Geometric processing of graphical objects

Jonas Gomes
Visgraf Laboratory
IMPA – Instituto de Matemática Pura e Aplicada
Rio de Janeiro, Brasil

Graphical Object

- Image, sound, volumes, surfaces, video ...

- Graphical object

$$f : U \subset \mathbf{R}^n \to \mathbf{R}^m$$

- Shape U

- Attribute function f

- Dimension of $\mathcal{O}$ = dimension of U

GO Examples

- Image

$$f : U \subset \mathbf{R}^2 \to \mathbf{R}$$

- Simple shape

- Complex attribute

GO Examples

- Volumetric object ($3D$ image)

$$f\colon U \subset \mathbf{R}^3 \to \mathbf{R}$$

  - Dimension of U is 3
  - f is the density function

- Video (Image sequence)

$$f\colon [a,b] \times V \subset \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R},\ V \subset \mathbf{R}^2$$

  - One-parameter family of images

$$f_t\colon V \subset \mathbf{R}^2 \to \mathbf{R}$$

  - $U = [a,b] \times V$
  - Dimension of U is 3
  - Video is a volumetric object

GO Analysis and Processing

- Attribute function
  - Function space
  - Operators, Filters and Transforms.

- Shape
  - Geometry
  - Topology

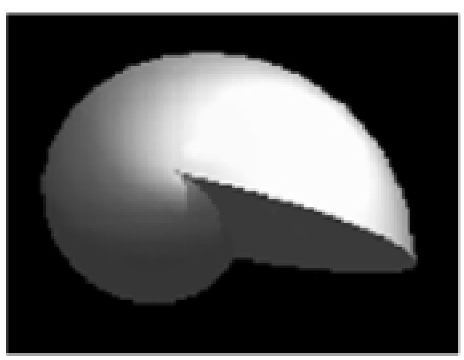
GO Analysis and Processing

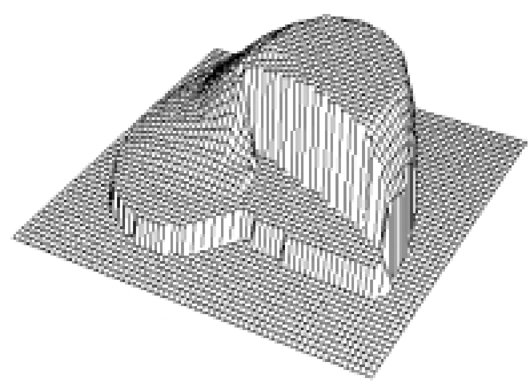
- Tradeoff between attribute and shape

- Image as a surface (Monge surface)

$$f\colon U \subset \mathbf{R}^2 \to \mathbf{R}$$

$$G(f) = \{(u, v, f(u, v)), \quad u, v \in U\} \subset \mathbf{R}^3$$





- Volumetric object as a hypersurface

$$f\colon U \subset \mathbf{R}^3 \to \mathbf{R}$$

$$G(f) = \{(x, y, z, f(x, y, z)), \quad x, y, z \in U\} \subset \mathbf{R}^4$$

GO Analysis and Processing

- Geometry and Topology tools
  - Differential geometry
  - Differential topology

- Needs differentiability
  - Ill-posed problem (noise)

- Regularization techniques
  - Scale spaces

Scale Spaces

- $f\colon \mathbf{R} \to \mathbf{R}, \qquad L\colon \mathbf{R} \times \mathbf{R}^{+} \to \mathbf{R}$

$$\begin{aligned} L(x,0) &= f(x) \\ L(x,t) &= g(x,t) * f(x) \\ &= \int_{\mathbf{R}} g(u,t) f(x-u) du \\ g(x,t) &= \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \quad (Gaussian) \end{aligned}$$

- Solution of the Heat equation

$$\frac{\partial L}{\partial t} = c \ \Delta L$$

- Scale decreases when $t \to \infty$.

