Introduction to Geometric Algebra

Lecture VI

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CNPq *Conselho Nacional de Desenvolvimento Científico e Tecnológico*

Lecture VI

Checkpoint

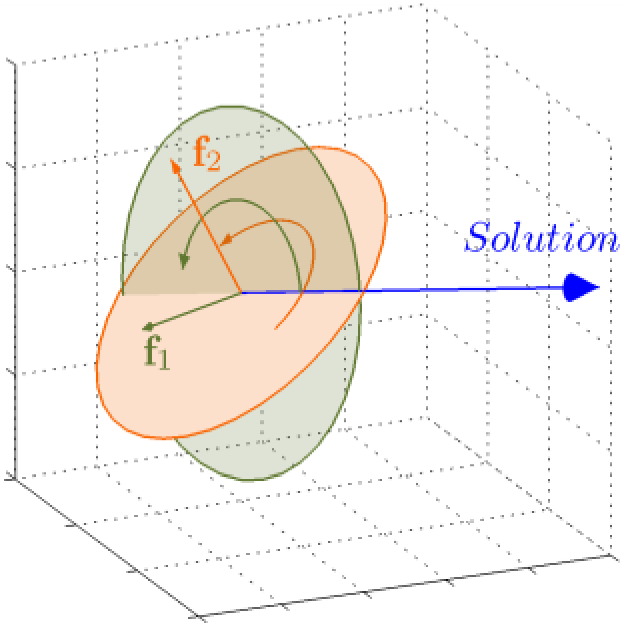
Checkpoint

- Euclidean vector space model of geometry
  - Euclidean metric
  - Blades are Euclidean subspaces
  - Versors encode reflections and rotations

Checkpoint

- Solving homogeneous systems of linear equations

Each equation of the system is the
dual of and hyperplane
that passes through the origin.



Checkpoint

- Rotation rotors as the exponential of 2-blades

$$\boldsymbol{R} = \exp\left(-\frac{\phi}{2}\mathbf{B}_{\langle 2 \rangle}\right) = \cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\mathbf{B}_{\langle 2 \rangle}$$

- The logarithm of rotors in 3-D vector space

$$\log\left(\boldsymbol{R}\right) = \frac{\left\langle \boldsymbol{R} \right\rangle_2}{\left\| \left\langle \boldsymbol{R} \right\rangle_2 \right\|} \tan^{-1}\left(\frac{\left\| \left\langle \boldsymbol{R} \right\rangle_2 \right\|}{\left\| \left\langle \boldsymbol{R} \right\rangle_0 \right\|}\right)$$

Checkpoint

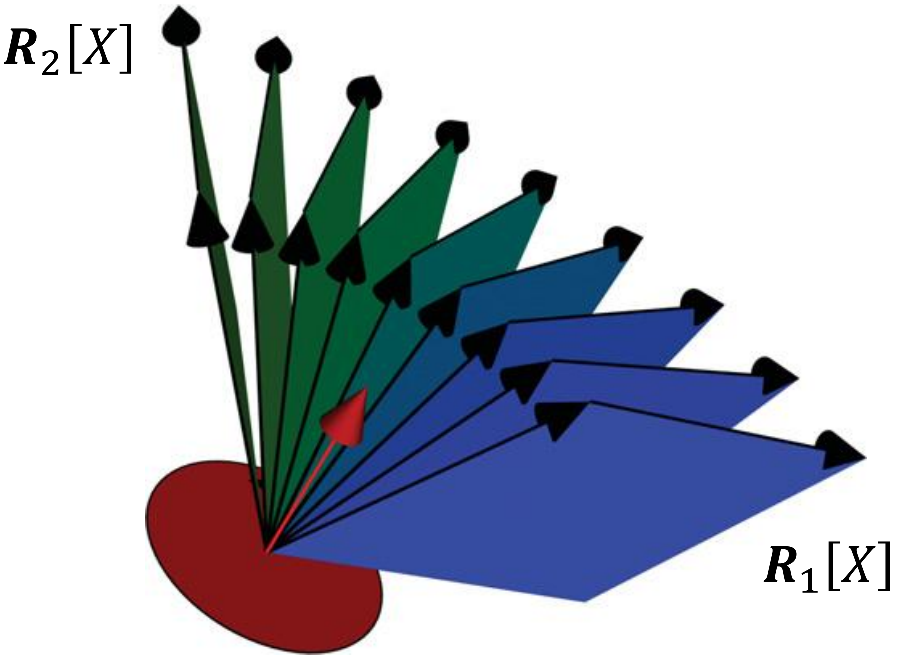
- Rotation interpolation

$$\boldsymbol{R} = \frac{\boldsymbol{R}_2}{\boldsymbol{R}_1}$$

Rotor to be interpolated

$$\boldsymbol{S} = \exp\left(\frac{\log(\boldsymbol{R})}{n}\right)$$

Rotation step (it is applied *n* times)

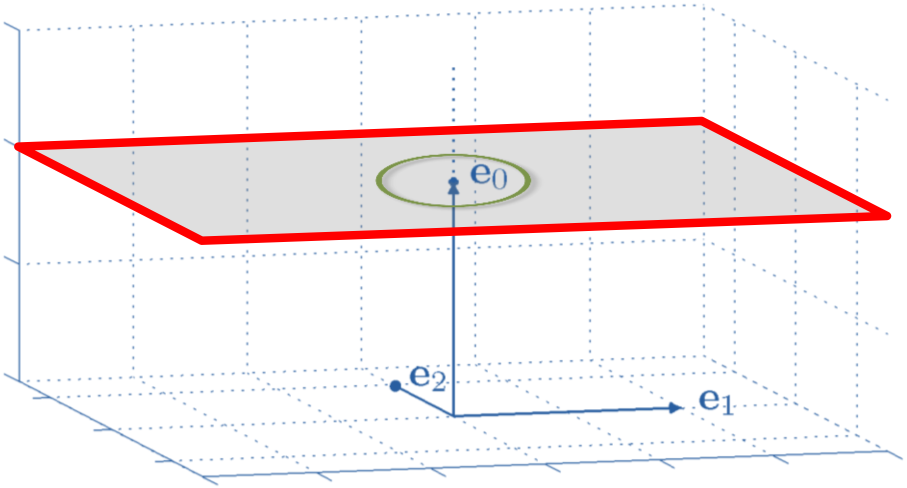


Adapted from L. Dorst, D. Fontijine, S. Mann. *Geometric algebra for computer science*. Morgan Kaufmann Publishers, 2007.

Checkpoint

- Homogeneous model of geometry
  - Euclidean metric
  - d-D base space, $(d+1)$-D representational space

The extra basis vector is interpreted as point at origin.

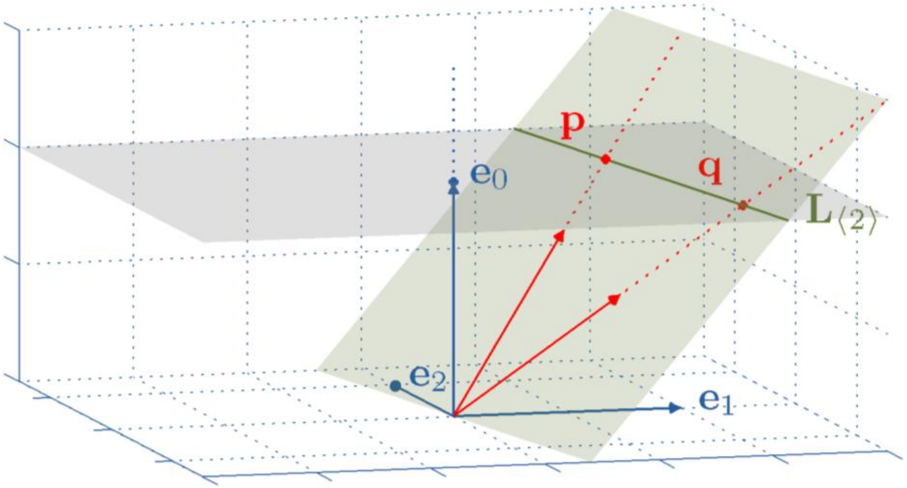


3-D Representational Space

Checkpoint

- Homogeneous model of geometry
  - Euclidean metric
  - d-D base space, $(d+1)$-D representational space
  - Blades are oriented flats or directions

$$\mathbf{F}_{\langle k+1 \rangle} = \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \cdots \wedge \mathbf{p}_{k+1}$$



3-D Representational Space

Checkpoint

- Homogeneous model of geometry
  - Euclidean metric
  - d-D base space, $(d+1)$-D representational space
  - Blades are flats or directions
  - Rotors encode rotations around the origin

$$\boldsymbol{R}\,\mathbf{X}_{\langle t \rangle}\,\boldsymbol{R}^{-1}$$

The rotation formula applies to
any blade (flat or direction).

It is the same for direct or dual blades.

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Checkpoint

- Homogeneous model of geometry
    - Euclidean metric
    - d-D base space, $(d+1)$-D representational space
    - Blades are flats or directions
    - Rotors encode rotations around the origin
    - Translation formula

The translation formula applies to any blade (flat or direction).

For dual elements the formula is slightly different.

$$\mathbf{X}_{\langle t\rangle} + \mathbf{t} \wedge \left(\mathbf{e}_0^{-1} \rfloor \mathbf{X}_{\langle t\rangle}\right)$$

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Checkpoint

- Homogeneous model of geometry
  - Euclidean metric
  - d-D base space, $(d+1)$-D representational space
  - Blades are flats or directions
  - Rotors encode rotations around the origin
  - Translation formula
  - Rigid body motion formula

It also applies to
any blade (flat or direction).

For dual elements the formula
is slightly different.

$$\boldsymbol{R}\,\mathbf{X}_{\langle t\rangle}\,\boldsymbol{R}^{-1}+\mathbf{t}\wedge\left(\mathbf{e}_0^{-1}\rfloor\left(\boldsymbol{R}\,\mathbf{X}_{\langle t\rangle}\,\boldsymbol{R}^{-1}\right)\right)$$

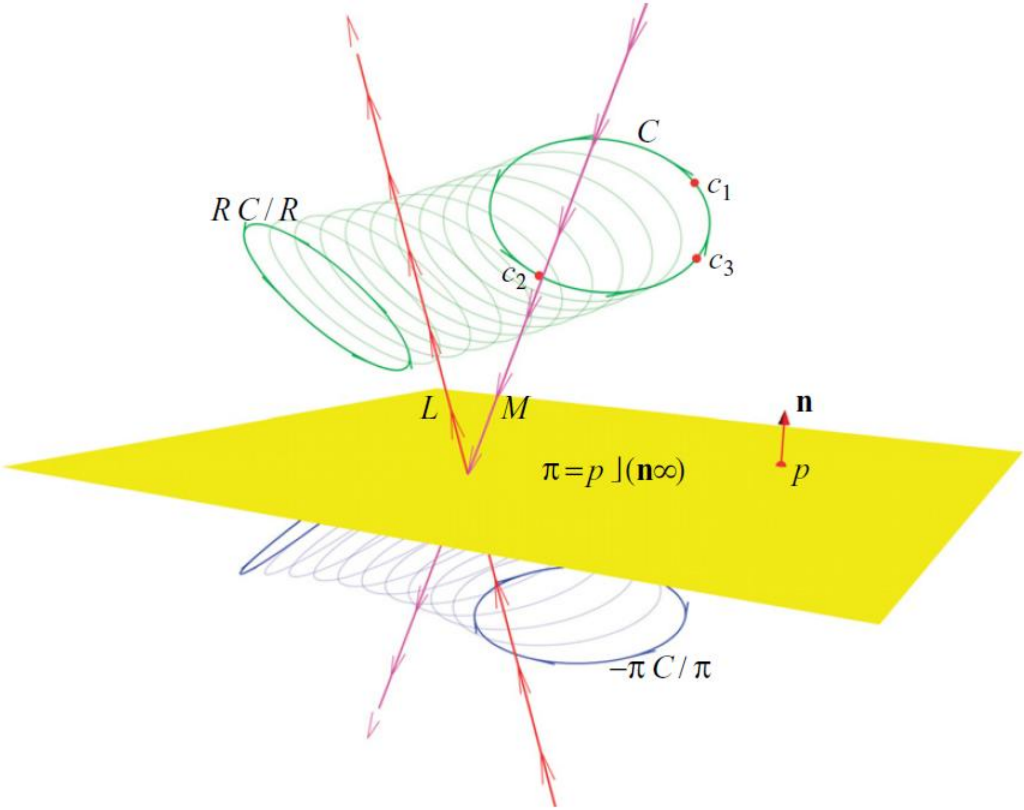
Today

- Lecture VI – Fri, January 22
  - Conformal model
  - Concluding remarks

Lecture VI

Conformal Model of Geometry

Motivational example



Adapted from L. Dorst, D. Fontijine, S. Mann. *Geometric algebra for computer science*. Morgan Kaufmann Publishers, 2007.

