

Geometric and Solid Modeling

Problems

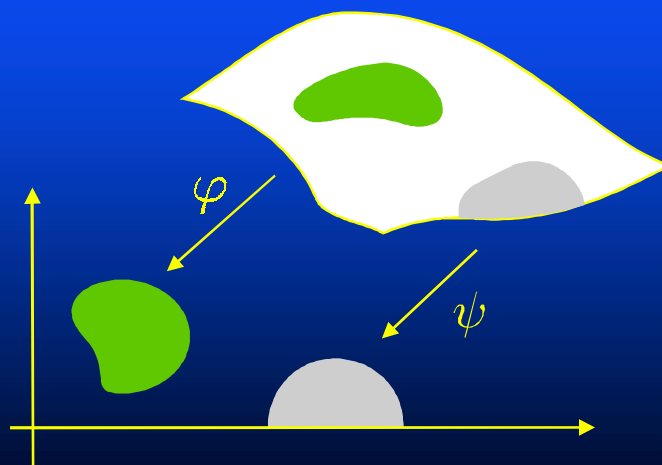
- **Define a Solid**
- **Define Representation Schemes**
- **Devise Data Structures**
- **Construct Solids**

Mathematical Models

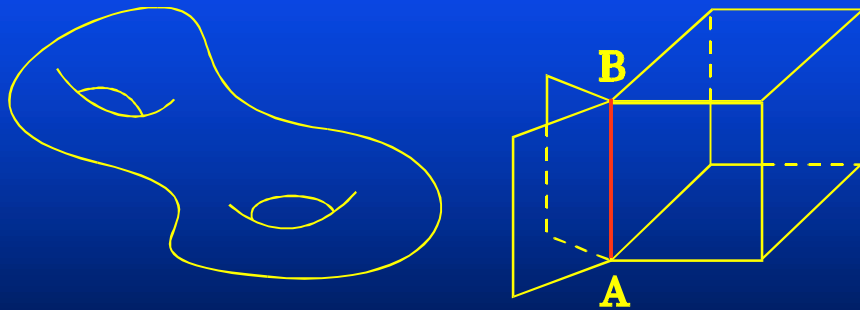
- Points
- Curves
- Surfaces
- Solids

“A shape is a set of Points”

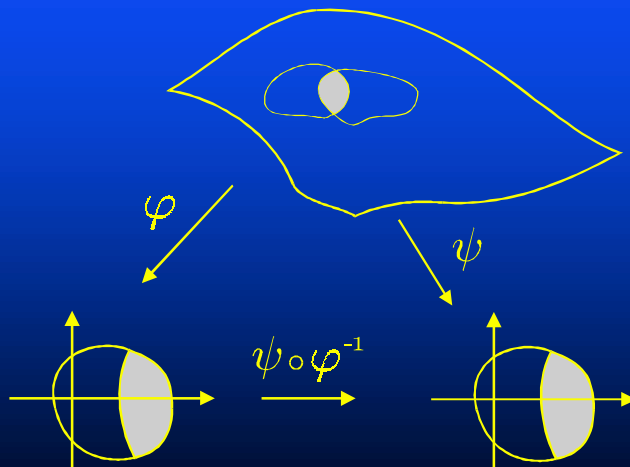
Manifold



Examples



Change of Coordinates



Classes of Manifolds

- Topological
- Combinatorial
- Differentiable
- Analytical
- Riemannian

Classes of Manifolds



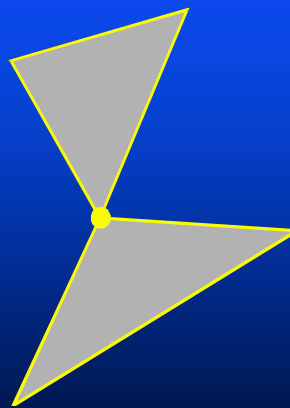
Manifolds and shapes

- Curves → 1D-manifolds
- Surfaces → 2D-manifolds
- Solids → 3D-manifolds with boundary
- “2D-solids” → 2D-manifolds with boundary