- In practice:
  Blur the object before differentiating

- Problem: Find the right size of scale

Differential Geometry

- Surface $M \subset \mathbf{R}^3$

- Gauss Map

$$N : M \to S^2, \quad p \mapsto n(p)$$

- Derivative of the Gauss map

$$N' : T_pM \to T_pM, \quad N'(v) = \frac{d}{dt}N(\alpha(t))\bigg|_{t=0}$$

- First fundamental form

$$I : T_pM \to T_pM, \quad I(v) = \langle v, v \rangle$$

- Second fundamental form

$$II : T_pM \to T_pM, \quad II(v) = \langle N'(v), v \rangle$$

- Surface reconstruction (Bonnet Theorem)

Differential Geometry

- Geometric interpretation
- First fundamental form: metric
- Second fundamental form: curvature
  - Normal curvatures, k_v
  - Principal curvatures, k_1, k_2
  - Gaussian Curvature

$$K = k_1 k_2$$

  - Mean curvature

$$H = k_1 + k_2$$

Image surface

- Monge surface

$$x(u,v) = (u, v, h(u,v))$$

- First fundamental form

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 + h_u^2 & h_u h_v \\ h_u h_v & 1 + h_v^2 \end{pmatrix}$$

- Second fundamental form

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \frac{1}{\sqrt{1 + h_u^2 + h_v^2}} \begin{pmatrix} h_{uu} & h_{uv} \\ h_{uv} & h_{vv} \end{pmatrix}$$

- Gaussian and Mean curvatures

$$K = \frac{\det B}{\det G}$$

$$H = \mathrm{trace}(BG^{-1})$$

Application: Geometric segmentation

- Image $h\colon U \subset \mathbf{R}^2 \to \mathbf{R}$

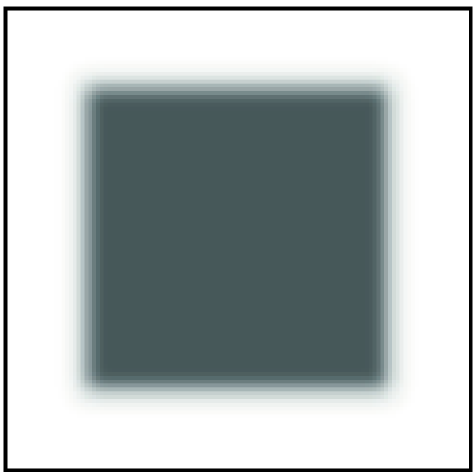
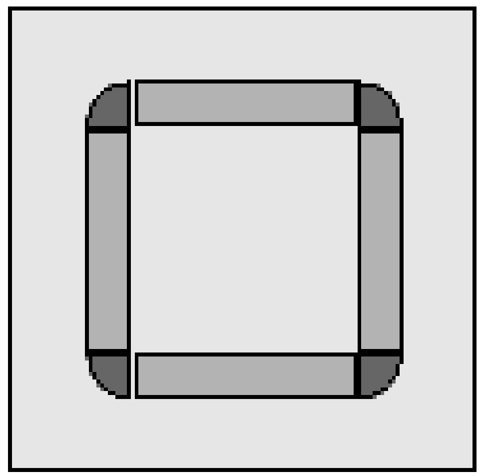
- $p \in g0D$ if II has rank 0 at $(p, h(p))$
  - Planar point
  - $H = K = 0$
  - Pieces of plane

- $p \in g1D$ if II has rank 1 at $(p, h(p))$
  - parabolic point
  - $K = 0$, $H \neq 0$
  - Pieces of cones or cylinders

- $p \in g2D$ if II has rank 2 at $(p, h(p))$
  - $K \neq 0$
  - Elliptic point, $K > 0$
  - Hyperbolic point, $K < 0$
  - Non-linear behavior in any direction