1. Create the circle through points $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$

$$C = c_1 \wedge c_2 \wedge c_3$$

2. Create a straight line **L**

$$L = a_1 \wedge a_2 \wedge \infty$$

3. Rotate the circle around the line and show **n** rotation steps

$$R = \exp(\phi\, L^*/2)$$

$$R^{1/N}\, C /\, R^{1/N}$$

4. Create a plane through point **p** and with normal vector **n**

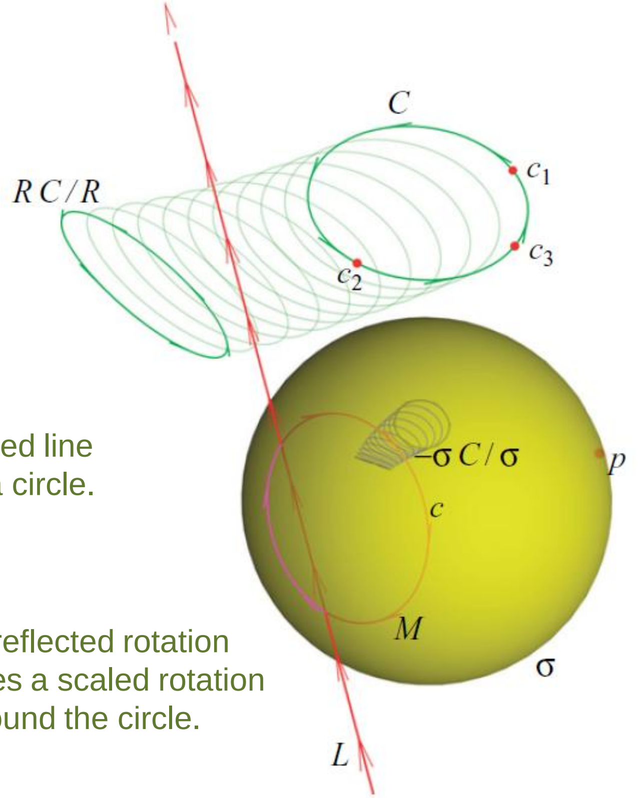
$$\pi = p\rfloor(\mathbf{n}\infty)$$

Dual plane

5. Reflect the whole situation with the line and the circlers in the plane

$$X \mapsto \;\; -\pi\, X/\pi$$

Motivational example



The reflected line becomes a circle.

The reflected rotation becomes a scaled rotation around the circle.

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1. Create the circle through points $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$

$$C = c_1 \wedge c_2 \wedge c_3$$

2. Create a straight line **L**

$$L = a_1 \wedge a_2 \wedge \infty$$

3. Rotate the circle around the line and show **n** rotation steps

$$R = \exp(\phi\, L^*/2)$$
$$R^{1/N}\, C / R^{1/N}$$

4. Create a sphere through point **p** and with center **c**

$$\sigma = p \rfloor (c \wedge \infty)$$

Dual sphere

The only thing that is different is that the plane was changed by the sphere.

5. Reflect the whole situation with the line and the circlers in the sphere

$$X \mapsto -\sigma X/\sigma$$

Points in a Euclidean space

- A Euclidean space has points at a well-defined distance from each other

$$d_E^2\left(\mathcal{P},\mathcal{Q}\right)=\left(\mathbf{p}-\mathbf{q}\right)^2=\left(\mathbf{p}-\mathbf{q}\right)\cdot\left(\mathbf{p}-\mathbf{q}\right)$$

- Euclidean spaces do not really have an origin
- It is convenient to close a Euclidean space by augmenting it with a point at infinity

The point at infinity is:

- The only point at infinity
- A point in common to all flats
- Invariant under the Euclidean transformations

Points in a Euclidean space

- A Euclidean space has points at a well-defined distance from each other

$$d_E^2(\mathcal{P},\mathcal{Q}) = (\mathbf{p}-\mathbf{q})^2 = (\mathbf{p}-\mathbf{q})\cdot(\mathbf{p}-\mathbf{q})$$

- Euclidean spaces do not really have an origin

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finity

The point at infinity is:

ly point at infinity

t in common to all flats

- Invariant under the Euclidean transformations

In the conformal model of geometry
these properties are
central because such model is designed
for Euclidean geometry.

Base space and representational space

- The arbitrary origin is achieved by assign an extra dimension to the d-dimensional base space
- The point at infinity is another extra dimension assigned to the d-dimensional base space

d-dimensional base space

$$\left\{\mathbf{o}, \boxed{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_d}, \infty\right\}$$

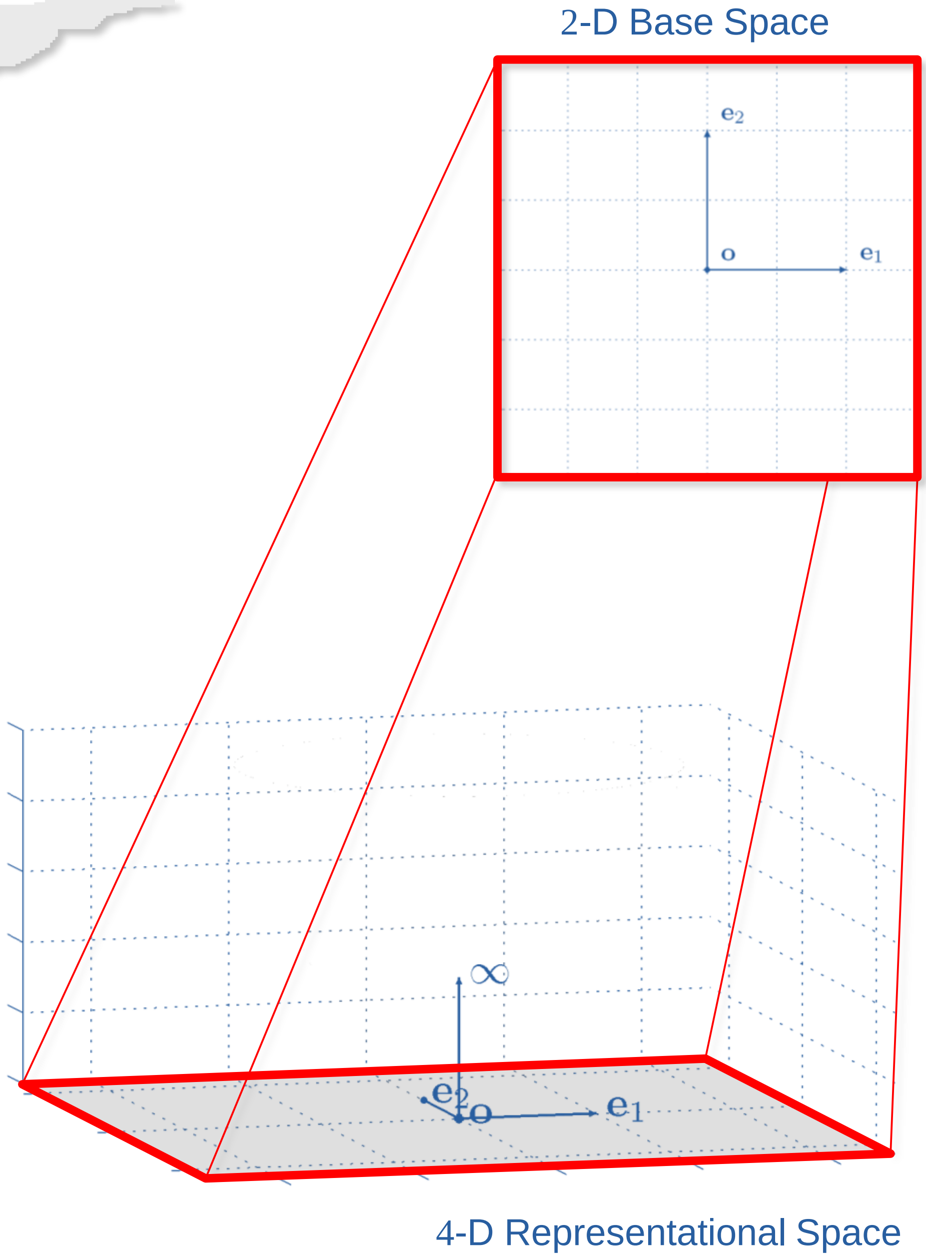
Point at origin

Point at infinity

$(d + 2)$-dimensional representational space

Example

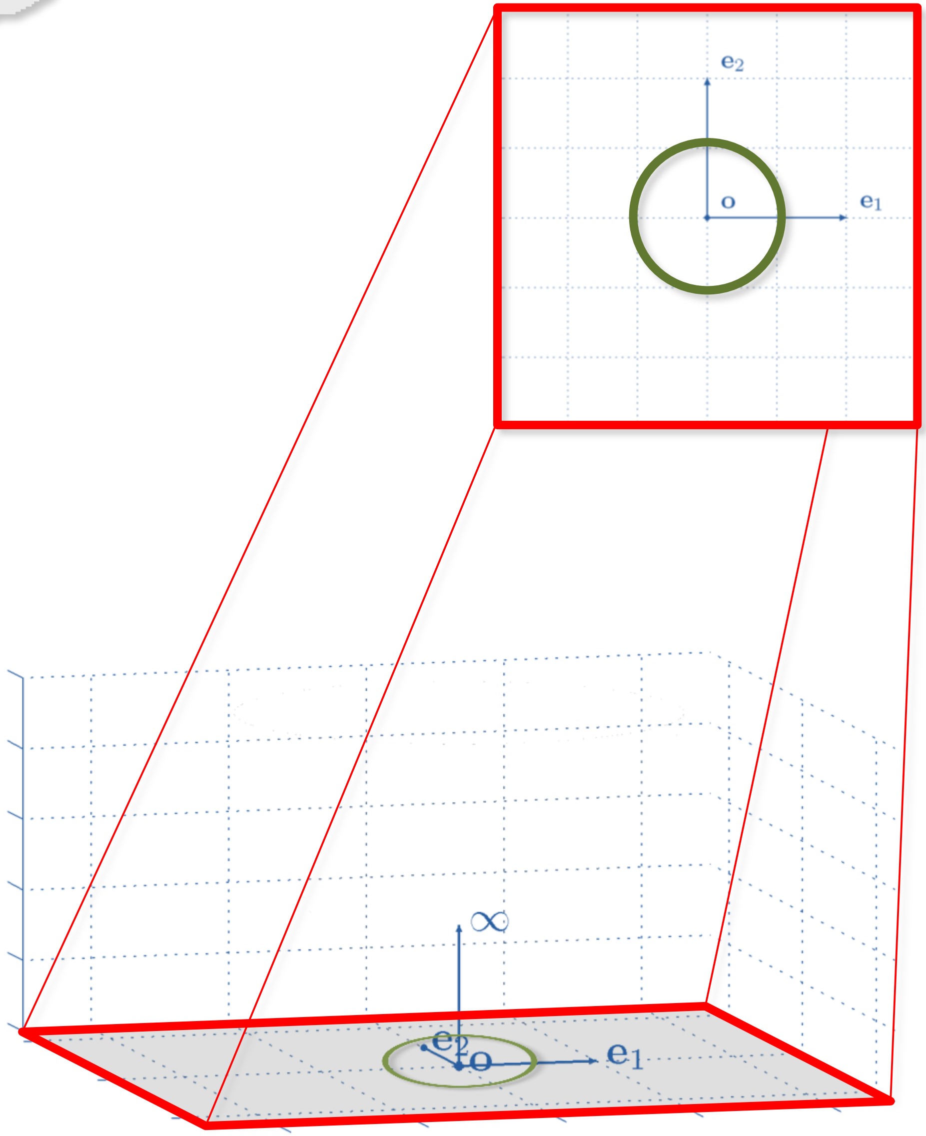
Here, the 4-D representational space is seem as homogeneous coordinates, where the **o** coordinate is set to one.



Example

Basis vector interpreted
as point at origin.

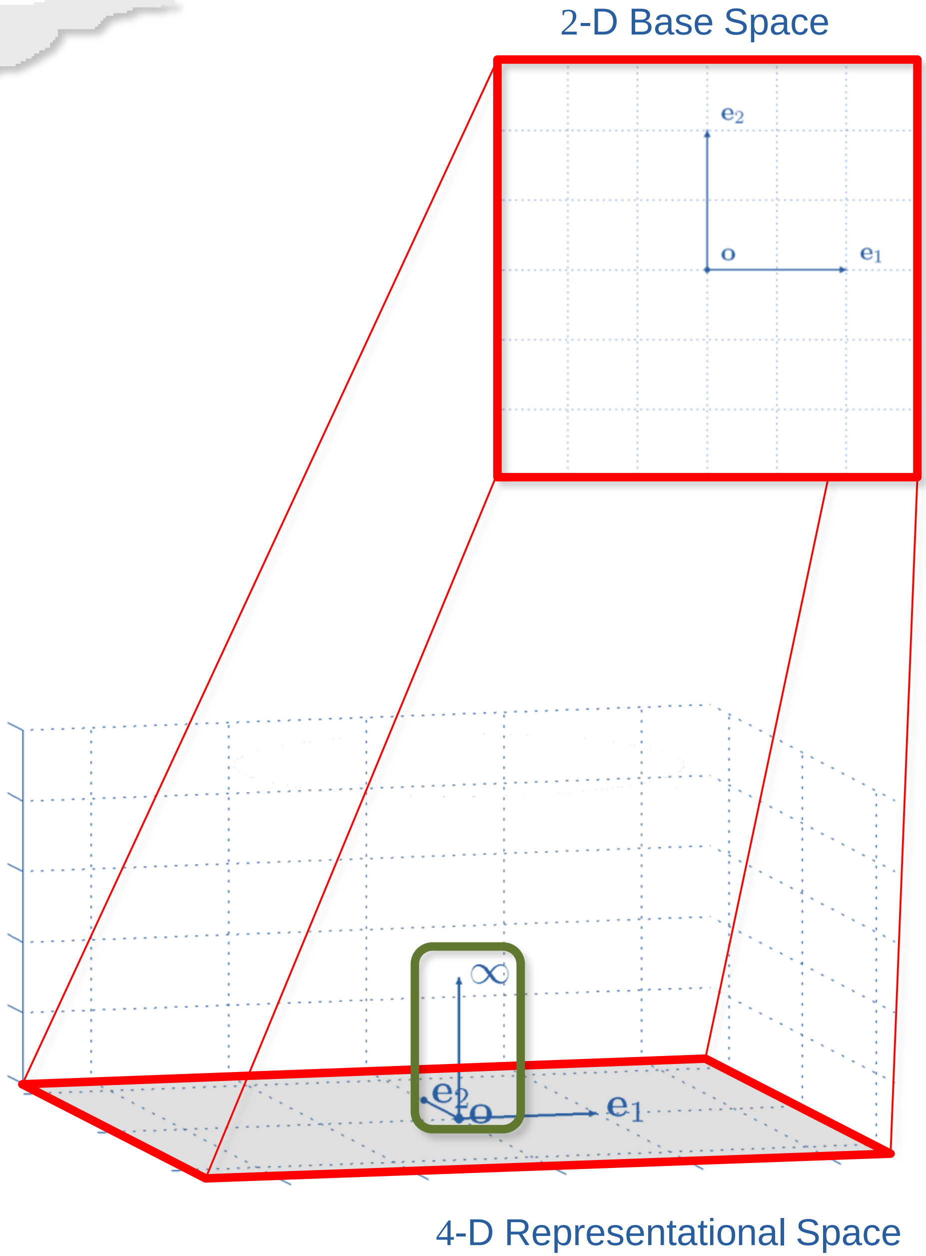
2-D Base Space



4-D Representational Space

Example

Basis vector interpreted
as point at infinity.