2-Dimensional Examples



manifold

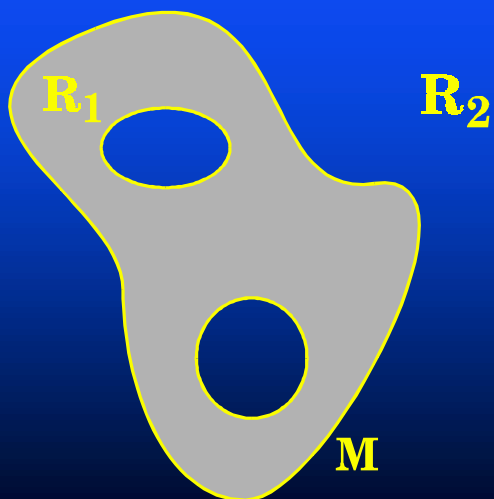


non-manifold

Surfaces and Solids

- The boundary of a solid is a compact surface
- *Jordan-Brouwer Theorem*
 - A compact curve defines a “2D-solid” in the plane
 - A compact surface defines a solid in the space

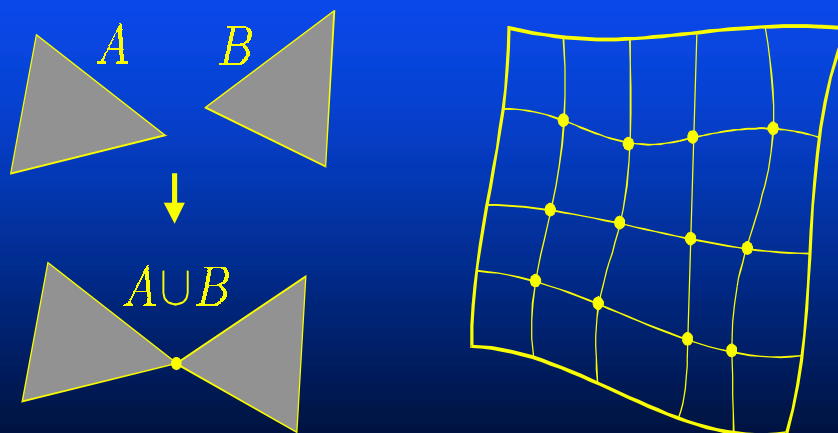
Jordan-Brouwer Theorem



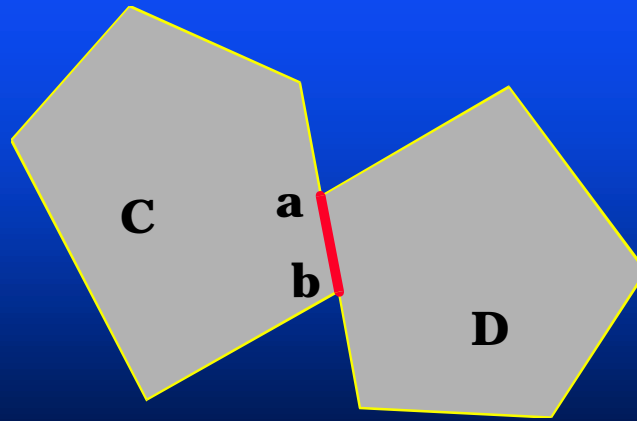
Limitation of Manifolds

- **Homogeneous Geometry.**
 - **No dangling edges and faces**
- **Non-closure under Boolean Operations.**
 - **Limitation in constructing models**
- **Triviality of Local Topology.**
 - **Unable represent internal structures**

Limitation of Manifolds



Non-homogeneous Topology



- Useful for auxiliary information

Non-manifold Shapes

- What is a non-manifold?
- Literature on Modeling
 - **Data structures**
 - **Skip one level of abstraction**

