

Subdivision Curves on Triangular Meshes

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Abstract. Subdivision curves have great importance for many CAD/CAM applications. In this paper we propose a simple method to define subdivision schemes on triangulations. It works by translating to the triangulation a perturbation of a planar binary subdivision. To reproduce this perturbation in the surface we use both, shortest and straightest geodesics, so we call this strategy *intrinsic projection method*. It can reproduce any binary subdivision scheme, regardless whether it is linear or not.

1 Introduction

Subdivision curves are very important for several science and engineering applications. In the Euclidean space \mathbb{R}^n , they are easy to define by just selecting a small set of control points. A drawing algorithm typically performs some subdivision steps, thus approximating the limit curve by a polygonal line in a matter of milliseconds. In this setting, there are many references on the convergence of these schemes.

Given their good properties and the advantages of using these curves, it is natural to extend them to non-euclidean geometries. In section 3 we review some of these extensions. In this paper we propose a simple method to define subdivision schemes on triangulations.

1.1 Overview

The paper is organized as follows: In section 2 we make a short review of subdivision curves on the euclidean space. Section 3 begins reviewing the different extensions of subdivision schemes to non-euclidean geometries, and the main contributions of this work are lately presented in this section, where we propose a general method to define subdivision schemes on triangulations. A small review of discrete geodesic computation is presented in section 3.3, while in section 5 we conclude with some remarks about this work.

2 Subdivision Curves

Subdivision schemes constitute an efficient tool for curve and surface design in Computer Graphics. In this section we make a brief review of subdivision schemes for curve design on \mathbb{R}^d . Given a polygonal line P , with vertices on \mathbb{R}^d

denoted by $\dots, P_{-1}, P_0, P_1, \dots$, a subdivision of P is defined as the limit of a sequence of refinements of it. For the sake of simplicity, we assume that polygons have infinite vertices.

The binary subdivision of P^0 (namely P^1) is obtained by first inserting a new point between two consecutive points of P^0 and then perturbing the position of some or all the points. The rules used to perturb the points are determined by the subdivision scheme. If we denote by S the refinement operator then we have $P^{j+1} = SP^j = S^j P^0 = S^j P$ and the limit curve is denoted by $S^\infty P$.

Subdivision schemes are classified according to the rules used to obtain P^{j+1} from P^j . Note that new vertices in P^{j+1} have odd indexes while old vertices correspond to even indexes in P^{j+1} . It is then natural to have different rules for even and odd vertices:

$$\begin{aligned} P_{2i}^{j+1} &= \dots + a_{-2}P_{i+1}^j + a_0P_i^j + a_2P_{i-1}^j + \dots \\ P_{2i+1}^{j+1} &= \dots + a_{-1}P_{i+1}^j + a_1P_i^j + a_3P_{i-1}^j + \dots \end{aligned} \quad (1)$$

In these two equations, the sequence of coefficients $a = \{a_i, i \in \mathbb{Z}\}$ is called *mask* and is supposed to have only a finite number of non-zero entries. These rules can be summarized in one equation,

$$P_i^{j+1} = (SP^j)_i = \sum_{k \in \mathbb{Z}} a_{i-2k} P_k^j, \quad i \in \mathbb{Z}. \quad (2)$$

A deeper study of linear stationary schemes can be found in the book of Cavaretta et. al. [2]. A collection of non-linear schemes was described by Dyn and Levin [4].

2.1 Examples

Chaikin's Scheme. Maybe the most intuitive way to smooth a polygonal line is to cut its corners. Chaikin's scheme does exactly that. Each refinement consists on cutting the consecutive edges of the polygon in the ratio $\frac{1}{4} : \frac{1}{2} : \frac{1}{4}$. Figure 1(a) shows some steps of the refinement of a polygon using Chaikin's rule. The following are the equations of this scheme:

$$P_{2i}^{j+1} = \frac{3}{4}P_i^j + \frac{1}{4}P_{i+1}^j, \quad P_{2i+1}^{j+1} = \frac{1}{4}P_i^j + \frac{3}{4}P_{i+1}^j. \quad (3)$$

Schemes like Chaikin's are called *approximant* because they do not interpolate the vertices of the control polygon, so that the resulting curves are just smooth approximations of the control polygon.