Euclidean points as null vectors

- Euclidean points in the base space are vectors in the representational space
- The inner product of such vectors is directly proportional to the square distance of the points

$$\mathbf{p} \cdot \mathbf{q} \boxed{\equiv} -\frac{1}{2} d_E^2 \left(\mathcal{P}, \mathcal{Q} \right)$$

For a unit finite point
and the point at infinity, $\mathbf{p} \cdot \infty = -1$.

Here, $\mathbf{p}$ and $\mathbf{q}$ are vectors in the representational space. They encode unit finite points $\mathcal{P}$ and $\mathcal{Q}$, respectively.

We know that $d_E^2 \left(\mathcal{P}, \mathcal{P} \right) = 0$.
As a consequence, $\mathbf{p} \cdot \mathbf{p} = 0$.

Non-Euclidean metric matrix

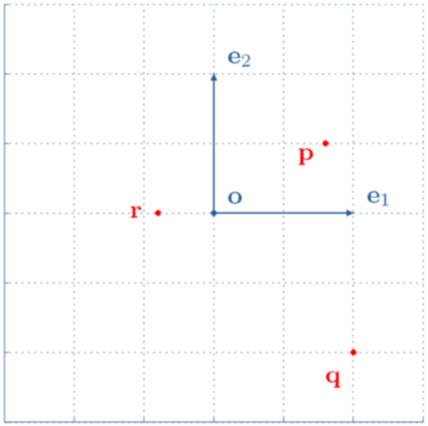
$$M = \begin{array}{cc} \begin{array}{cccccc} \mathbf{o} & \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_d & \boldsymbol{\infty} \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} & \begin{array}{c} \mathbf{o} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_d \\ \boldsymbol{\infty} \end{array} \end{array}$$

Unit finite point

$$\mathbf{u} = \mathbf{o} + \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \cdots + \alpha_d \mathbf{e}_d + \frac{1}{2}\left(\sum_{i=1}^{d} \alpha_i^2 \right) \boldsymbol{\infty}$$

Finite points

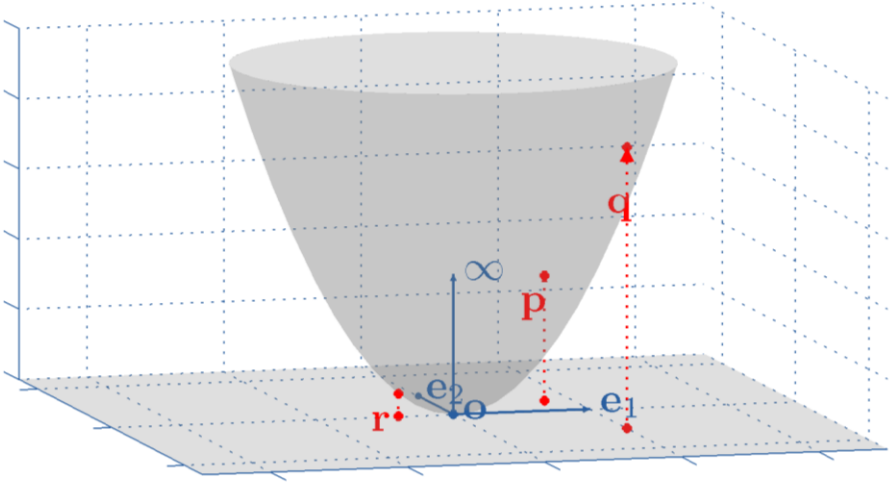
2-D Base Space



General finite point

$$\mathbf{p} = \gamma\left(\mathbf{o} + \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \cdots + \alpha_d\mathbf{e}_d + \frac{1}{2}\left(\sum_{i=1}^{d}\alpha_i^2\right)\boldsymbol{\infty}\right)$$

Euclidean points define
a paraboloid in the ∞-direction



4-D Representational Space

Lecture IV

Conformal Primitives

Conformal primitives

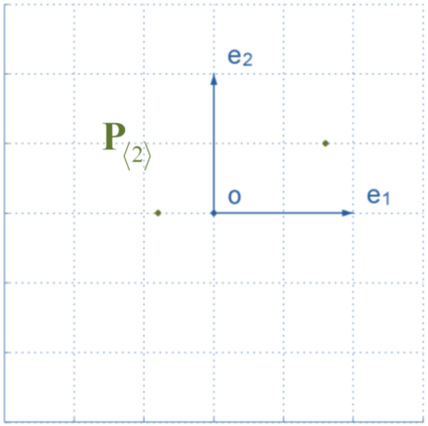
- Oriented rounds
    - Point pair, circle, sphere, etc.
- Oriented flats
    - Straight line, plane, etc.
- Frees
    - Directions
- Tangents
    - Directions tangent to a round at a given location

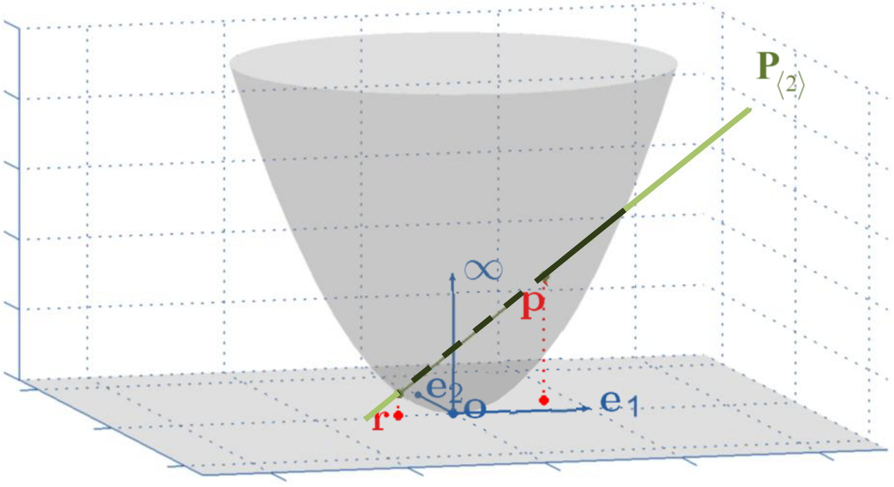
Oriented rounds

- They are built as the outer product of finite points
- Examples
    - Point pair (0-sphere)

$$\mathbf{P}_{\langle 2 \rangle} = \mathbf{p} \wedge \mathbf{r}$$

2-D Base Space





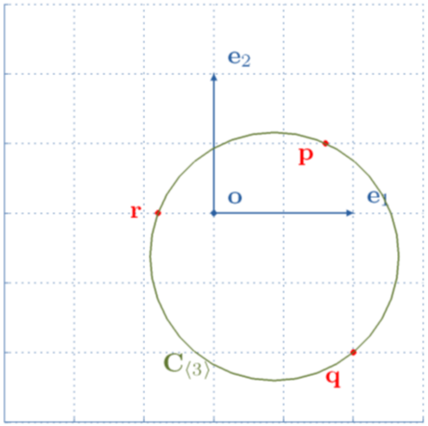
4-D Representational Space

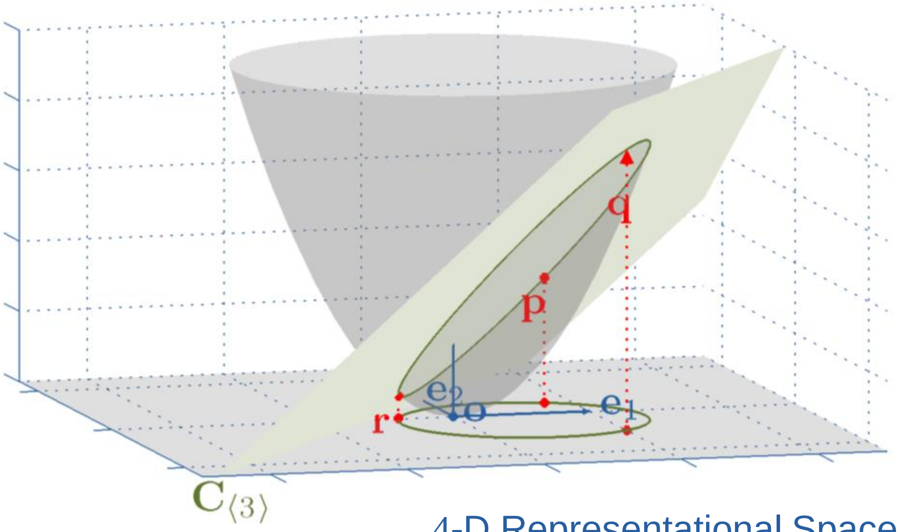
Oriented rounds

- They are built as the outer product of finite points
- Examples
    - Point pair (0-sphere)
    - Circle (1-sphere)

$$C_{\langle 3 \rangle} = \mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r}$$

2-D Base Space





4-D Representational Space

Oriented rounds

- They are built as the outer product of finite points

- Examples
    - Point pair (0-sphere)
    - Circle (1-sphere)
    - Sphere (2-sphere)
    - etc.

k-Sphere from k+2 finite points

$$\mathbf{S}_{\langle k+2\rangle} = \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \cdots \wedge \mathbf{p}_{k+2}$$

k-Sphere with center point $\mathbf{c}$, radius $\boldsymbol{\rho}$,
and the direction of the carrier flat

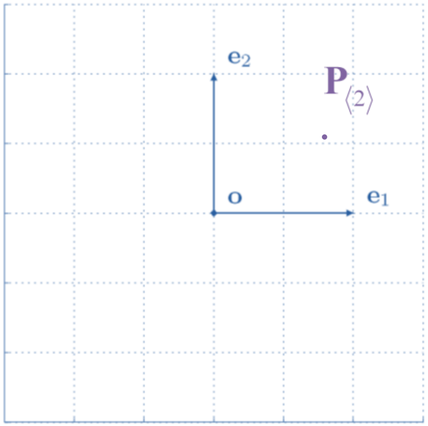
$$\mathbf{S}_{\langle k+2\rangle} = \left(\mathbf{c} + \frac{1}{2}\rho^2 \boldsymbol{\infty}\right) \wedge \left(-\mathbf{c} \rfloor \left(\hat{\mathbf{A}}_{\langle k\rangle} \boldsymbol{\infty}\right)\right)$$

(d-1)-Sphere around $\mathbf{c}$ through $\mathbf{p}$

$$\mathbf{S}_{\langle d+1\rangle} = \mathbf{p} \wedge \left(\mathbf{c} \wedge \boldsymbol{\infty}\right)^{-*}$$