Geometric segmentation

- Example: Blurred square

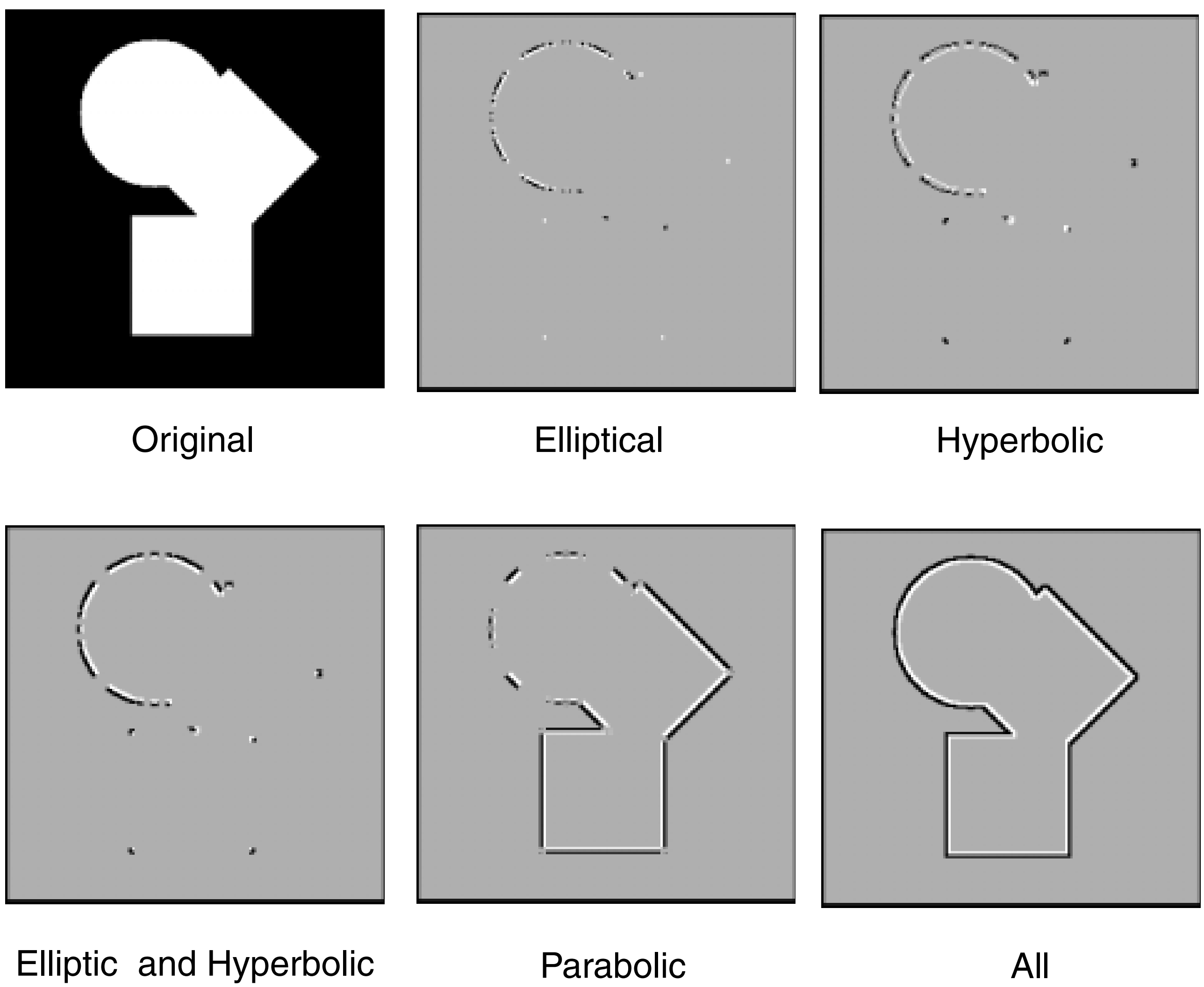



- Parabolic and elliptic points
  - High changes in curvature
  - High frequencies

