$$

$$\mathbf{A}_{\langle k\rangle} \subset \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_d\right)$$

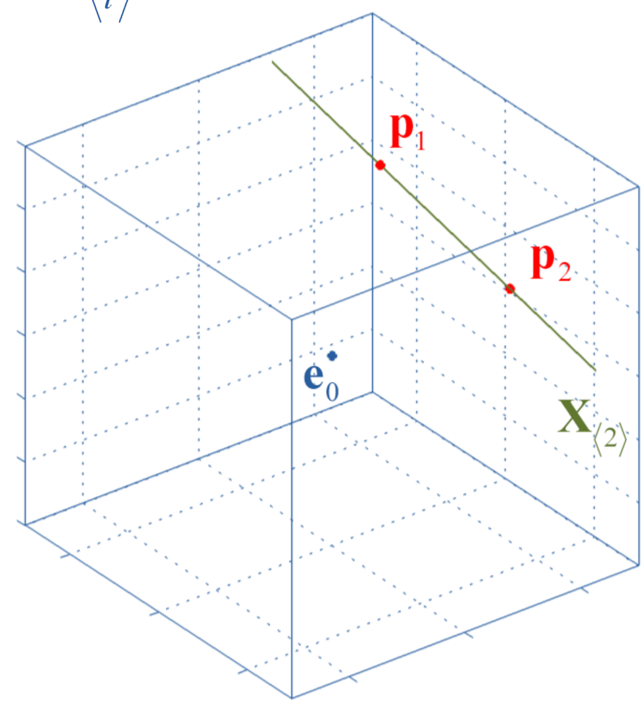
Hyperplane ((d-1)-flat) from unit normal and distance from the origin

$$\mathbf{H}_{\langle d\rangle} = \left(-\mathbf{n} + \delta\, \mathbf{e}_0^{-1}\right)^{-*}$$

$$\mathbf{n} \subset \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_d\right)$$

Parameters of $\mathbf{X}_{\langle t \rangle}$

	Finite Flat	Direction
Condition:	$\mathbf{e}_0 \rfloor \mathbf{X}_{\langle t \rangle} \neq 0$	$\mathbf{e}_0 \rfloor \mathbf{X}_{\langle t \rangle} = 0$
Direction $\mathbf{A}_{\langle t-1 \rangle}$:	$\mathbf{e}_0^{-1} \rfloor \mathbf{X}_{\langle t \rangle}$	$\mathbf{X}_{\langle t \rangle}$
Moment $\mathbf{M}_{\langle t \rangle}$:	$\mathbf{e}_0^{-1} \rfloor \left(\mathbf{e}_0 \wedge \mathbf{X}_{\langle t \rangle} \right)$	
Support vector $\mathbf{s}$:	$\mathbf{M}_{\langle t \rangle} / \mathbf{A}_{\langle t-1 \rangle}$	
Unit support point $\mathbf{e}_0 + \mathbf{s}$:	$\mathbf{X}_{\langle t \rangle} / \mathbf{A}_{\langle t-1 \rangle}$	



3-D Base Space

Applying a rotation around the origin to $\mathbf{X}_{\langle t \rangle}$

$$\boxed{R}\,\mathbf{X}_{\langle t \rangle}\;R^{-1}$$

A rotor in $\left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_d\right)$

The rotation formula applies to any blade (flat or direction).

It is the same for direct or dual blades.

$$\begin{aligned}
\mathbf{X}_{\langle t \rangle} = \left(\mathbf{e}_0 + \mathbf{s}\right) \wedge \mathbf{A}_{\langle t-1 \rangle} \mapsto \mathit{Rotation}\left[\mathbf{X}_{\langle t \rangle}\right] &= \left(\mathbf{e}_0 + R\,\mathbf{s}\,R^{-1}\right) \wedge \left(R\,\mathbf{A}_{\langle t-1 \rangle}\,R^{-1}\right) \\
&= \left(R\left(\mathbf{e}_0 + \mathbf{s}\right)R^{-1}\right) \wedge \left(R\,\mathbf{A}_{\langle t-1 \rangle}\,R^{-1}\right) \\
&= R\left(\left(\mathbf{e}_0 + \mathbf{s}\right) \wedge \left(\mathbf{A}_{\langle t-1 \rangle}\right)\right) R^{-1} \\
&= R\,\mathbf{X}_{\langle t \rangle}\,R^{-1}
\end{aligned}$$