Regular Sets

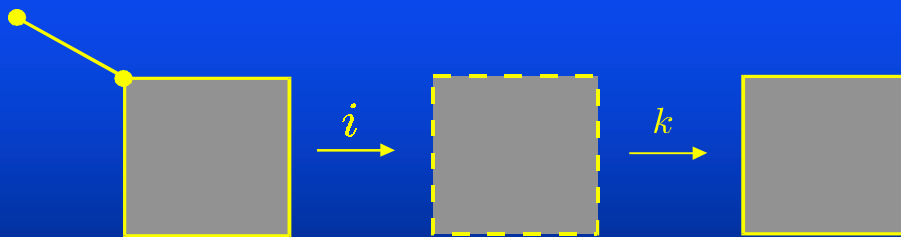
- R-Sets (A. Requicha, 1977)
- Regularity

$$X = k(i(X))$$

i - topological interior operator

k - topological closure operator

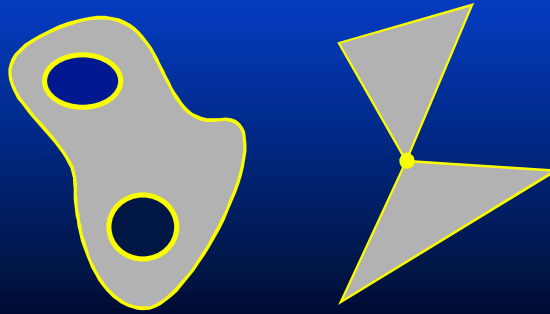
Regular Sets



- $i \rightarrow$ homogeneous topology
- $k \rightarrow$ constructs boundary

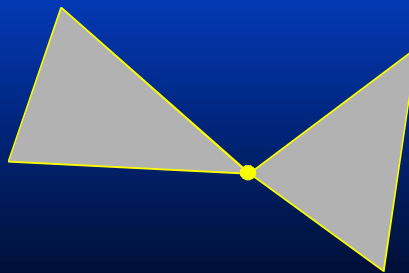
R-Sets and Shapes

“A *solid* is an R-Set in 3-dimensional euclidean space”



R-Sets and Manifolds

- Manifold solid \longrightarrow R-Set solid
- R-Set solid $\not\longrightarrow$ Manifold solid



Limitation of R-Sets

- **Homogeneous Geometry**
- **Difficult to define non-solid shapes**
 - **curves, surfaces**
- **Boundary access difficult**

Space Decompositions

Subdivision + Structure



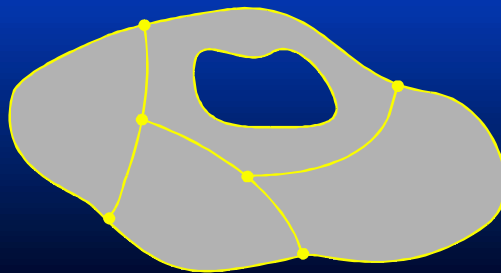
Geometry + Topology

Space Decompositions

- Space partition
- Cell Decomposition
 - Affine Cell decomposition
 - Simplicial decomposition
 - Triangulation
- Stratifications

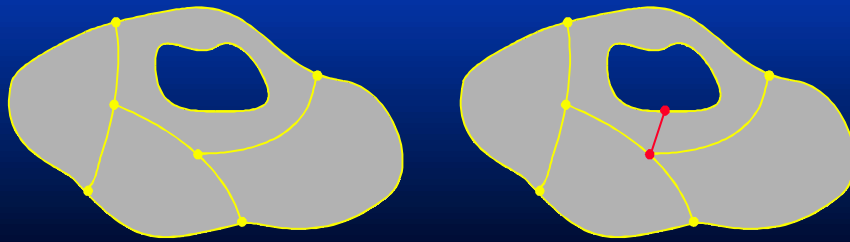
Space Partition

- Collection of sets U_i , $i=1, \dots, n$
- $X = \bigcup U_i$
- $U_i \cap U_j = \emptyset$ if $i \neq j$



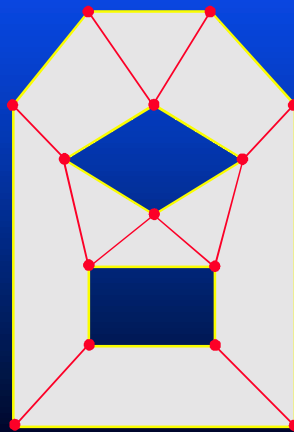
Cell Decomposition

- Each partition is a point or an open disk
- The boundary has only lower dimensional cells