- Non-linear high pass filters

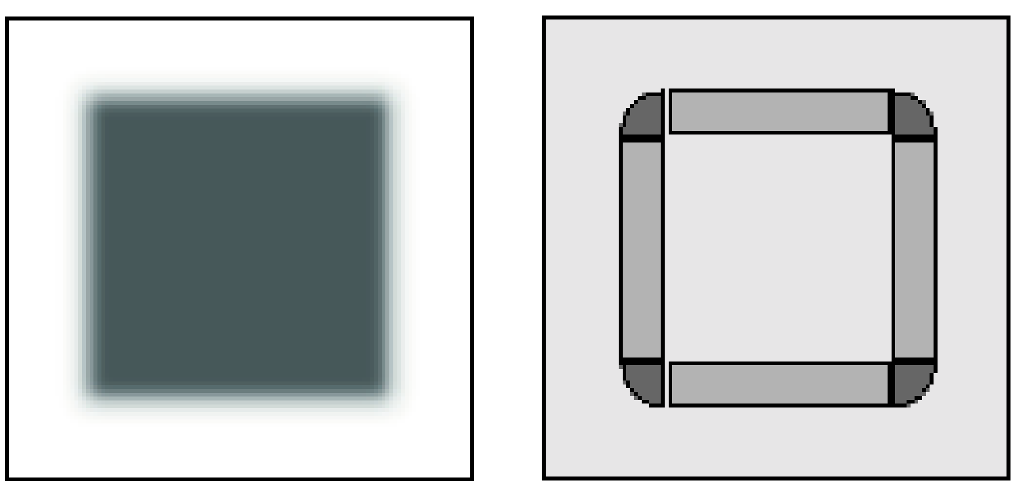
Geometric Segmentation

- Non-linear high pass filters



Perceptual Segmentation

- Image $h\colon U \subset \mathbf{R}^2 \to \mathbf{R}$,
  - $p \in U$
  - V a neighborhood of p in U
- $p \in 0D$ if h is constant on V
- $p \in 1D$ if V is foliated by parallel line segments
- $p \in 2D$ if it is neither $0D$ or $1D$
- Example (Blurred square)



Geometric and Perceptual Segmentations

- Relationship

$$0D \subset g0D$$
$$1D \subset g0D \cup g1D$$
$$2D \subset g1D \cup g2D$$

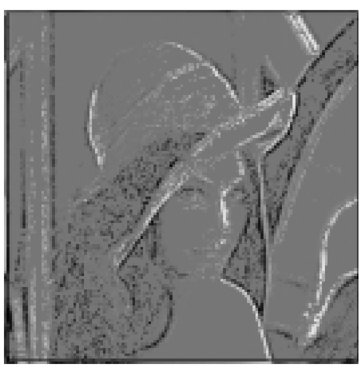
- Reconstruction problem
  - From perceptual segmentation
  - From geometric segmentation

Reconstruction

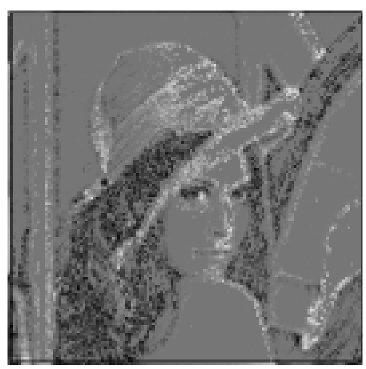
- Reconstruction from elliptic points
  - Possible but unstable (noise)
  - Scale space techniques
- Reconstruction from elliptic + parabolic points
  - Possible and stable
- Reconstruction from boundary
  - D. Marr conjecture
  - Problem: Precise definition of boundary
  - Boundary = Elliptic + parabolic points