Applying a translation to $\mathbf{X}_{\langle t \rangle}$

$$\mathbf{X}_{\langle t \rangle} + \boxed{\mathbf{t}} \wedge \left(\mathbf{e}_0^{-1} \rfloor \mathbf{X}_{\langle t \rangle} \right)$$

A translation vector in $\left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_d \right)$

$$\begin{aligned}
\mathbf{X}_{\langle t \rangle} = \left(\mathbf{e}_0 + \mathbf{s} \right) \wedge \mathbf{A}_{\langle t-1 \rangle} \mapsto \mathit{Translation}\left[\mathbf{X}_{\langle t \rangle} \right] &= \left(\mathbf{e}_0 + \left(\mathbf{s} + \mathbf{t} \right) \right) \wedge \mathbf{A}_{\langle t-1 \rangle} \\
&= \left(\left(\mathbf{e}_0 + \mathbf{s} \right) + \mathbf{t} \right) \wedge \mathbf{A}_{\langle t-1 \rangle} \\
&= \left(\mathbf{e}_0 + \mathbf{s} \right) \wedge \mathbf{A}_{\langle t-1 \rangle} + \mathbf{t} \wedge \mathbf{A}_{\langle t-1 \rangle} \\
&= \mathbf{X}_{\langle t \rangle} + \mathbf{t} \wedge \mathbf{A}_{\langle t-1 \rangle} \\
&= \mathbf{X}_{\langle t \rangle} + \mathbf{t} \wedge \left(\mathbf{e}_0^{-1} \rfloor \mathbf{X}_{\langle t \rangle} \right)
\end{aligned}$$

The translation formula applies to any blade (flat or direction).

For dual elements the formula is slightly different.

Applying a rigid body motion to $\mathbf{X}_{\langle t \rangle}$

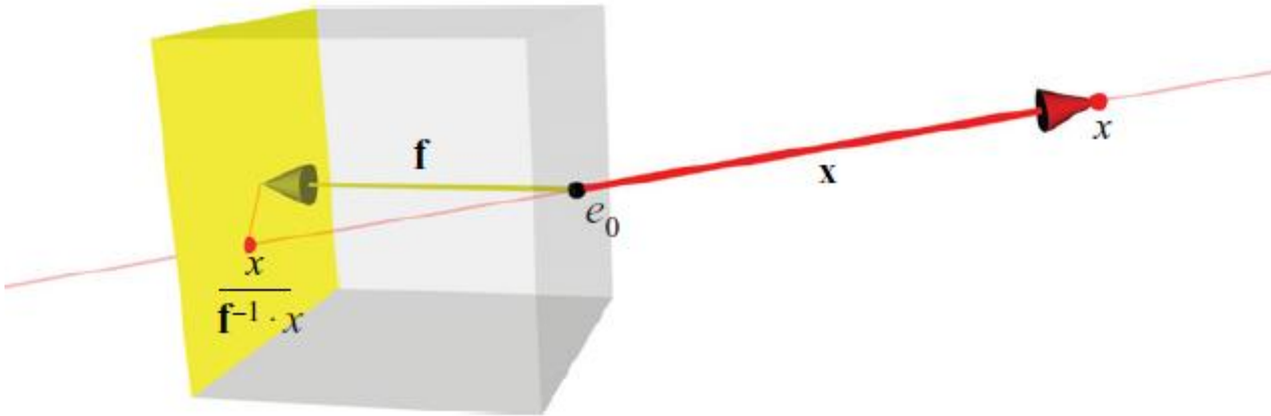
$$\boldsymbol{R}\,\mathbf{X}_{\langle t \rangle}\,\boldsymbol{R}^{-1} + \mathbf{t} \wedge \left(\mathbf{e}_0^{-1} \rfloor \left(\boldsymbol{R}\,\mathbf{X}_{\langle t \rangle}\,\boldsymbol{R}^{-1} \right) \right)$$

The most common way to characterizing
a rigid body motion is a rotation around the origin,
followed by a translation.

It also applies to
any blade (flat or direction).

For dual elements the formula
is slightly different.