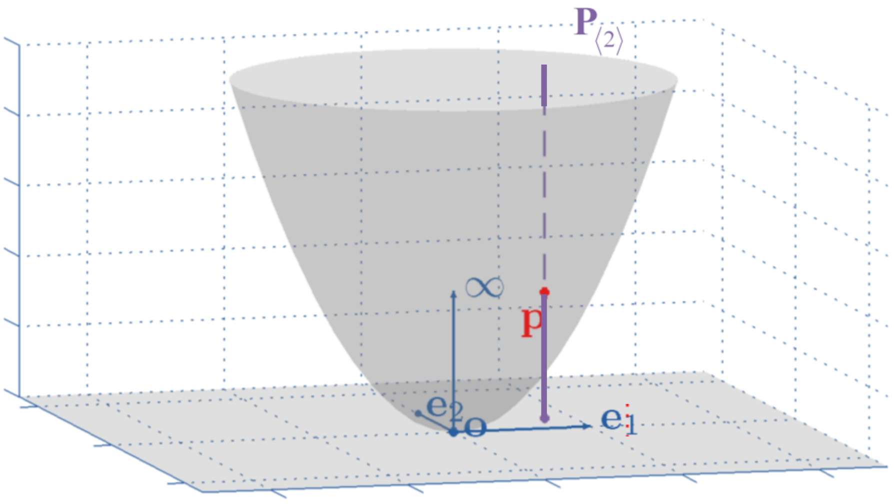
Oriented flats

2-D Base Space



- They are built as the outer product of finite points and the point at infinity
- Examples
  - Flat point (0-flat)

$$\mathbf{P}_{\langle 2\rangle} = \mathbf{p} \wedge \infty$$

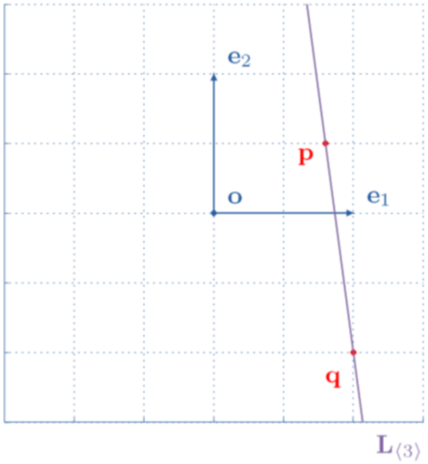


4-D Representational Space

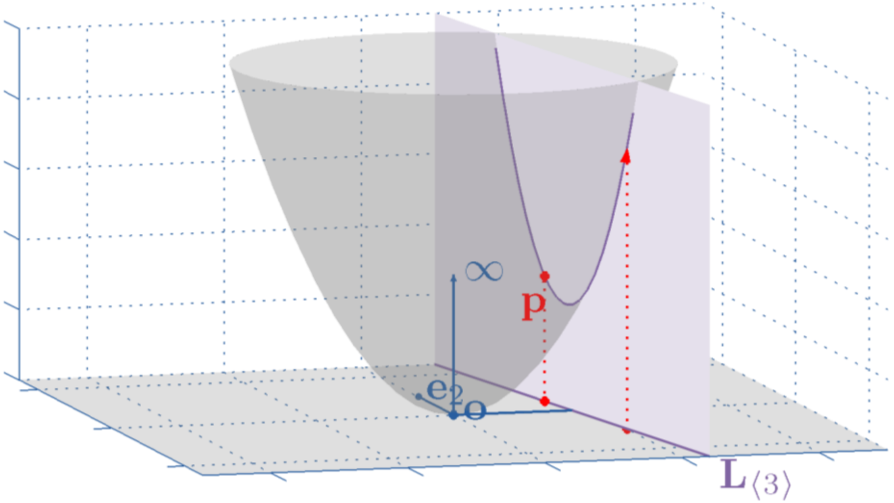
Oriented flats

- They are built as the outer product of finite points and the point at infinity
- Examples
    - Flat point (0-flat)
    - Straigh line (1-flat)

2-D Base Space



$$\mathbf{L}_{\langle 3 \rangle} = \mathbf{p} \wedge \mathbf{q} \wedge \infty$$



4-D Representational Space

Oriented flats

- They are built as the outer product of finite points and the point at infinity
- Examples
  - Flat point (0-flat)
  - Straigh line (1-flat)
  - Plane (2-flat)
  - etc.

k-Flat from *k*+1 finite points

$$\mathbf{F}_{\langle k+2 \rangle} = \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \cdots \wedge \mathbf{p}_{k+1} \wedge \infty$$

k-Flat from support point and *k*–D direction

$$\mathbf{F}_{\langle k+2 \rangle} = \mathbf{p} \wedge \mathbf{A}_{\langle k \rangle} \wedge \infty$$

Hyperplane from unit normal and distance from the origin

$$\mathbf{H}_{\langle d+1 \rangle} = \left(\mathbf{n} + \delta \infty \right)^{-*}$$

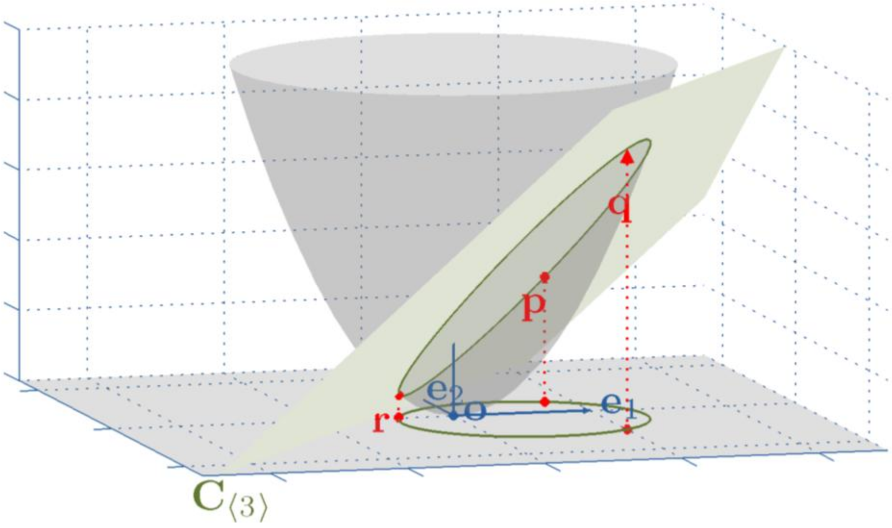
Mid-hyperplane between unit $\mathbf{p}$ and $\mathbf{q}$

$$\mathbf{H}_{\langle d+1 \rangle} = \left(\mathbf{p} + \mathbf{q} \right)^{-*}$$

Hyperplane with normal $\mathbf{n}$, through $\mathbf{p}$

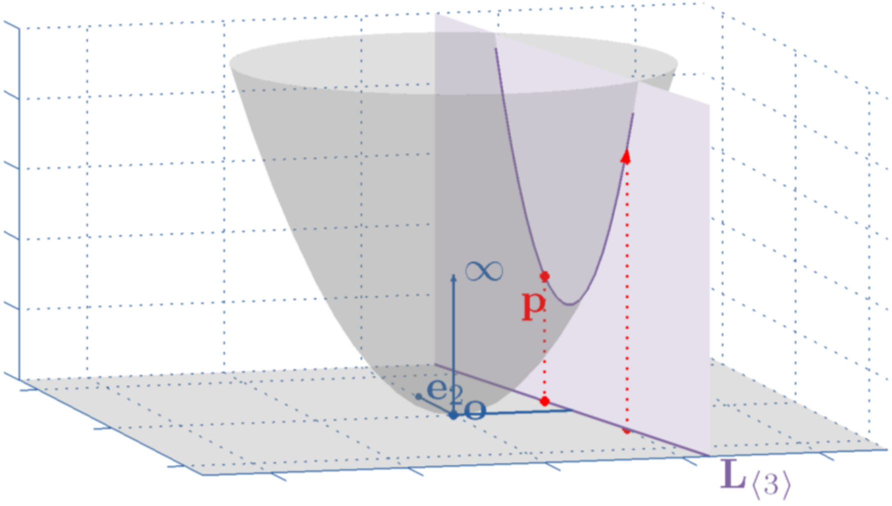
$$\mathbf{H}_{\langle d+1 \rangle} = \mathbf{p} \wedge \left(\mathbf{n} \wedge \infty \right)^{-*}$$

Flats are rounds with infinite radius



$$\mathbf{C}_{\langle 3\rangle} = \mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r}$$

$$\mathbf{L}_{\langle 3\rangle} = \mathbf{p} \wedge \mathbf{q} \wedge \infty$$



Frees

- A free element is interpreted as a direction
- A free is built as the outer product of vectors in the base space and the point at infinity

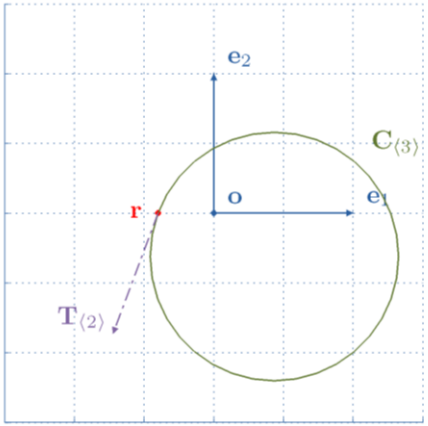
$$\mathbf{D}_{\langle k+1 \rangle} = \mathbf{A}_{\langle k \rangle} \wedge \infty$$

where $\mathbf{A}_{\langle k \rangle} \subseteq \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_d \right)$

They are invariant to translation because they are perpendicular to the assumed origin vector.

Tangents

2-D Base Space



- They are subspaces tanget to the paraboloid defined by the finite points

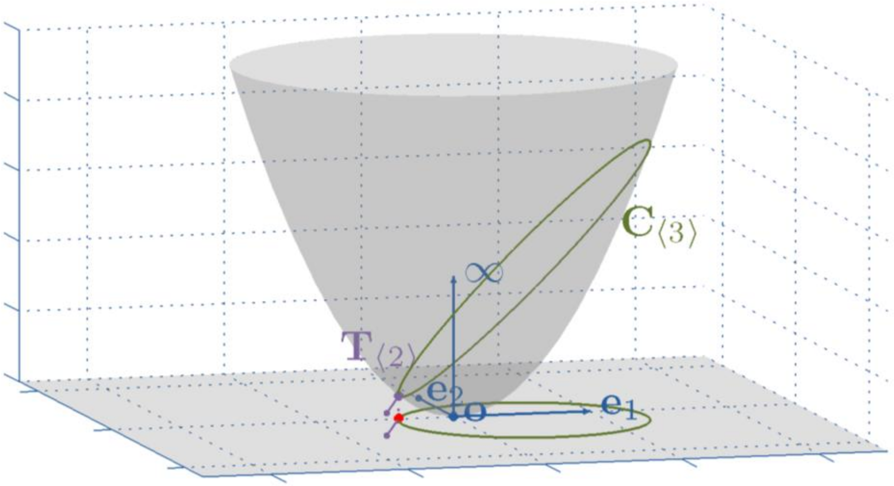
- Point-like interpretation and also direction information

Tangent to a round at the point $\mathbf{p}$

$$\mathbf{T}_{\langle k-1\rangle} = \mathbf{p} \rfloor \widehat{\mathbf{X}}_{\langle k\rangle}$$

Tangent at $\mathbf{p}$ with a given direction

$$\mathbf{T}_{\langle k-1\rangle} = \mathbf{p} \wedge \left(-\mathbf{p} \rfloor \left(\widehat{\mathbf{A}}_{\langle k\rangle} \wedge \infty \right) \right)$$



4-D Representational Space

Lecture VI

Universal Orthogonal Transformations

Euclidean transformations as versors

- Euclidean transfromations preserve the point at infinity, i.e.,

$$\hat{V}\,\infty\,V^{-1}=\infty$$

- The condition on a versor to be Euclidean is

$$\infty\,\rfloor V=0$$

- The simplest and most general Euclidean versor is

$$\mathbf{h}=\mathbf{n}+\delta\,\infty$$

This vector is a dual hyperplane.
As an 1-versor it encodes a reflection.

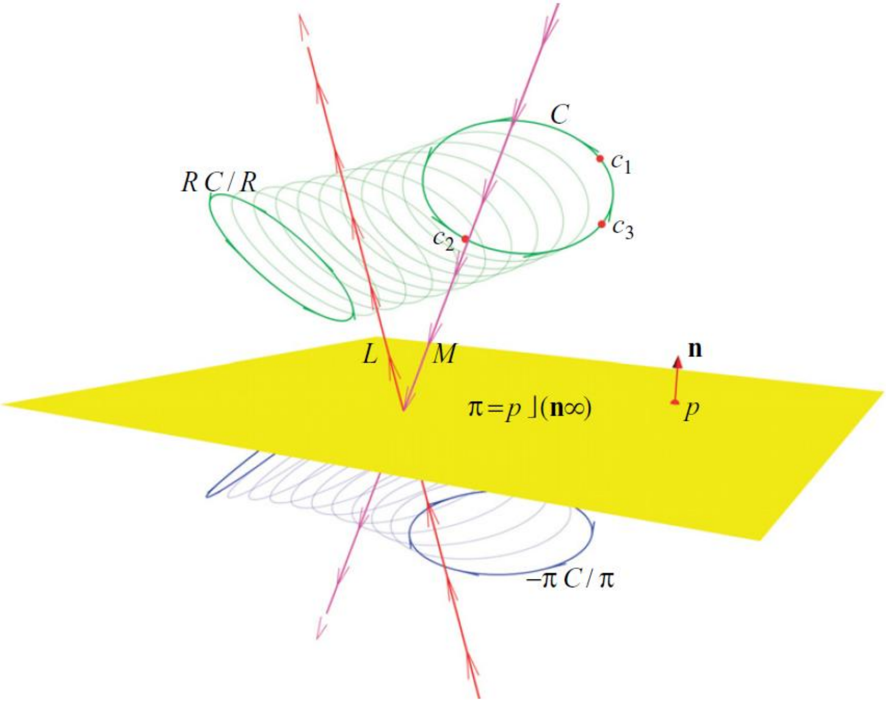
Reflection versor

- The dual of hyperplanes and hyperspheres act as mirrors

$$\mathbf{h} = \mathbf{H}^{*}_{\langle d+1 \rangle}$$

$$X' = \mathbf{h}\,\widehat{X}\,\mathbf{h}^{-1}$$

All Euclidean transformations can be made by multiple reflections in well-chosen planes.