Reconstruction

- Reconstruction of Lena



1D



2D

1D + 2D

Volumetric Processing

- Volumetric object

$$f\colon U \subset \mathbf{R}^3 \to \mathbf{R}$$

- Monge hypersurface

$$(x, y, z) \mapsto (x, y, z, f(x, y, z)) \subset \mathbf{R}^4$$

- Gauss map

$$N\colon U \to S^3 \subset \mathbf{R}^4$$

- Second fundamental form

$$II_p(v) = \langle N'_p(v), v \rangle$$

- II_p is a symmetric form

Volumetric processing

- Basis of eigenvalues $\{e_1, e_2, e_3\}$ of T_pM

$$N'_p(e_1) = \lambda_1 e_1$$
$$N'_p(e_2) = \lambda_2 e_2$$
$$N'_p(e_3) = \lambda_3 e_3$$

- Matrix of II_p

$$II_p = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

- Gaussian curvature

$$K = \det(II_p) = \lambda_1 \lambda_2 \lambda_3$$

- Mean curvature

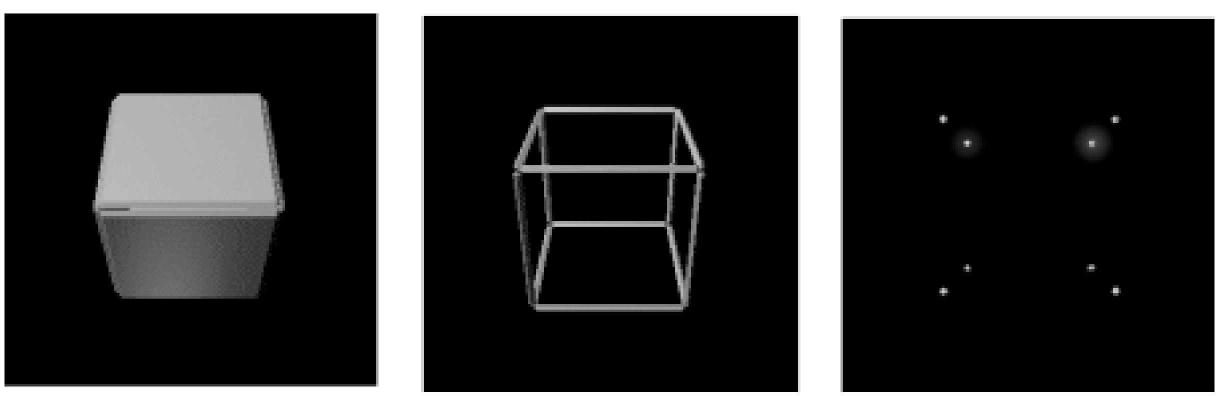
$$H = \mathrm{trace}(II_p) = \lambda_1 + \lambda_2 + \lambda_3$$

- Scalar curvature

$$S = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3$$

Geometric classification

- p is $g0D$ is II_p has rank 0
- p is $g1D$ is II_p has rank 1
- p is $g2D$ is II_p has rank 2
- p is $g3D$ is II_p has rank 3
- Example (Blurred cube)