Affine Cell Decomposition

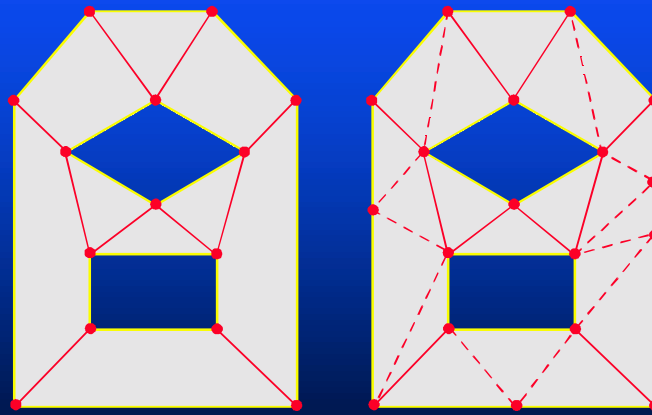
- Cells with linear structure



Simplicial Decomposition

- Simplex
- Each cell is a *simplex*
- Generalizes triangulation

Simplicial and affine decompositions

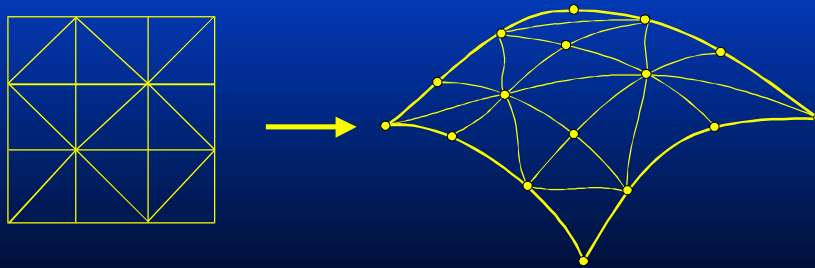


affine

simplicial

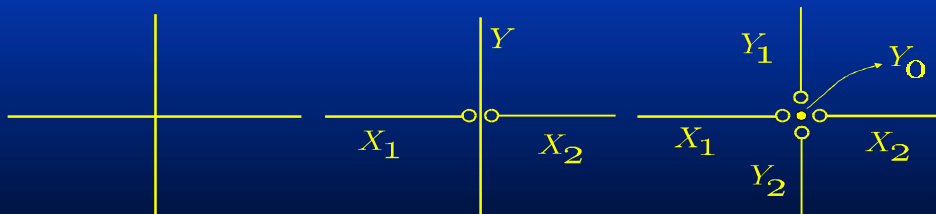
Triangulation

- Affine
- Topological



Stratification

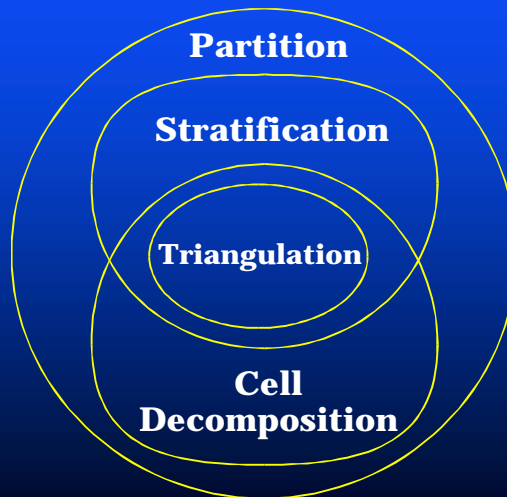
- Each partition set is a manifold
- Whitney Regularity
 - boundary conditions