Adapted from L. Dorst, D. Fontijine, S. Mann. *Geometric algebra for computer science*. Morgan Kaufmann Publishers, 2007.

Translation rotor

- The double reflection on two paralell planes with same orientation make a translation

Translation vector

Using the dual of the planes as mirrors:

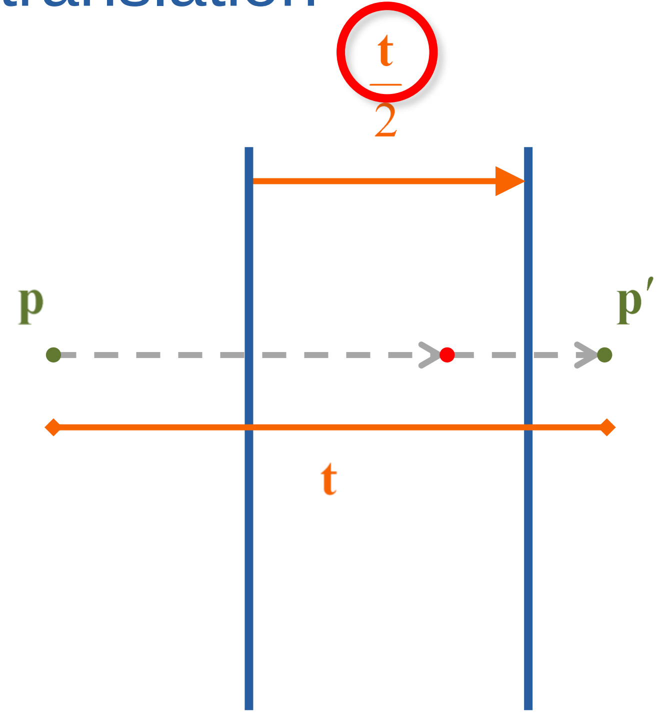
$$\left(\mathbf{n}+\delta_2\,\infty\right)\left(\mathbf{n}+\delta_1\,\infty\right)=1-\left(\delta_2-\delta_1\right)\mathbf{n}\,\infty$$

$$=1-\frac{1}{2}\mathbf{t}\,\infty$$

$$\equiv T$$

Unit normal vector
in base space

$$\text{where } \mathbf{t}=2\left(\delta_2-\delta_1\right)\mathbf{n}$$



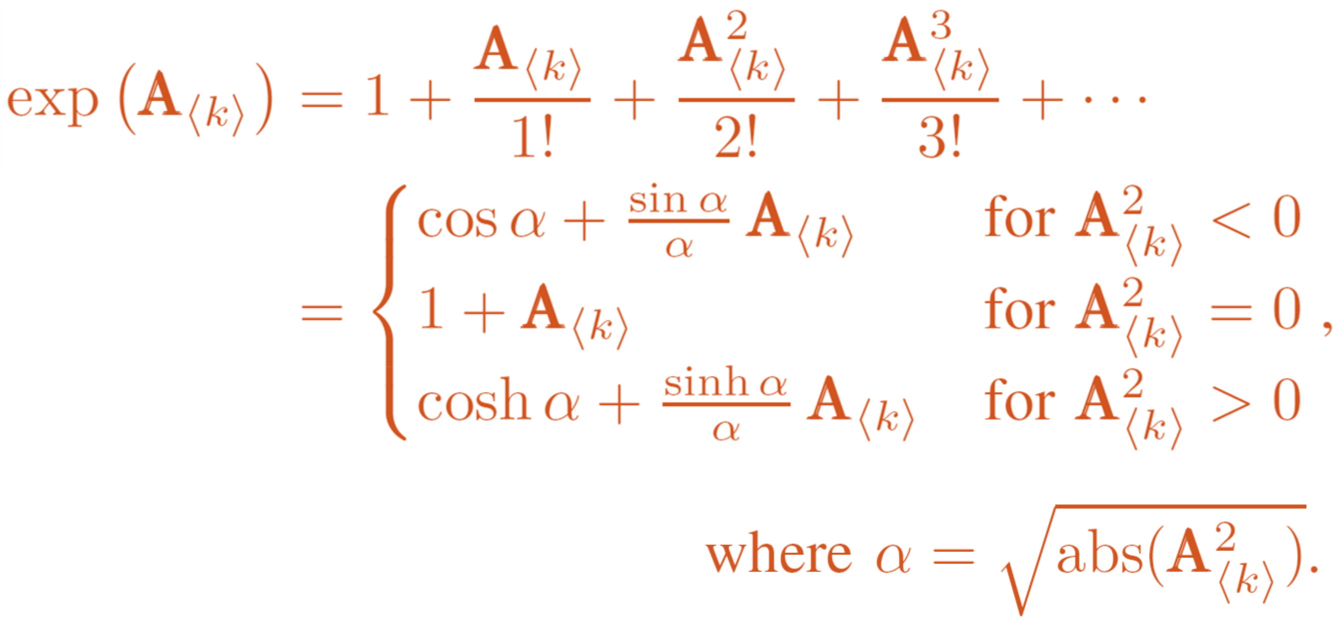
Translation rotor

- The double reflection on two paralell planes with sam

Using the dual

$(\mathbf{n}+\delta_2\,\infty)(\mathbf{n}+$

The exponential of *k*-blades for arbitrary metric spaces



$$\exp\left(\mathbf{A}_{\langle k\rangle}\right) = 1 + \frac{\mathbf{A}_{\langle k\rangle}}{1!} + \frac{\mathbf{A}^2_{\langle k\rangle}}{2!} + \frac{\mathbf{A}^3_{\langle k\rangle}}{3!} + \cdots$$

$$= \begin{cases} \cos\alpha + \frac{\sin\alpha}{\alpha}\mathbf{A}_{\langle k\rangle} & \text{for } \mathbf{A}^2_{\langle k\rangle} < 0 \\ 1 + \mathbf{A}_{\langle k\rangle} & \text{for } \mathbf{A}^2_{\langle k\rangle} = 0 \\ \cosh\alpha + \frac{\sinh\alpha}{\alpha}\mathbf{A}_{\langle k\rangle} & \text{for } \mathbf{A}^2_{\langle k\rangle} > 0 \end{cases},$$

$$\text{where } \alpha = \sqrt{\text{abs}(\mathbf{A}^2_{\langle k\rangle})}.$$

Translation rotor

- The double reflection on two paralell planes with same orientation make a translation

Using the dual of the planes as mirrors:

$$(\mathbf{n}+\delta_2\,\infty)(\mathbf{n}+\delta_1\,\infty)=1-(\delta_2-\delta_1)\,\mathbf{n}\,\infty$$

$$=1-\frac{1}{2}\mathbf{t}\,\infty$$

$$\equiv \boldsymbol{T}$$

Unit normal vector
in base space

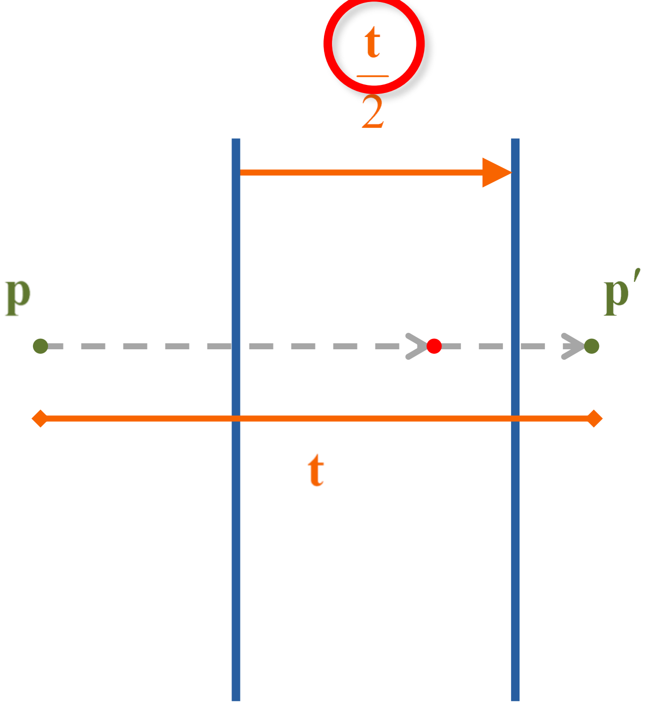
$$\text{where } \mathbf{t}=2(\delta_2-\delta_1)\,\mathbf{n}$$

Exponential form:

$$\boldsymbol{T}=\exp\left(-\frac{1}{2}\mathbf{t}\,\infty\right)=1-\frac{1}{2}\mathbf{t}\,\infty$$

$$X'=\boldsymbol{T}\;X\;\boldsymbol{T}^{-1}$$

Translation vector



Rotation rotor

- The double reflection on two non-paralell planes through the origin make a rotation

Using the dual of the planes as mirrors:

$$(\mathbf{n}_2 + \delta_2 \infty)(\mathbf{n}_1 + \delta_1 \infty) = \mathbf{n}_2\ \mathbf{n}_1$$

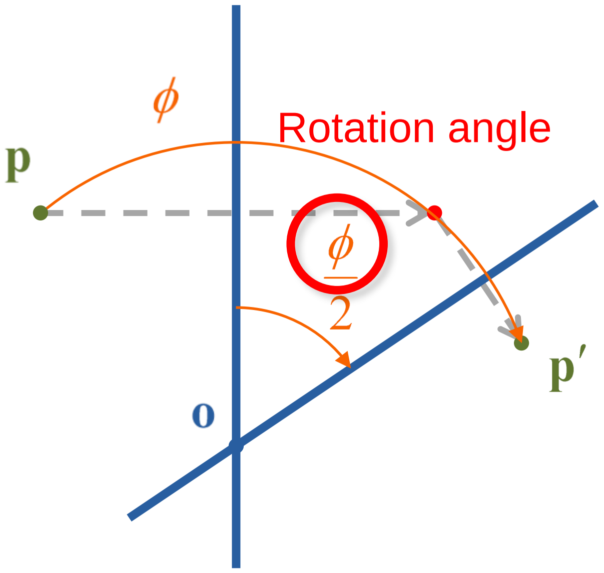
$$= \mathbf{n}_2 \cdot \mathbf{n}_1 + \mathbf{n}_2 \wedge \mathbf{n}_1$$

$$= \cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\mathbf{B}_{\langle 2 \rangle}$$

$$\equiv \boldsymbol{R}$$

Unit normal vector in base space

The distance from the origin is zero



Exponential form:

$$\boldsymbol{R} = \exp\left(-\frac{\phi}{2}\mathbf{B}_{\langle 2 \rangle}\right) = \cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\mathbf{B}_{\langle 2 \rangle}$$

$$X' = \boldsymbol{R}\ X\ \boldsymbol{R}^{-1}$$

General rigid body motion

- It can be composed by first doing a rotation in the origin and following it by a translation

$$\boldsymbol{M} = \exp\left(-\frac{1}{2}\mathbf{t}\,\infty\right)\exp\left(-\frac{\phi}{2}\mathbf{B}_{\langle 2\rangle}\right)$$

Translation
(or combined translations)

Rotation
(or combined rotations)

←

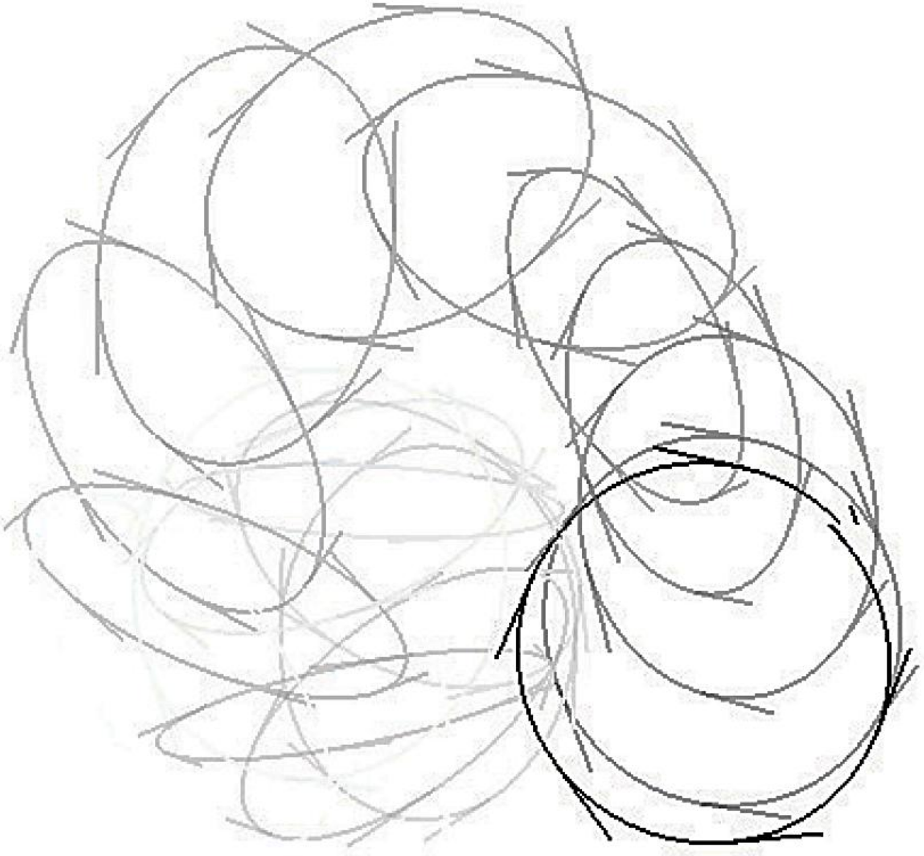
Transformations are applied
from the right to the left