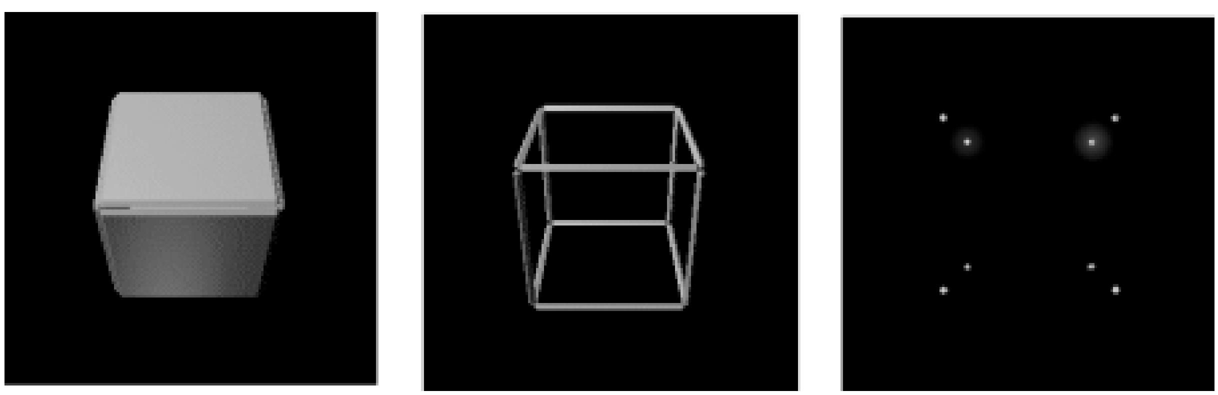


- $g0D$ points are called *planar points*

Geometric classification

- Relation between rank and curvatures
  - II_p has rank 0 $\Leftrightarrow H = K = S = 0$
  - II_p has rank 1 $\Leftrightarrow K = S = 0$ and $H \neq 0$
  - II_p has rank 2 $\Leftrightarrow K = 0$ and $S \neq 0$
  - II_p has rank 3 $\Leftrightarrow K \neq 0$

- High-pass non-linear filters (edges)



- Scale invariance

- Reconstruction from non-planar points
  - Possible and stable

An application: Video cut segmentation

- Shot
  - A video sequence with a continuous camera motion
- Cut
  - editing operation of joining shots without transition effects

1 2 3 4 5 6 7 8 9 10 11 12

- Video cut segmentation
  - Separate video shots in a video sequence

Video cut segmentation

- Some applications
  - video indexing
  - video query
  - video mosaics

- Problems
  - Real time (for some applications)
  - Presence of noise
  - False cuts

- Approach to solve the problem
  - Classification problem
  - Discriminating function
  - Use feature space

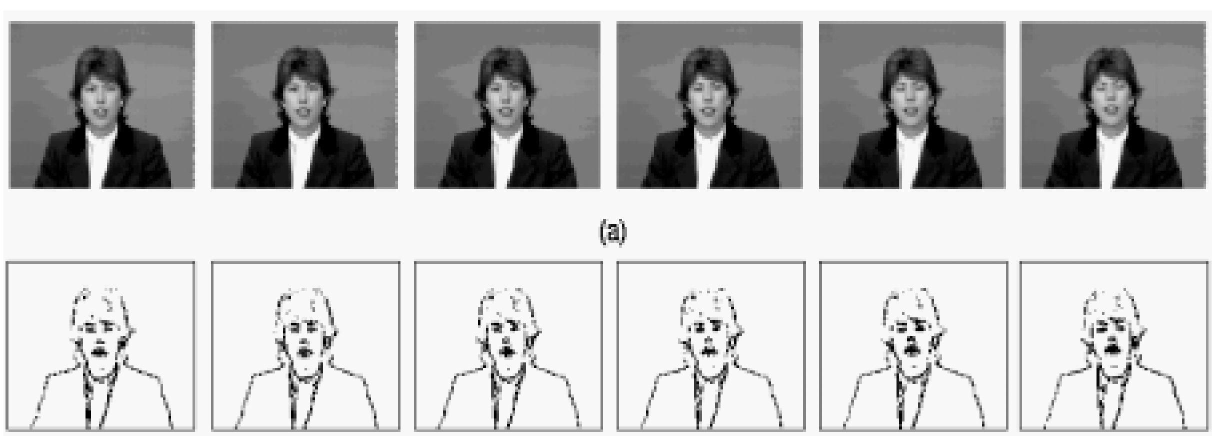
Video cut detection

- Feature operator: Planar filter
  - $f : U \subset \mathbf{R}^3 \to \mathbf{R}$ volumetric object.
  - For each $p \in U$

$$F(p) = \begin{cases} 0, & \text{if } p \text{ is planar} \\ 1, & \text{if } p \text{ is } g1D,\ g2D \text{ or } g3D \end{cases}$$