Space Decompositions

- Uniform
- Adaptive
- Arithmetic Structure
 - **Boolean Algebra**

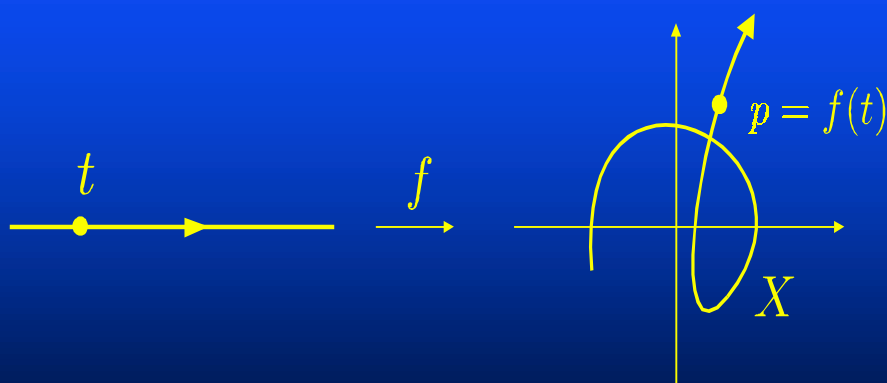
Space Decompositions



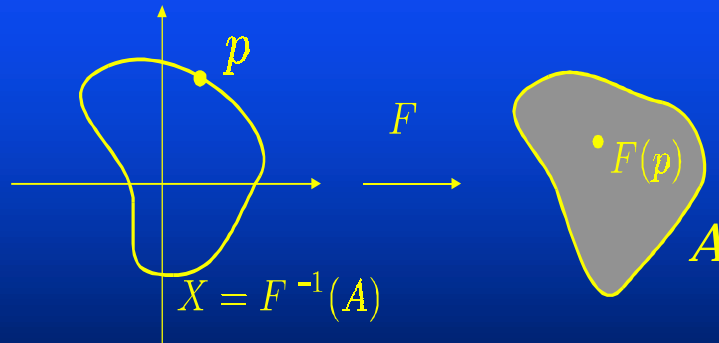
Mathematical Descriptions

- Equations to Describe Sets
- Parametric
- Implicit

Parametric description



Implicit description



- **Common case:** $A = \{0\}$

Mathematical descriptions

- **Advantages**
- **Disadvantages**
- **Description extent**
 - **Local**
 - **Global**
- **Decompositions and extent**

Algebraic descriptions

- **Implicit description**
- **polynomial primitive**

$$F: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$F(x_1, \dots, x_n) = \sum_{j=1}^m a_j x_1^{e_{1,j}} x_2^{e_{2,j}} \dots x_n^{e_{n,j}}$$

$$a_j \in \mathbb{R} \quad e_{i,j} \in \mathbb{Z}^+$$

Polynomial primitive

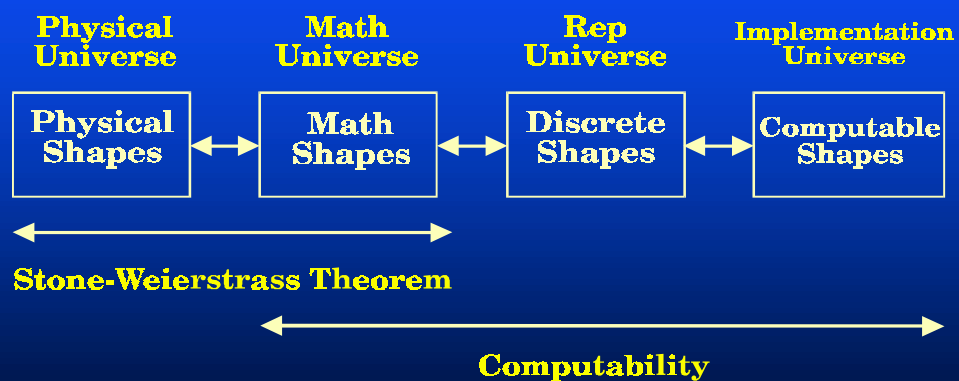
$$F > \mathbf{0}, \quad F < \mathbf{0}, \quad F \geq \mathbf{0}, \quad F \leq \mathbf{0}, \quad F = \mathbf{0}$$



Semi-algebraic sets

- $\cap, \cup, -$ of polynomial primitives
- Boolean algebra
- Polynomial shapes
- Piecewise algebraic shapes

Why piecewise algebraic?