Interpolation of rigid body motions

The logarithm of rigid body motions
is defined for 3-dimensional
base space.

$$S = \exp\left(\frac{\log(M)}{n}\right)$$

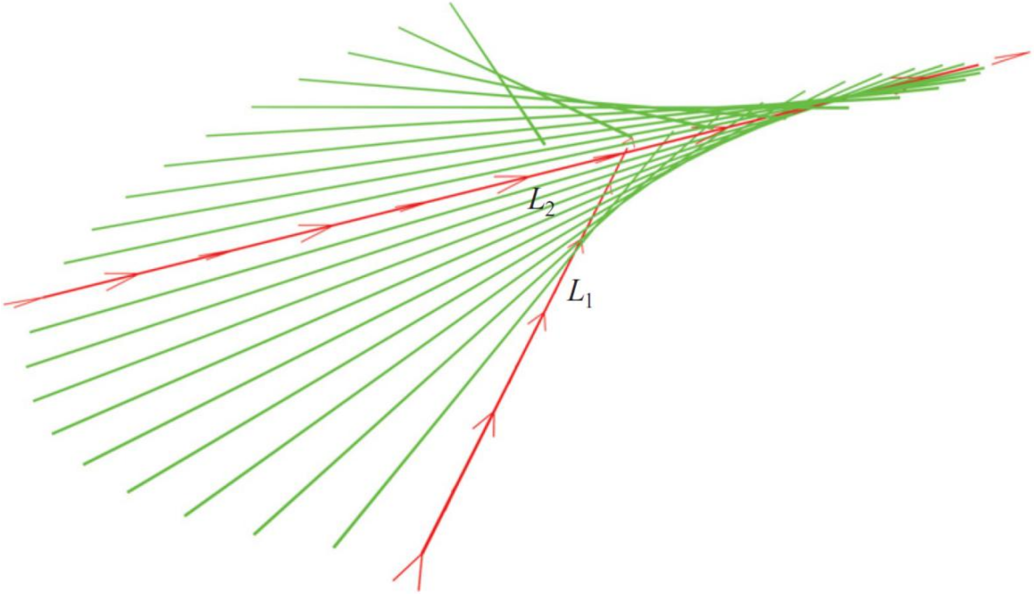
Motion step



Adapted from L. Dorst, D. Fontijine, S. Mann. *Geometric algebra for computer science*. Morgan Kaufmann Publishers, 2007.

Interpolation of rigid body motions

The square of a rigid body motion can be computed as the rate of two flats.



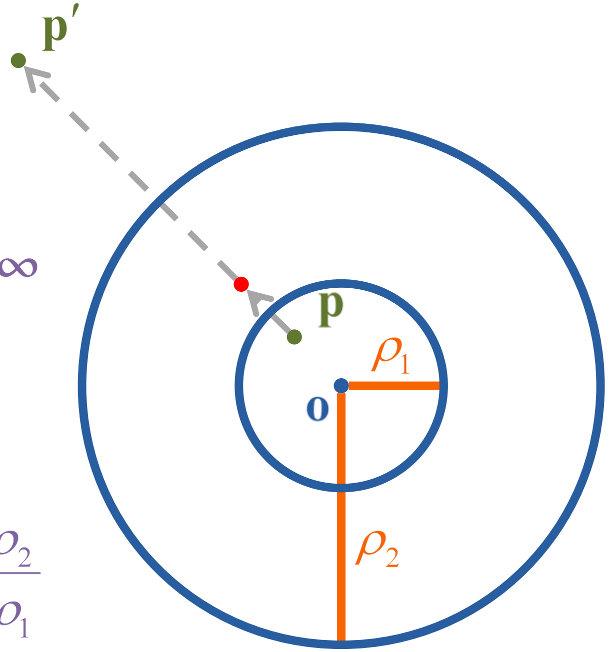
$$S = \exp\left(\frac{1}{2n}\log\left(\frac{\mathbf{L}_2}{\mathbf{L}_1}\right)\right)$$

Motion step

Adapted from L. Dorst, D. Fontijine, S. Mann. *Geometric algebra for computer science*. Morgan Kaufmann Publishers, 2007.

Positive scaling rotor

- The double reflection on two concentric spheres make a positive scale



Using the dual of the spheres as mirrors:

$$\left(\boxed{\mathbf{o}} - \frac{1}{2}\rho_2^2\,\infty\right)\left(\boxed{\mathbf{o}} - \frac{1}{2}\rho_1^2\,\infty\right) = \left(\rho_1^2 + \rho_2^2\right) - \frac{1}{2}\left(\rho_1^2 - \rho_2^2\right)\mathbf{o}\wedge\infty$$

Centered on the origin

$$= \cosh\left(\frac{\gamma}{2}\right) + \sinh\left(\frac{\gamma}{2}\right)\mathbf{o}\wedge\infty$$

$$\equiv \boldsymbol{S} \qquad \text{where } \exp\left(\frac{\gamma}{2}\right) = \frac{\rho_2}{\rho_1}$$

The scaling factor is

$$\sigma = \frac{\rho_2^2}{\rho_1^2} = \exp(\gamma)$$

Positive scaling rotor

- The double reflection on two concentric spheres make a p

Using the dual

$\left(\mathbf{o}-\frac{1}{2}\rho_2^2\,\infty\right)\Big($

The exponential of *k*-blades for arbitrary metric spaces

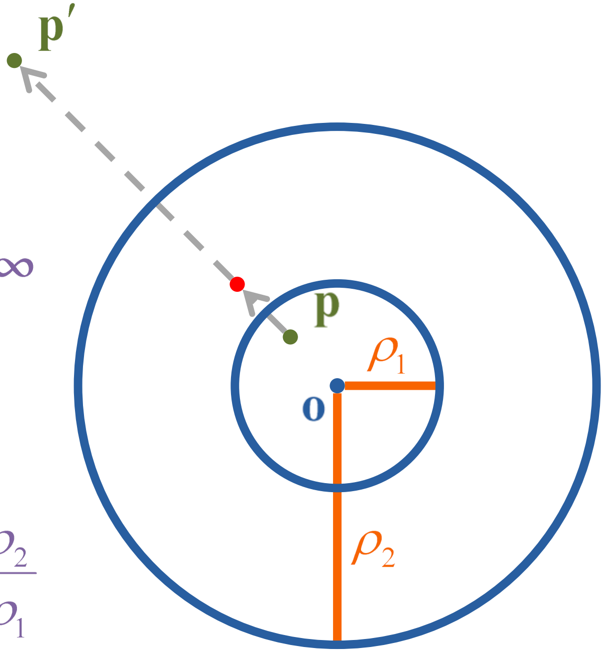
$$\exp\left(\mathbf{A}_{\langle k\rangle}\right)=1+\frac{\mathbf{A}_{\langle k\rangle}}{1!}+\frac{\mathbf{A}^2_{\langle k\rangle}}{2!}+\frac{\mathbf{A}^3_{\langle k\rangle}}{3!}+\cdots$$

$$=\begin{cases}\cos\alpha+\frac{\sin\alpha}{\alpha}\mathbf{A}_{\langle k\rangle} & \text{for } \mathbf{A}^2_{\langle k\rangle}<0\\ 1+\mathbf{A}_{\langle k\rangle} & \text{for } \mathbf{A}^2_{\langle k\rangle}=0\\ \cosh\alpha+\frac{\sinh\alpha}{\alpha}\mathbf{A}_{\langle k\rangle} & \text{for } \mathbf{A}^2_{\langle k\rangle}>0\end{cases},$$

$$\text{where } \alpha=\sqrt{\operatorname{abs}(\mathbf{A}^2_{\langle k\rangle})}.$$

Positive scaling rotor

- The double reflection on two concentric spheres make a positive scale



Using the dual of the spheres as mirrors:

$$\left(\mathbf{o}-\frac{1}{2}\rho_2^2\,\infty\right)\left(\mathbf{o}-\frac{1}{2}\rho_1^2\,\infty\right)=\left(\rho_1^2+\rho_2^2\right)-\frac{1}{2}\left(\rho_1^2-\rho_2^2\right)\mathbf{o}\wedge\infty$$

Centered on the origin

$$=\cosh\left(\frac{\gamma}{2}\right)+\sinh\left(\frac{\gamma}{2}\right)\mathbf{o}\wedge\infty$$

$$\equiv\boldsymbol{S}$$

$$\text{where }\exp\left(\frac{\gamma}{2}\right)=\frac{\rho_2}{\rho_1}$$

Exponential form:

$$\boldsymbol{S}=\exp\left(-\frac{\gamma}{2}\mathbf{o}\wedge\infty\right)=\cosh\left(\frac{\gamma}{2}\right)+\sinh\left(\frac{\gamma}{2}\right)\mathbf{o}\wedge\infty$$

$$X'=\boldsymbol{S}\;X\;\boldsymbol{S}^{-1}$$

The scaling factor is

$$\sigma=\frac{\rho_2^2}{\rho_1^2}=\exp\left(\gamma\right)$$

General positive scaled rigid body motion

- It can be composed by doing a rotation in the origin, a positive scaling, and following them by a translation

$$\boldsymbol{M} = \exp\left(-\frac{1}{2}\mathbf{t}\,\infty\right)\exp\left(-\frac{\gamma}{2}\mathbf{o}\wedge\infty\right)\exp\left(-\frac{\phi}{2}\mathbf{B}_{\langle 2\rangle}\right)$$

Translation
(or combined translations)

Positive scaling
(or combined scalings)

Rotation
(or combined rotations)

Rotation and scaling in the origin commute

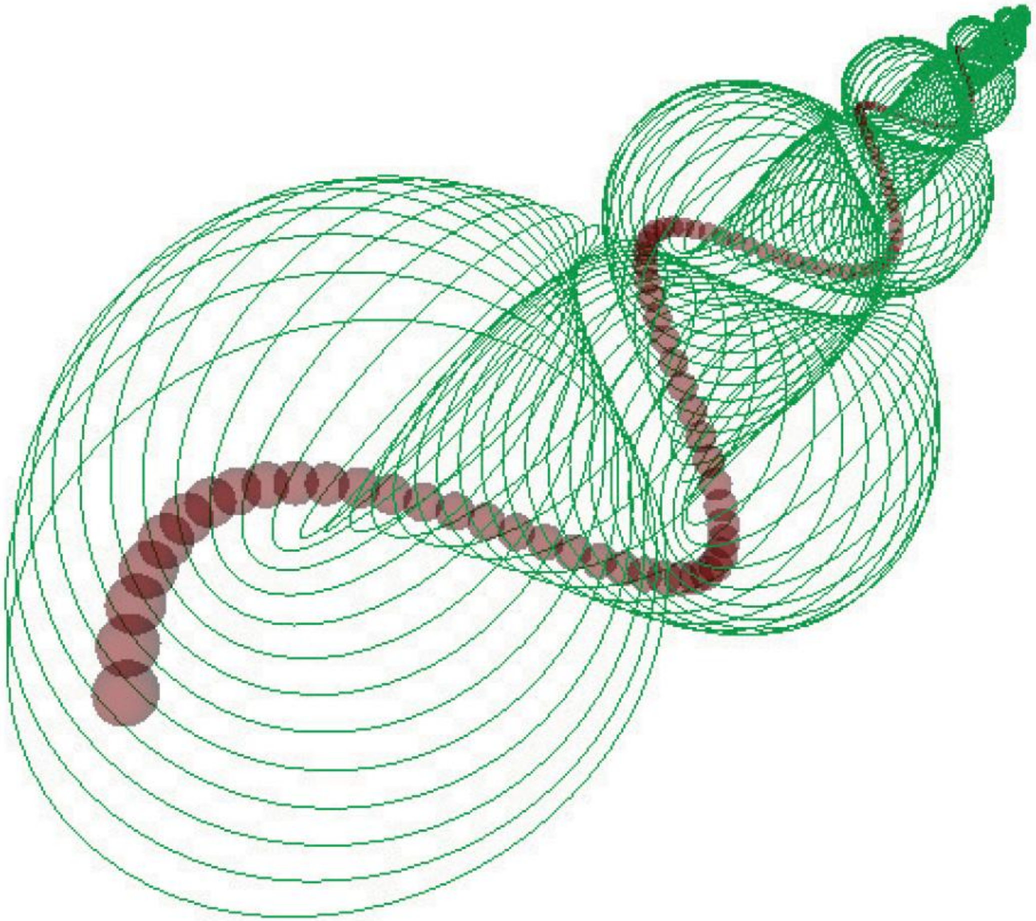
The exponential form of orthogonal transformations is easy to remember.

Interpolation of positive scaled rigid body motions

The logarithm of positive scaled rigid body motions is defined for 3-dimensional base space.

$$\boldsymbol{S} = \exp\left(\frac{\log(\boldsymbol{M})}{n}\right)$$

Motion step



Adapted from L. Dorst, D. Fontijine, S. Mann. *Geometric algebra for computer science*. Morgan Kaufmann Publishers, 2007.