- High pass filter

- Example

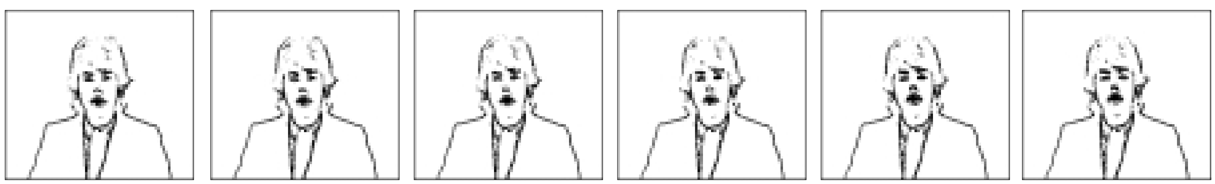


- Not an edge detector of each frame
  - Takes into account temporal variation

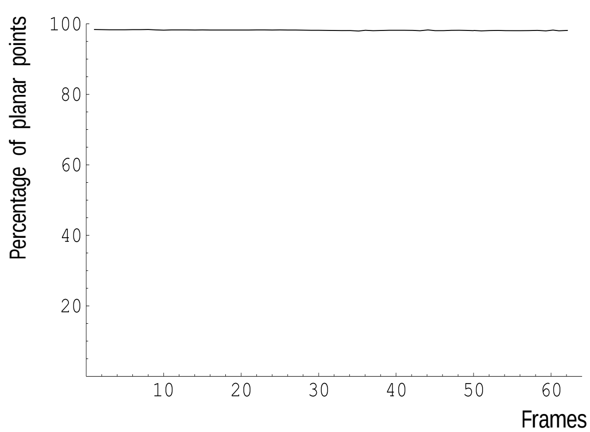
Video cut detection

- Discriminating function
  - Percentage of planar points

(a)



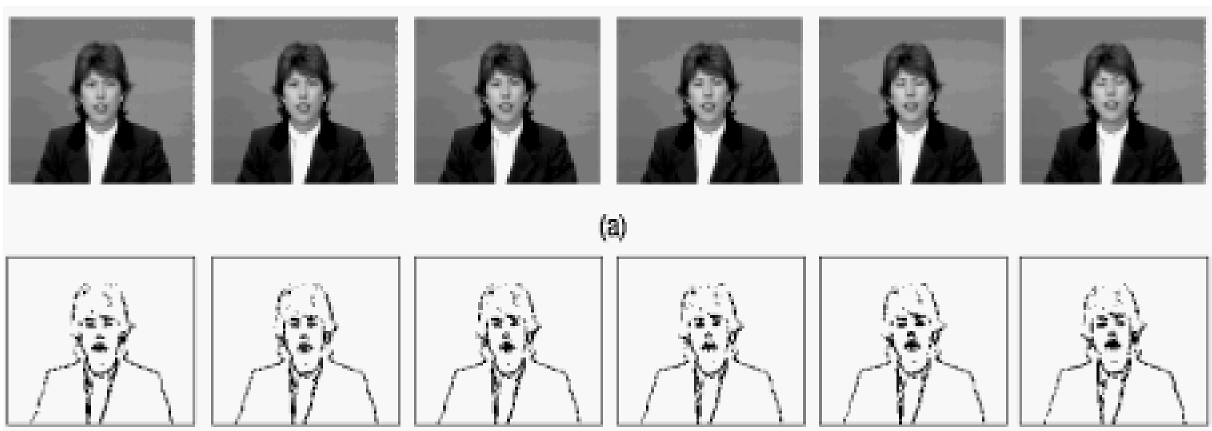
(b)



(c)