4-point Scheme. Interpolant schemes are those which interpolate the vertices of the control polygon. Consequently, every new vertex belongs to the curve $S^\infty P$. The more popular interpolant scheme is the so-called 4-point scheme [5]. Its rules are:

$$P_{2i}^{j+1} = P_i^j, \quad P_{2i+1}^{j+1} = \left(\frac{1}{2} + w\right)(P_i^j + P_{i+1}^j) - w(P_{i-1}^j + P_{i+2}^j) \quad (4)$$

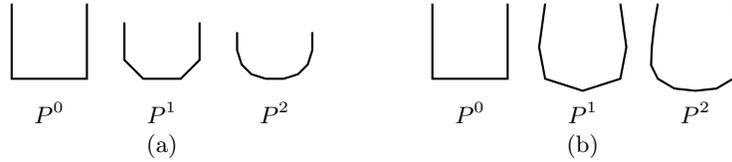


Fig. 1. Refinement of a polygon using (a) Chaikin's scheme, and (b) 4-point scheme.

where w is a tension parameter. Figure 1(b) shows some steps of the 4-point scheme.

3 Subdivision Curves on Manifolds

The problem of designing curves on manifolds is very interesting and have various applications. Some variants of de Casteljau's Algorithm [3] have been applied to define curves on Riemannian manifolds and Lie groups [10, 13], and also on triangulations [7]. Hofer and Pottmann [12] defined spline curves by minimizing some energy functionals. Bonneau and Hahmann [1] give some strategies to smooth curves on triangulations.

Recently, a general framework for linear subdivision on smooth manifolds have been described by Helmut and Pottmann [18]. The convergence and smoothness analysis of this curves can be found in the works of Wallner and Dyn [16, 17]. They generalize the linear schemes to manifolds in two different ways. Their first method is to substitute linear average by geodesic average. This method is very good because it is completely intrinsic although for some schemes it would have to compute many geodesics. The second method is to perform each subdivision step in the ambient space, projecting the new points into the manifold. This approach is more efficient but depending on the geometry could conduce to wrong or unexpected results.

We propose a third method to generalize subdivision schemes to manifolds. It uses geodesic averages but it is not based on them, so it can reproduce non-linear schemes as well, it also performs fewer geodesic computations. In the other hand, it does a sort of intrinsic projection that eliminates the problem of getting the wrong projection when the surface has a complicated geometry.

3.1 Revisiting Curve Subdivision

Consider the following subdivision scheme:

$$P_{2i}^{j+1} = P_i^j, \quad P_{2i+1}^{j+1} = \frac{(P_i^j + P_{i+1}^j)}{2}. \quad (5)$$

This scheme performs a linear subdivision consisting just on inserting new points at midpoints of each pair of consecutive old points. Lets call this scheme of B .

Every binary subdivision scheme can be written as a perturbation of B , regardless of it being linear or not. We have now a uniform description for all binary schemes. For example, given a scheme S and a set of control points P , defining $v_i = (SP)_i - (BP)_i$ we can write

$$(SP)_i = (BP)_i + v_i. \quad (6)$$

Now each subdivision of a control polygon can be regarded as a two step process, where we perform one step of subdivision given by B , and one step of perturbation given by the set of vectors v_i .

This approach to subdivision schemes may not be of great use in the case of traditional schemes on the euclidean space, particularly linear schemes. However, it will be very useful to reproduce planar subdivision schemes on manifolds.

3.2 Intrinsic Projection Method

Since triangle meshes are manifolds of dimension two, we are going to define our subdivision schemes by imitating planar schemes. Given a triangle mesh \mathcal{M} , a control polygon P with vertices on \mathcal{M} , and a subdivision scheme S , we define the scheme $S_{\mathcal{M}}$ as a two step process as done above.

Step 1: binary subdivision. To perform this step we use the scheme $B_{\mathcal{M}}$, defined similar to B :

$$P_{2i}^{j+1} = P_i^j, \quad P_{2i+1}^{j+1} = \text{midpoint of } \mathbf{geodesic}(P_i^j, P_{i+1}^j). \quad (7)$$

The difference between this scheme and B is the substitution of linear interpolation by geodesic interpolation. See section 3.3 for a review of discrete geodesic computation.

Step 2: perturbation. To perform the perturbation step we need to compute the vectors v_i . Also the resulting point $(S_{\mathcal{M}}P)_i = (B_{\mathcal{M}}P)_i + v_i$ must belong to \mathcal{M} . It is very hard to fulfill such restriction, so we proceed in a different way. To compute the perturbation we find first a vector t_i tangent to \mathcal{M} at $(B_{\mathcal{M}}P)_i$. After that, the new point $(S_{\mathcal{M}}P)_i$ belongs to the unique straightest geodesic line departing from $(B_{\mathcal{M}}P)_i$ in the direction of t_i and is distant to $(B_{\mathcal{M}}P)_i$ by the magnitude of t_i . See section 3.3 for references on discrete tangent vectors and the unicity of straightest geodesic.