Pinhole camera



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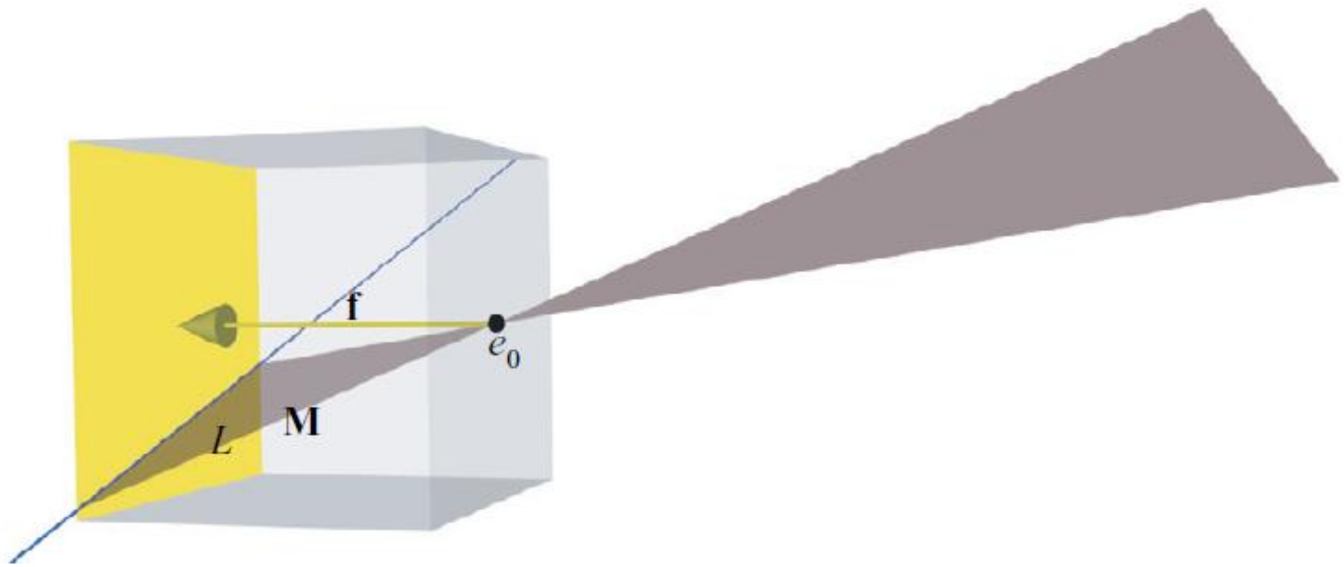
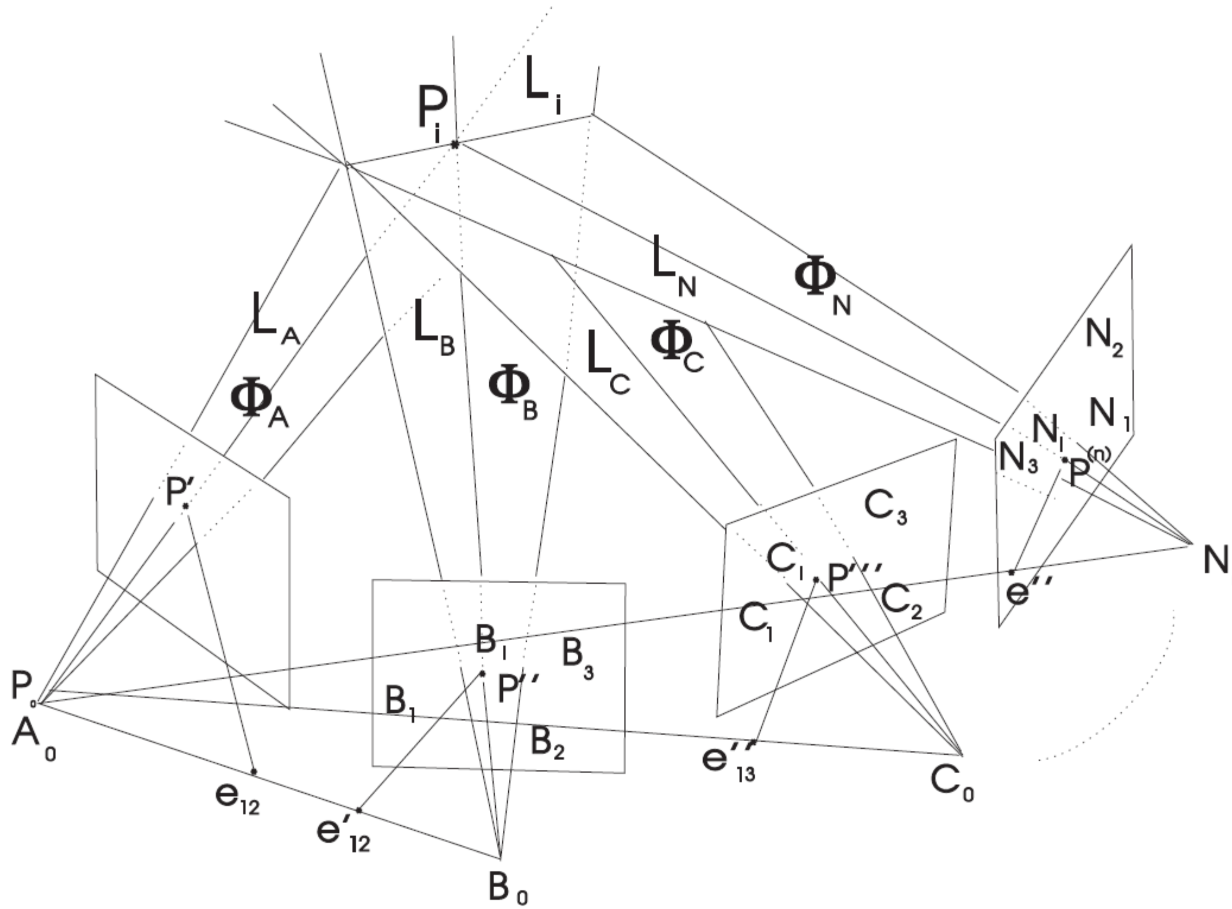
Pinhole camera