Semi-algebraic sets

- **Decidability**
 - **Point membership classification**
 - **Existence of a procedure**
- **Definability**
 - **Quantifier elimination**
 - **Tarsky's theorem**

Representation in Geometric and Solid Modeling

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Representation schemes

- Pure primitive instancing
- Constructive schemes
- Decomposition schemes
- Hybrid schemes

Pure Primitive Schemes

- Primitives
 - Families of objects
- Parameters
- Space transformations
- Example
 - *(sphere, x, y, r)*

Pure Primitive Schemes

- **Range of Representation**
 - **Depends on the primitives**
- **Robust but very limited**

Constructive Schemes

- **Primitives**
- **Space transformations**
- **Set operations**
- **Basic blocks to complex objects**

Constructive Solid Geometry

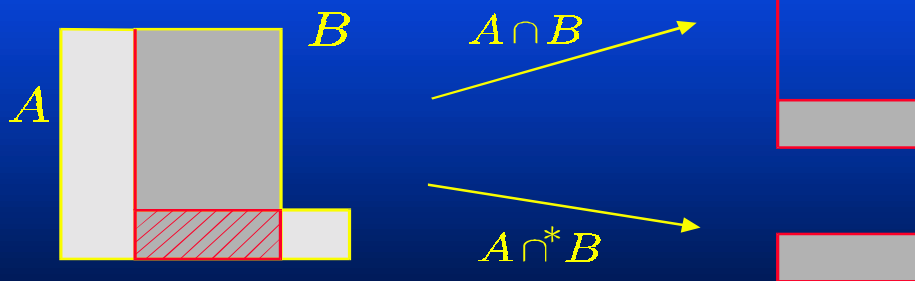
- Regular sets
- Regularized set operations

$$X \cap^* Y = ki(X \cap Y)$$

$$X \cup^* Y = ki(X \cup Y)$$

i - interior operator
k - closure operator

Constructive Solid Geometry



Constructive Solid Geometry

- Range of representation
 - Solids defined as *regular sets*
- Homogeneous geometry

CNRG

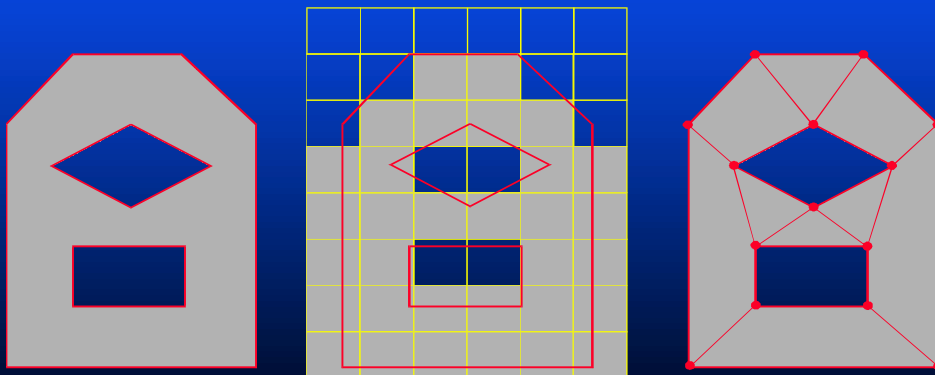
- Constructive non-regularized geometry
- *Rossignac & Requicha, 1990*
 - Extension of CSG
 - Arbitrary partitions
 - Non-homogeneous geometry