Rest now to compute the vectors t_i . The idea is simple, we are going to create a plane control polygon \tilde{P} whose sides have the same length as the sides (geodesic curves) of P , and whose angles are proportional to those of P . If a vertex of P coincides with a mesh vertex, the total angle α at that mesh vertex will probably be different from 2π , so the corresponding angle in \tilde{P} will be proportional to it. For instance, if the angle of P at that vertex is β , the corresponding angle at \tilde{P} will be $(2\pi/\alpha)\beta$. For vertices of P inside a face or edge of \mathcal{M} we have $\alpha = 2\pi$

so the angle in \tilde{P} will be the same. To compute the vectors t_i we first compute the v_i 's using (6) for \tilde{P} :

$$v_i = (S\tilde{P})_i - (B\tilde{P})_i. \quad (8)$$

Now t_i is obtained from v_i by changing the coordinates to the tangent plane to \mathcal{M} at P_i .

3.3 Implementation

The implementation of the intrinsic projection method is very simple. In this section we do a small description of the details involved, and show the results of the implementation of Chaikin and 4-point schemes.

To perform the binary subdivision step on the polygon P , we just compute the shortest geodesic line joining every two consecutive vertices of P , and insert its midpoint between the two points. Thus obtaining $B_{\mathcal{M}}P$.

To perform the perturbation step, for each vertex $(B_{\mathcal{M}}P)_i$, take the portion of P that influence it and construct a plane polygon \tilde{P}_i , and compute the corresponding plane perturbation vector v_i . Observe that the amount of vertices of P influencing a vertex of $B_{\mathcal{M}}P$ depends on the subdivision scheme.

The more delicate part is the mapping of v_i to a tangent vector t_i . We proceed as follows:

- v_i :** Let v be the vector $\tilde{P}_{i+1} - \tilde{P}_i$. Vector v_i is given as a pair (ρ_i, l_i) , where ρ_i is the anticlockwise angle between the vectors v and v_i , and l is the size of v_i .
- t_i :** To obtain t_i , find the direction at $(B_{\mathcal{M}}P)_i$ on M that makes an anticlockwise angle proportional to ρ_i with the geodesic segment joining $(B_{\mathcal{M}}P)_i$ and $(B_{\mathcal{M}}P)_{i+1}$.
- perturbation:** The perturbation is given by the extreme of the straightest geodesic beginning at $(B_{\mathcal{M}}P)_i$ in the direction of t_i , whose length is l_i .

Discrete Geodesics. We are interested on designing curves on triangulations, so we need to compute discrete geodesics. Locally shortest curves [8, 9, 15] solve the boundary value problem, whereas straightest geodesics [11] solve the initial value problem. In this setting, a direction is given as a tangent vector to the surface; the definition of tangent vector for triangulated surfaces [11] can be summarized in the following

Definition 1. *A vector v is tangent to the mesh \mathcal{M} at point P , if there exists a positive number ε such that the point $P + \varepsilon v$ belongs to a face f containing P .*

Note that –by the previous definition– the set of all tangent vectors to \mathcal{M} at P do not necessarily belongs to a plane. This happens exactly when P lies in an edge or vertex of \mathcal{M} . In those cases, the tangent vectors belong to the support planes of the faces incident to P .

Chaikin's Scheme. The Chaikin's scheme has the particularity that every time we compute a v_i vector, its associated angle ρ_i equals 0. Also $l_i = 0.25l$, where l is the length of the corresponding shortest geodesic segment. In this particular case, we can substitute the old points by the points corresponding to geodesic interpolation with parameters 0.25 and 0.75. See Fig. 2 and 3 for examples.

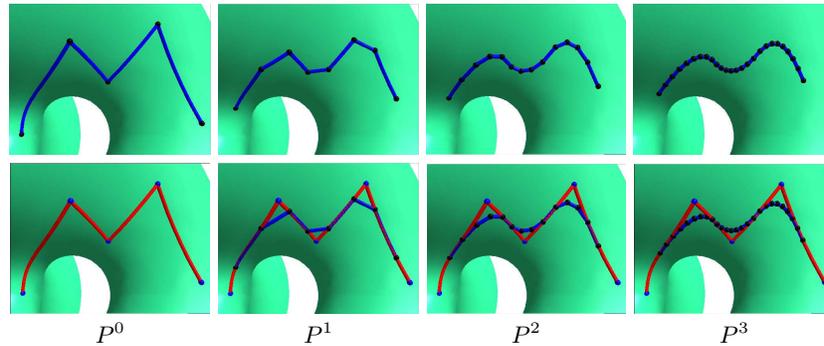


Fig. 2. First steps of Chaikin's subdivision scheme.