Transversion rotor

- The double reflection on two spheres with a common point make a transversion
- The reflection in the unit sphere, followed by a translation, and by another reflection in the unit sphere also make a transversion

Using the dual of the unit sphere and a translation:

$$\left(\mathbf{o}-\frac{1}{2}\infty\right)\left(1-\frac{1}{2}\mathbf{t}\,\infty\right)\left(\mathbf{o}-\frac{1}{2}\infty\right)=1+\mathbf{o}\,\mathbf{t}$$

Translation vector in base space

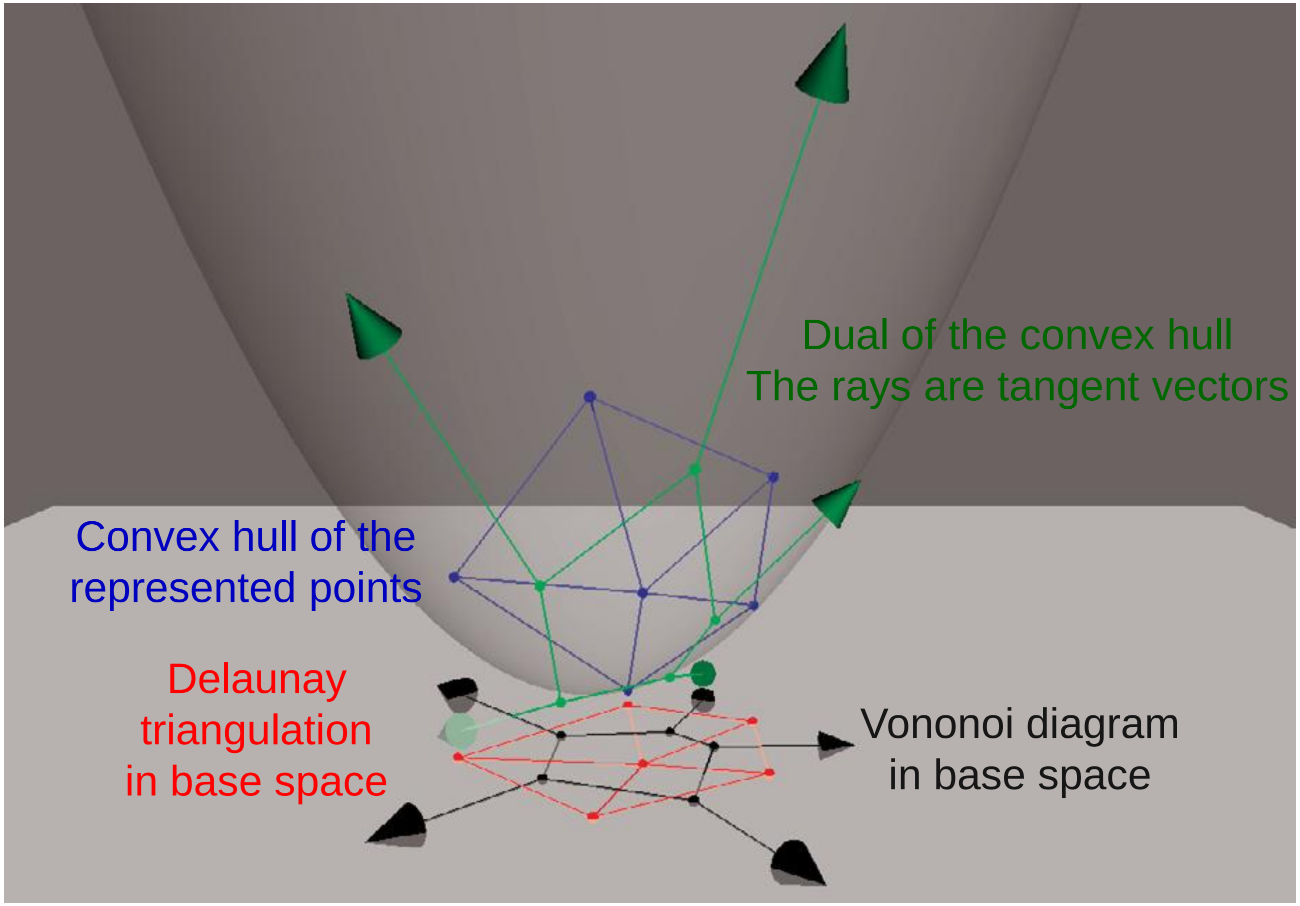
$$=\exp\left(\mathbf{o}\,\mathbf{t}\right)$$

A closed-form solution to the logarithm of a general conformal transformation also involving transversion is not yet known.

Lecture VI

Some Applications

Voronoi diagram and Delaunay triangulation



Dual of the convex hull
The rays are tangent vectors

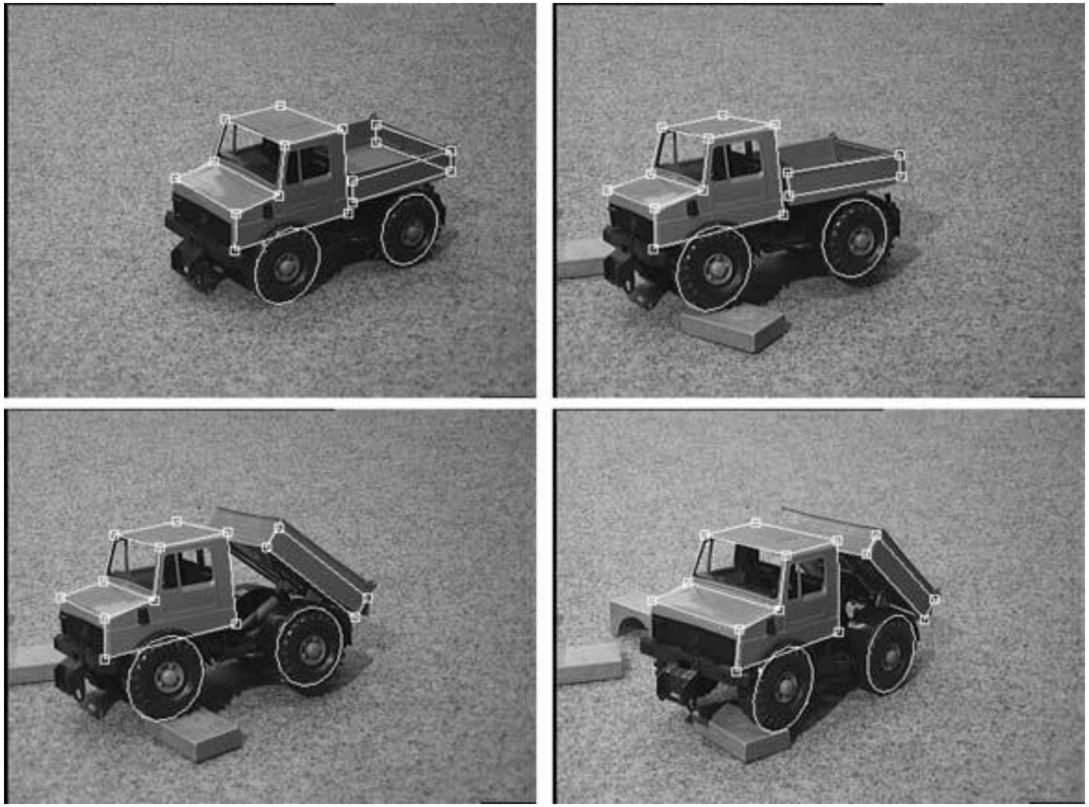
Convex hull of the
represented points

Delaunay
triangulation
in base space

Vononoi diagram
in base space

Adapted from L. Dorst, D. Fontijine, S. Mann. *Geometric algebra for computer science*. Morgan Kaufmann Publishers, 2007.

2-D/3-D pose estimation of different corresponding entities

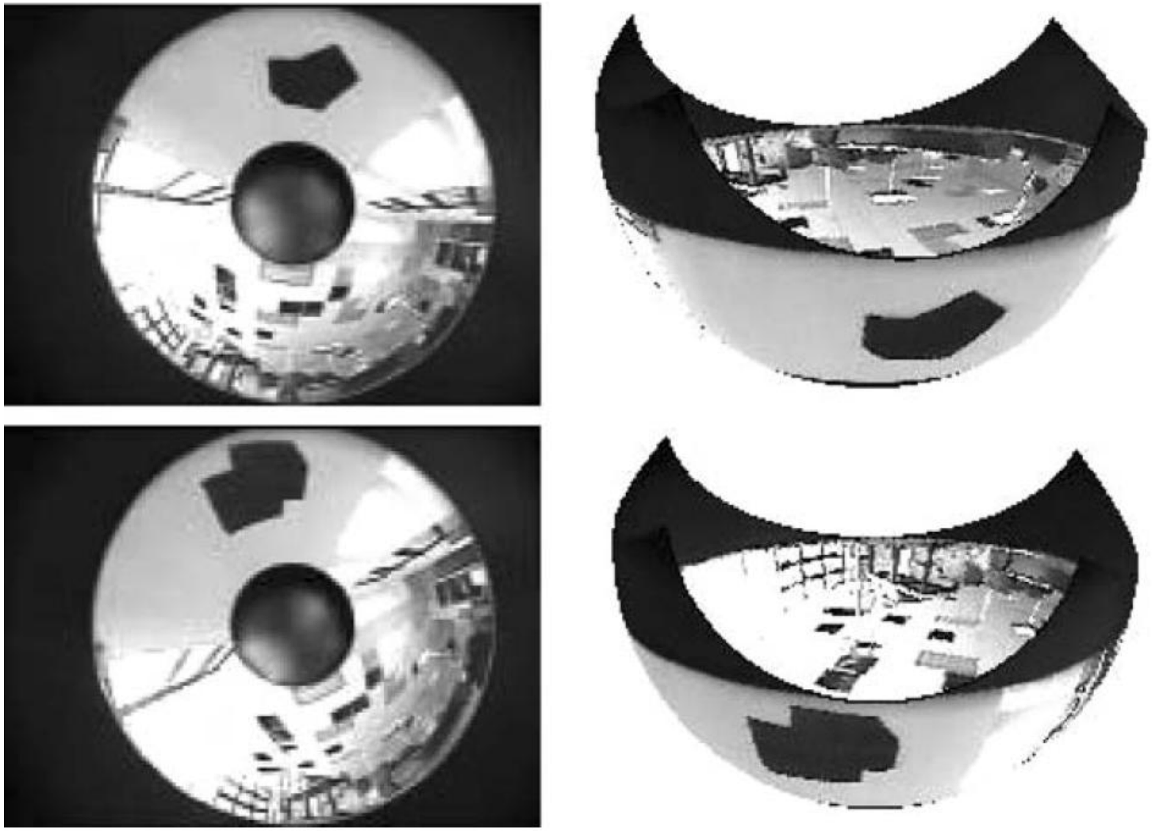


B. Rosenhahn, G. Sommer (2005) Pose estimation in conformal geometric algebra part II: real-time pose estimation using…, J. Math. Imaging Vis., 22:1, pp. 49–70.

Inverse kinematics of a human-arm-like robot

D. Hildenbrand *et al.* (2005), Advanced geometric approach for graphics and visual guided robot object manipulation, in Proc. of the Int. Conf. Robot. Autom., pp. 4727–4732.

Omnidirectional robot vision

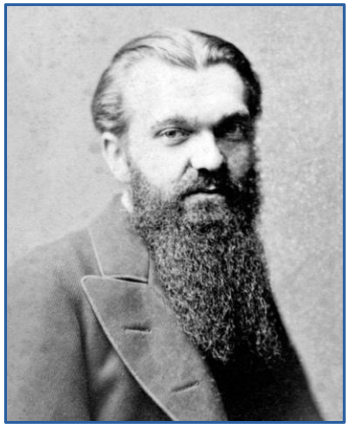


C. Lopez-Franco, E. Bayro-Corrochano (2006), Omnidirectional robot vision using conformal geometric computing, J. Math. Imaging Vis., 26:3, pp. 243-260.

Higher dimensional fractals modeling

J. Lasenby *et al.* (2006), Higher dimensional fractals in geometric algebra, Cambridge University Engineering Department, Tech. Rep. CUED/F-INFENG/TR.556.