Decomposition Schemes

- Pieces + Structures
- Space based
- Object based
- Adaptive
- Uniform

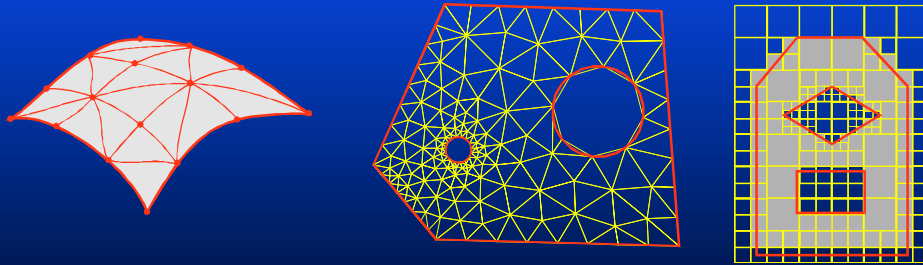
Decomposition Schemes

Space and Object Based

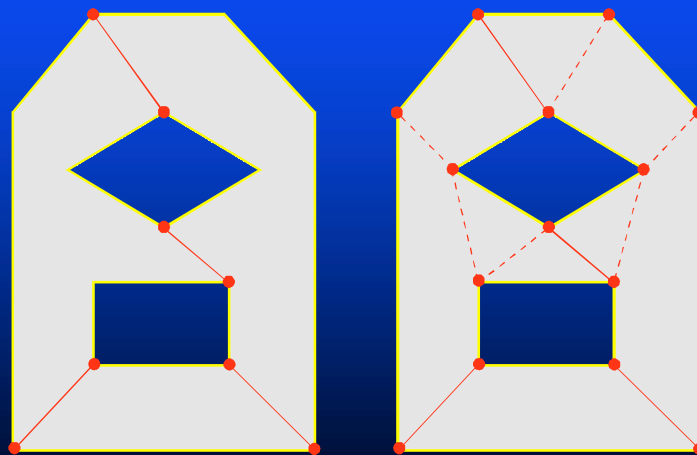


Decomposition Schemes

Uniform and adaptive



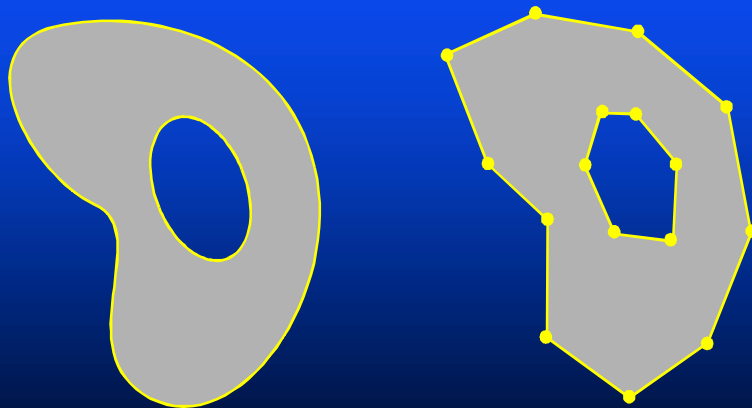
Cell Decomposition



Boundary Based Decompositions

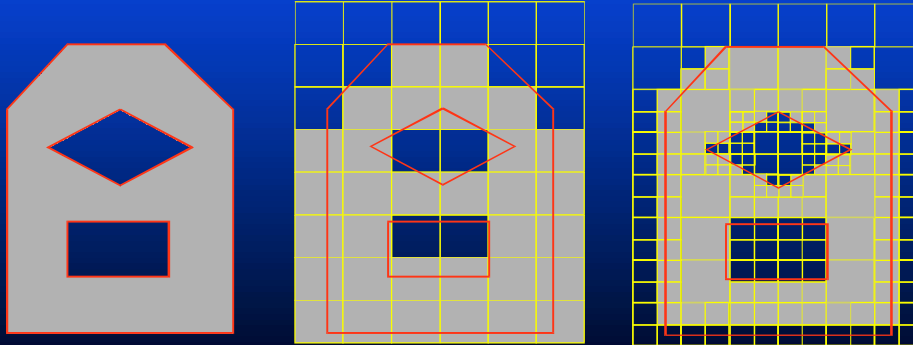
- The boundary defines the solid
 - *Jordan-Brouwer theorem*
- Boundary Representation
- Decomposition scheme?

Boundary Based Decompositions



Spatial Subdivision Representation

- Subdivides the ambient space



SGC

- Selective Geometric Complex
- *Rossignac, O'Connor, 1990*
 - Object based
 - Extends cell decomposition
 - “Cells” with arbitrary topology
 - Semi-algebraic support

Hybrid Schemes

- Advantages
- Disadvantages
- Example
 - Adaptive Spatial Subdivision
+
Boundary representation

Representation Conversion

- CSG to Boundary Representation
- Boundary Representation to CSG
- Rep-X \longleftrightarrow Rep-Y
 - Theory
 - Applications