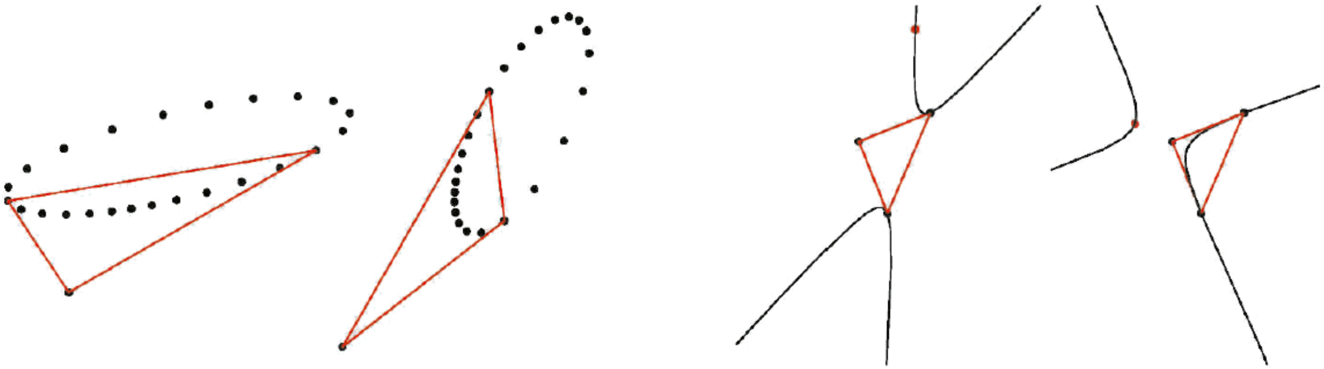
Adapted from L. Dorst, D. Fontijine, S. Mann. *Geometric algebra for computer science*. Morgan Kaufmann Publishers, 2007.

The analysis of the projective structure of uncalibrated cameras



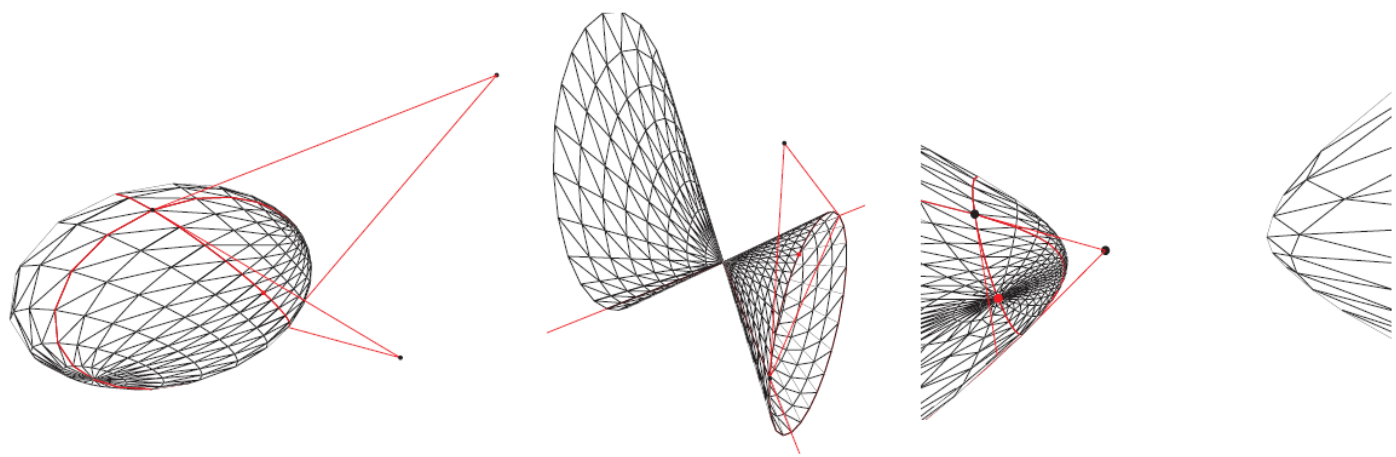
E. Bayro-Corrochano, *et al.* (1996) Geometric algebra: a framework for computing point and line correspondences and projective…, in Proc. of the 13th ICPR, pp. 334–338.