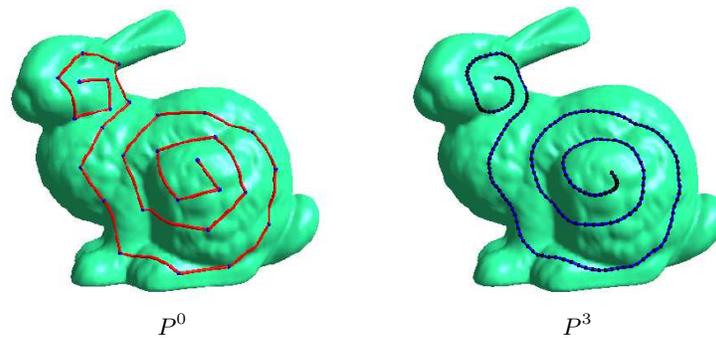


Fig. 3. Curve generated using Chaikin's scheme.

4-point Scheme. Since the 4-point scheme is interpolant, old points does not need to be perturbed. For the perturbation of each new point we must use four old points. Using these old points we can compute the planar perturbation (ρ, l) of B , and translate it to the \mathcal{M} using straightest geodesics. See Fig. 4 and 5 for some examples.

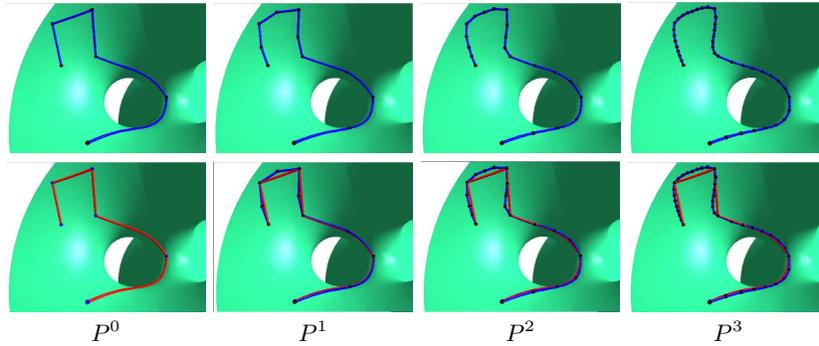


Fig. 4. First steps of 4-point subdivision scheme.

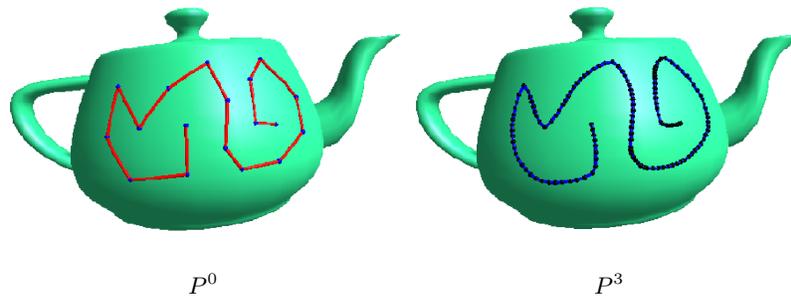


Fig. 5. Curve generated using 4-point scheme.

4 Mesh Segmentation with Subdivision Curves

In some problems related to 3D shape analysis, it is very important to segment a surface [6, 14]. Feature lines are the natural candidates to boundary of the segmented regions. Subdivision curves provide a flexible tool to describe those feature lines, since few points are sufficient to describe the lines. Besides, a potencial user could modify them locally, performing a fine tuning of the shape of the final curve. The smooth appearance of the lines thus obtained, improves the visual quality of the segmentation and contributes for further processing of the segmented meshes.

5 Conclusion

We have presented the intrinsic projection method to reproduce planar subdivision schemes on manifold triangulations. Since this method is based on planar schemes, it coincides with those schemes on planar meshes. Its extensions to smooth manifolds is straightforward. A nice feature of the method is its capability of reproducing non-linear schemes in the same way as the linear case.

The intrinsic projection method can be easily adapted to smooth manifolds. It is possible to compute geodesics solving both, the boundary value and initial value problem on those manifolds. See [18] for instance. In this context would be very nice to study the smoothness properties of $S^\infty P$.

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