Video cut detection

- Percentage of planar points



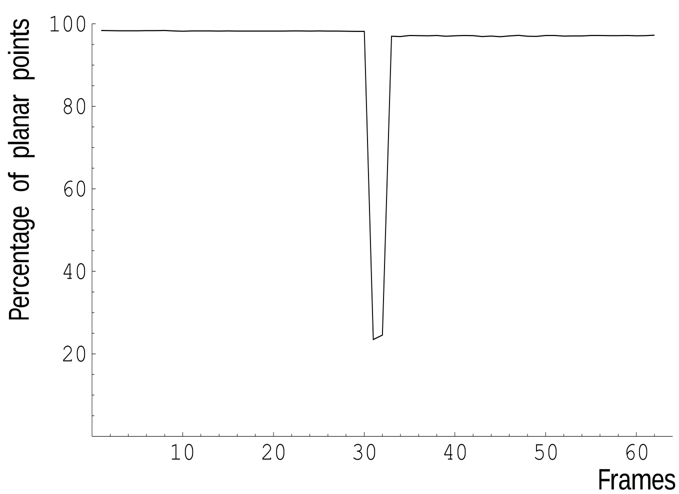
$frame$	$\#g3D$	$\#g2D$	$\#g1D$	$\#g0D$	$\%g0D$
2	143	154	960	78607	97.04
3	234	199	872	78559	96.98
4	152	142	812	78758	97.23
5	185	160	810	78709	97.17
6	140	183	847	78694	97.15
7	147	135	841	78741	97.21
8	221	199	827	78617	97.05
9	157	159	862	78686	97.14
10	117	152	844	78751	97.22
11	103	171	866	78724	97.19

Video cut detection

- A cut frame reduces the number of planar points
- Example

(a)

(b)

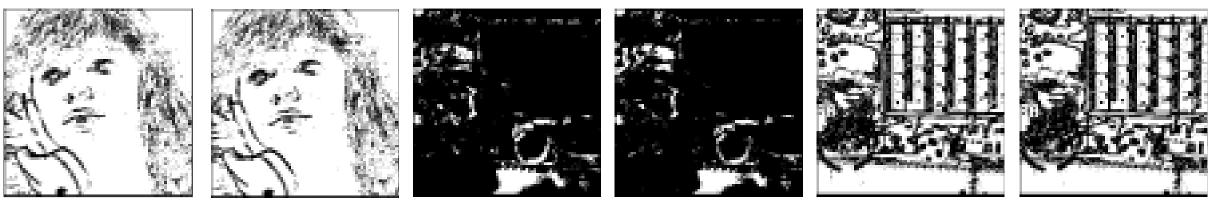


(c)

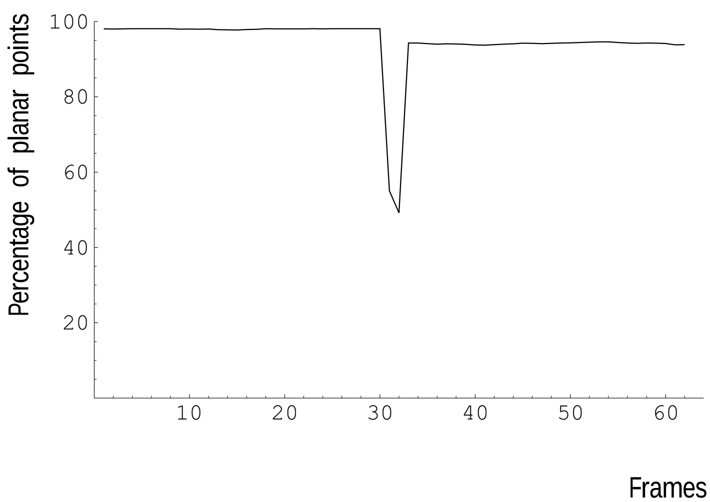
Video cut detection

- Example

(a)



(b)



(c)

Video cut detection

- Example with noisy sequences