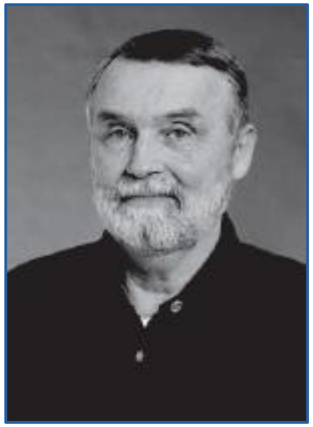
Credits



William K. Clifford
(1845-1879)

Paul Dirac
(1902-1984)

Wolfgang E. Pauli
(1900-1958)



David O. Hestenes
(1933-)

Hestenes, D. (2001) *Old wine in new bottles: a new algebraic framework for computational geometry*. **In: Geometric algebra with applications in science and engineering**, Boston: Birkhäuser, 3-17

Minkowsky space

- It has been well studied to represent space-time in relativity

The negative dimension is employed to represent time.

$$M = \begin{array}{c} \begin{array}{cccccc} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_d & \mathbf{e}_+ & \mathbf{e}_- \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 \end{array}\right) \begin{array}{c} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_d \\ \mathbf{e}_+ \\ \mathbf{e}_- \end{array} \end{array}$$

The conformal model is just another way to see it:

$$\mathbf{o} = \frac{1}{2}\left(\mathbf{e}_+ + \mathbf{e}_-\right)$$

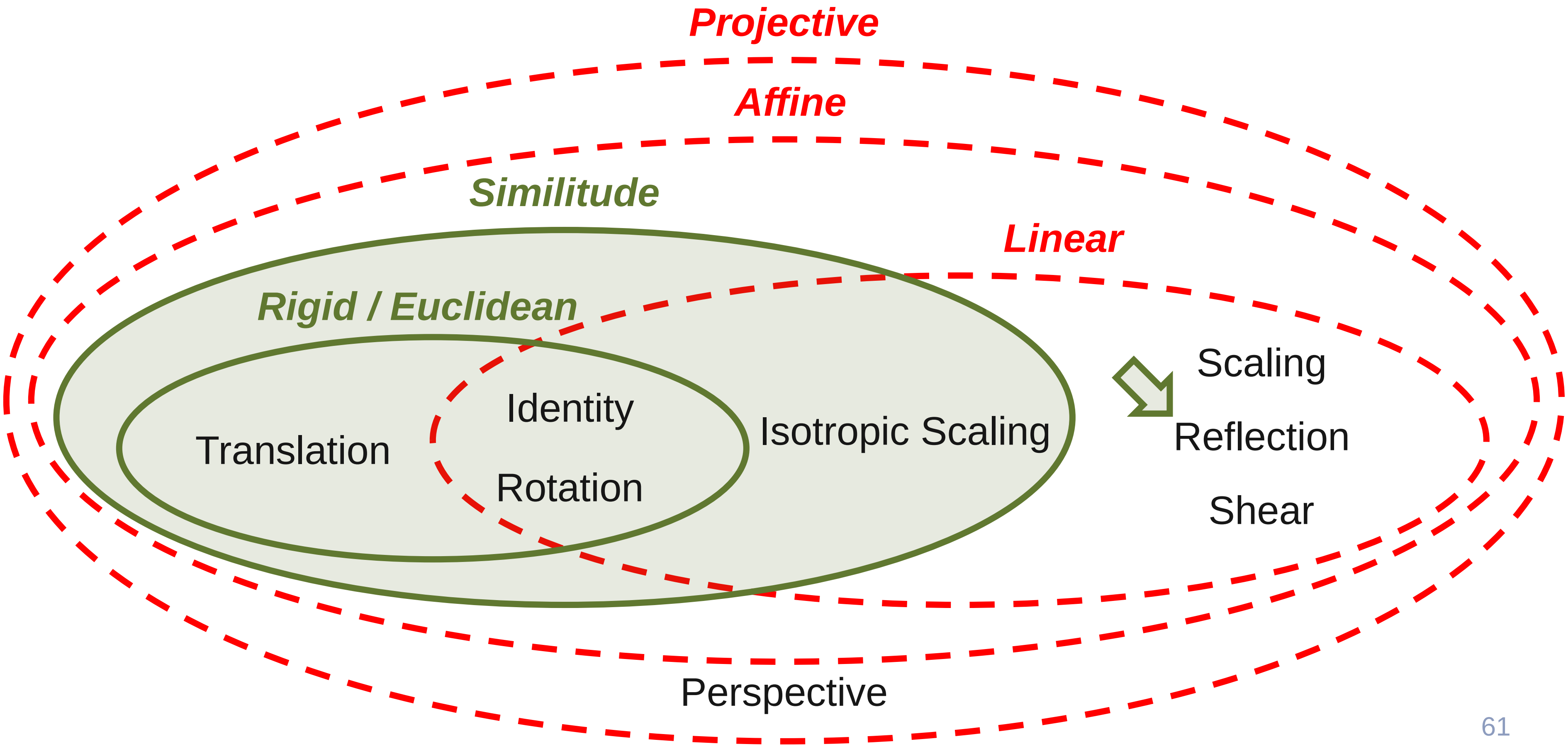
$$\infty = \mathbf{e}_- - \mathbf{e}_+$$

Lecture VI

So, what is next?

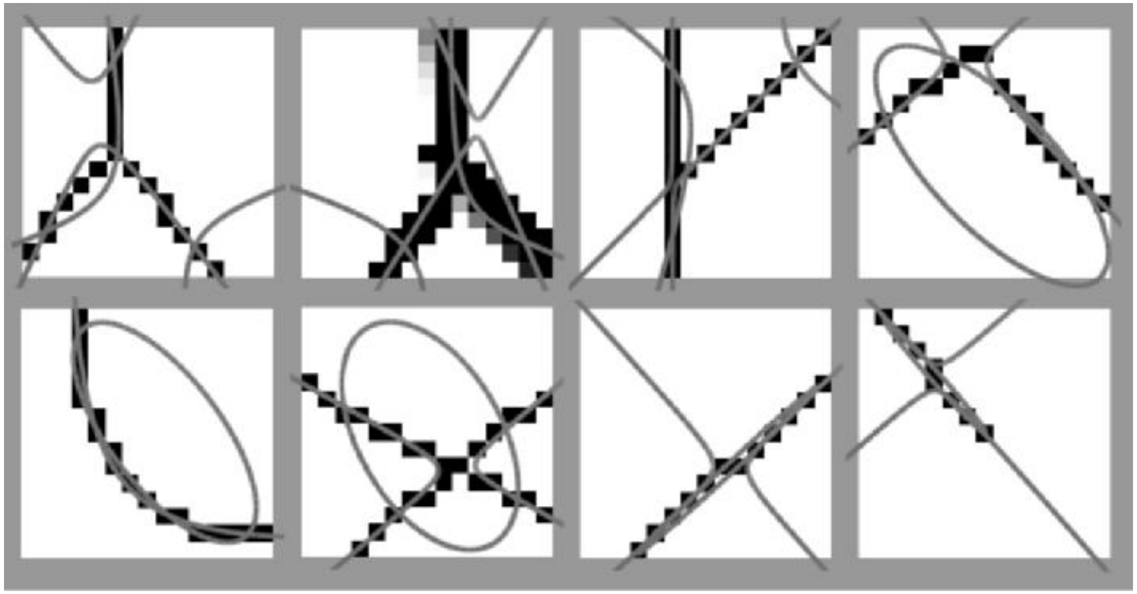
Drawbacks

- There are some limitations yet
  - Versors do not encode all projective transformations



However, there are other models of geometry

- Conic space and conformal conic space
- Created by Perwass to detect corners, line segments, lines, crossings, y-junctions and t-junctions in images



C. B. U. Perwass (2004) Analysis of local image structure using…, Instituts für Informatik und Praktische Mathematik der Universität Kiel, Germany, Tech. Rep. Nr. 0403.

Drawbacks

- Efficient implementation of GA is not trivial
  - Multivectors may be big (2^n coefficients)
  - Storage problems
  - Numerical instability

- Custom hardware is optimized for linear algebra
  - There is an US patent on the conformal model

A. Rockwood, H. Li, D. Hestenes (2005) System for encoding and manipulating models of objects, U.S. Patent 6,853,964.

Concluding remarks

- Consistent framework for geometric operations
  - Geometric elements as primitives for computation
  - Geometrically meaningful products
- Extends the same solution to
  - Higher dimensions
  - All kinds of geometric elements
- An alternative to conventional geometric approach
- It should contribute to improve software development productivity and to reduce program errors

Introduction to Geometric Algebra

Extra III

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Visgraf - Summer School in Computer Graphics - 2010

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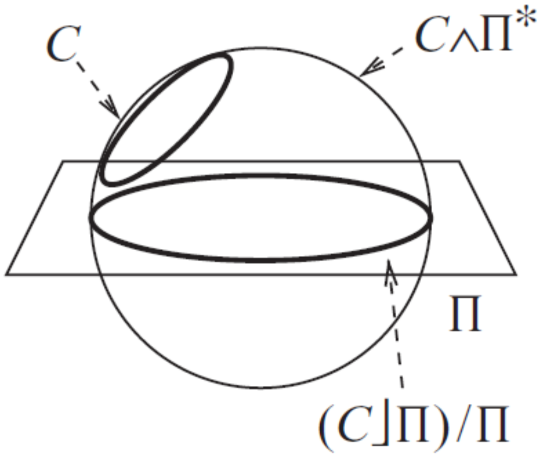
UFRGS UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL

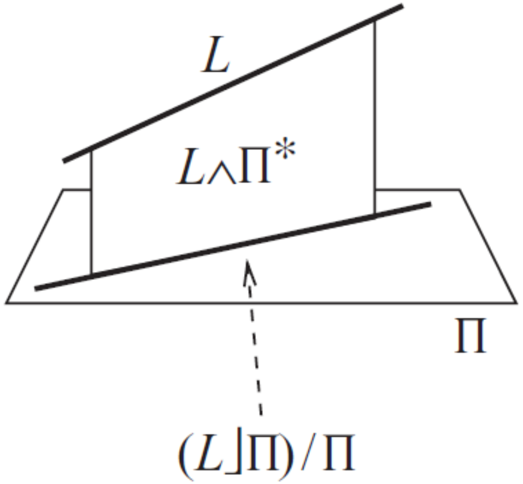
CNPq Conselho Nacional de Desenvolvimento Científico e Tecnológico

Orthogonal projection behavior

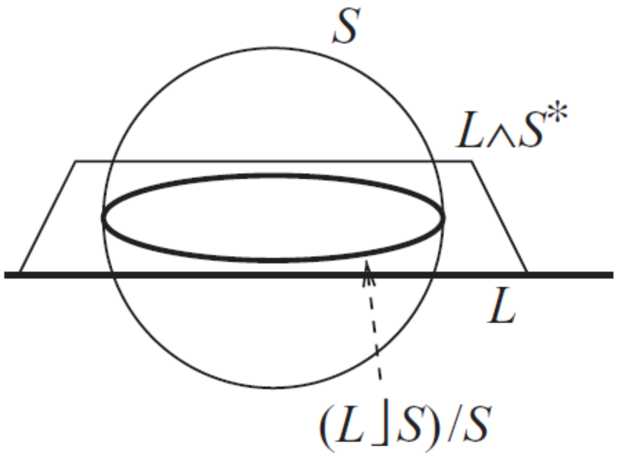
The projection of a flat
produces the expected element



Circle projected
ontho a plane



Straight line projected
ontho a plane

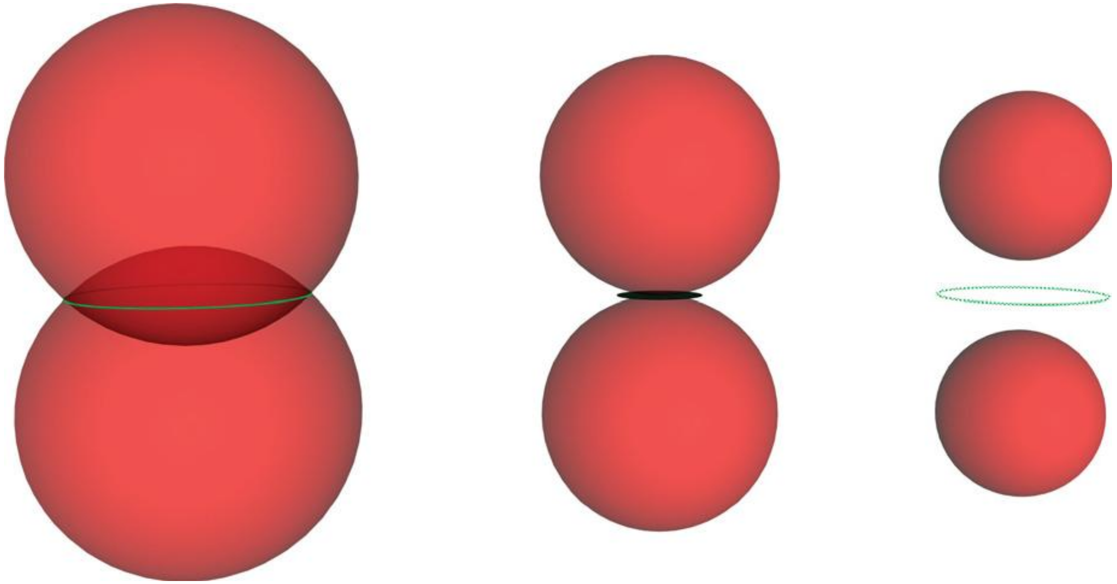


Straight line projected
ontho a sphere

An ellipse is not represented by
a blade in the conformal model

Adapted from L. Dorst, D. Fontijine, S. Mann. *Geometric algebra for computer science*. Morgan Kaufmann Publishers, 2007.

Intersection of two spheres



Real Circle $\lambda > 0$

Tangent Space $\lambda = 0$

Imaginary Circle $\lambda < 0$

$$\mathbf{C}_{\langle 3\rangle} = \mathbf{A}_{\langle 4\rangle} \cap \mathbf{B}_{\langle 4\rangle}$$

Scalar used for testing

$$\lambda = \mathbf{C}_{\langle 3\rangle}^{2}$$

Intersection point

$$\mathbf{p} = \mathbf{C}_{\langle 3\rangle} \infty \mathbf{C}_{\langle 3\rangle}$$

It also holds for sphere and plane!

Adapted from L. Dorst, D. Fontijine, S. Mann. *Geometric algebra for computer science*. Morgan Kaufmann Publishers, 2007.