Automatic tessellation of quadric surfaces



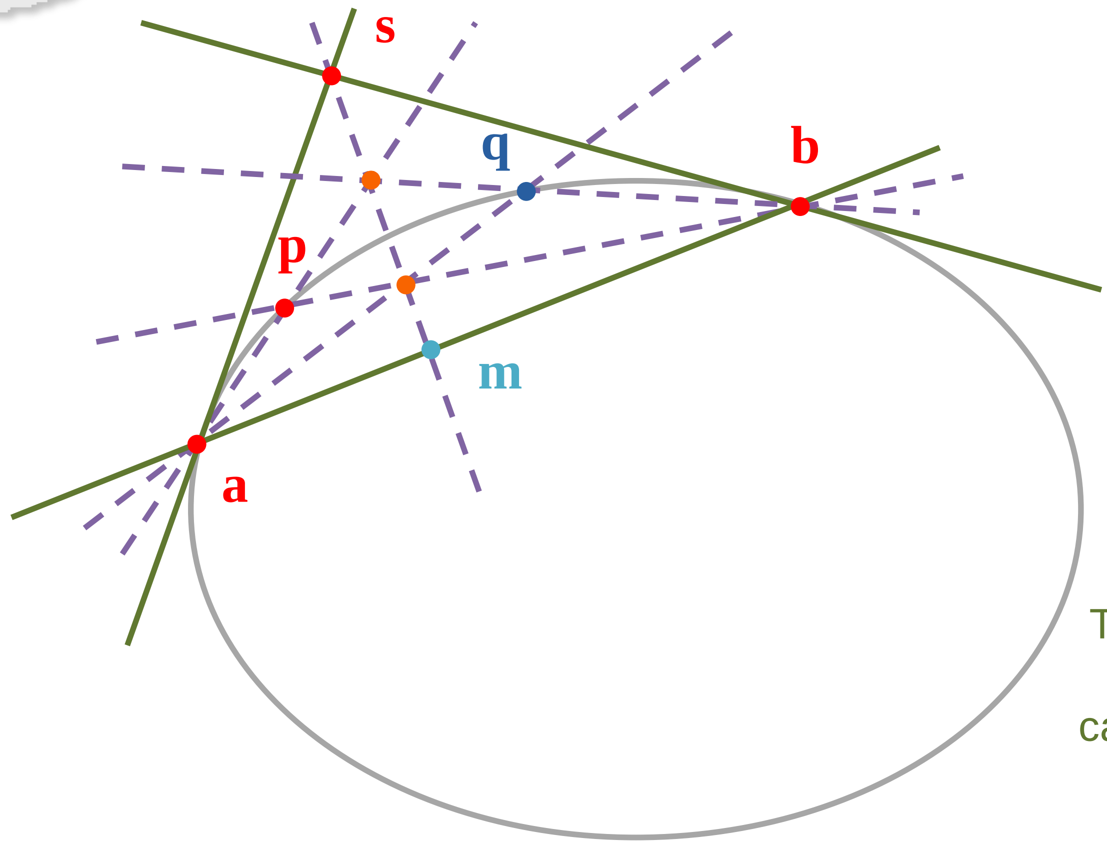
2-D

3-D



F. Jourdan *et al.* (2004), Automatic tessellation of quadric surfaces using Grassmann-Cayley algebra, in Proc. Int. Conf. Comput. Vis. Graph., pp. 674–682.

How to draw a conic section



The geometric meaning of each step can be written directly as algebraic equations.

$$q = \Big(b \wedge \big((a \wedge p) \vee (m \wedge s)\big)\Big) \vee \Big(a \wedge \big((b \wedge p) \vee (m \wedge s)\big)\Big)$$