Data Structures for Geometric and Solid Modeling

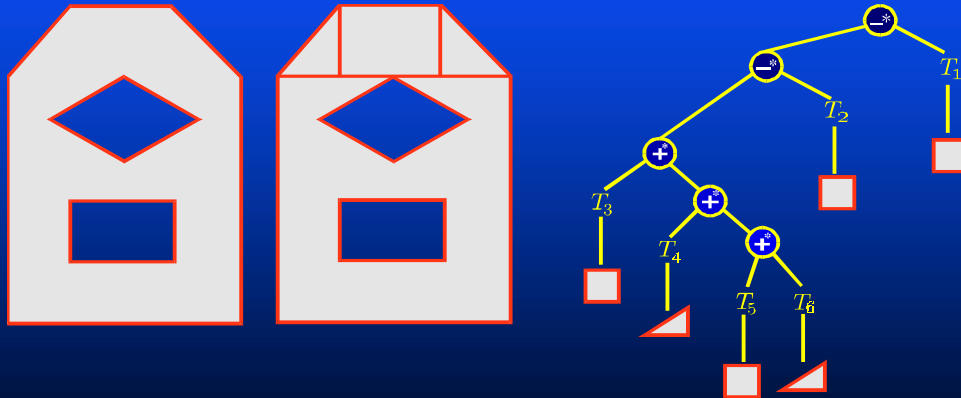
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Constructive Representations

- **Set operations**
- **Transformations**
- **Constructive Solid Geometry**
 - **CSG tree**
 - **Operations in the nodes**
 - **Transformations in the nodes**
 - **Primitives in the leaves**

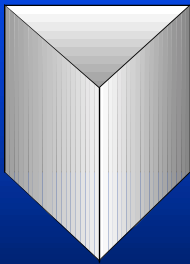
CSG Tree



Spatial Data Structure

- Flat
 - List of cells
- Hierarchical
 - Topological graph

Geometrical Hierarchy



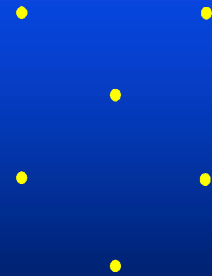
Shell



Faces

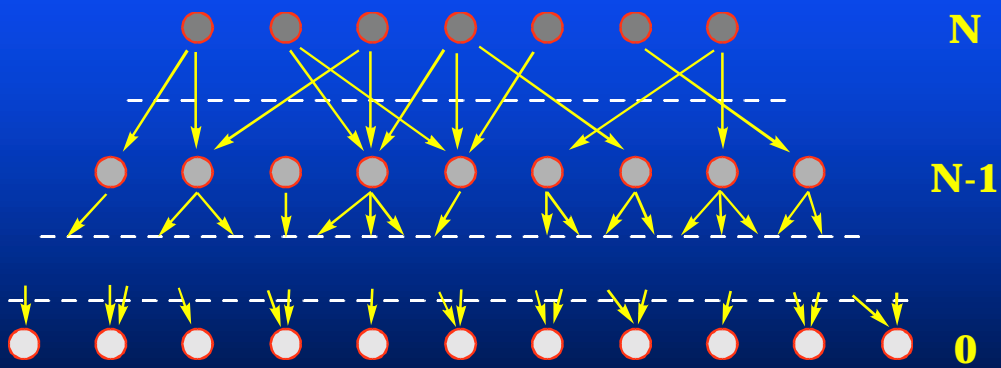


Edges



Vertices

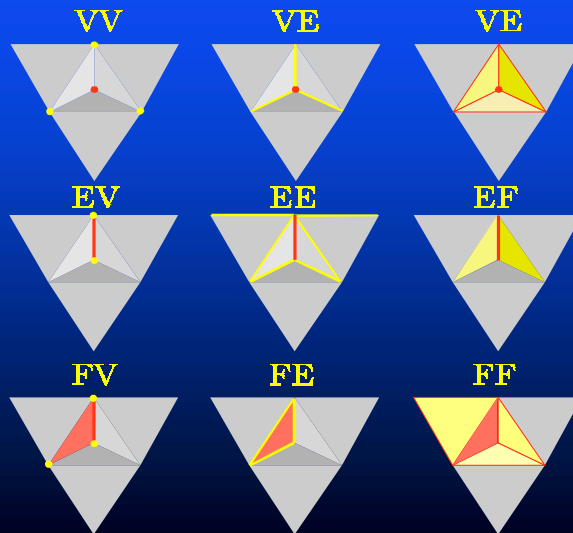
Topological Graphs



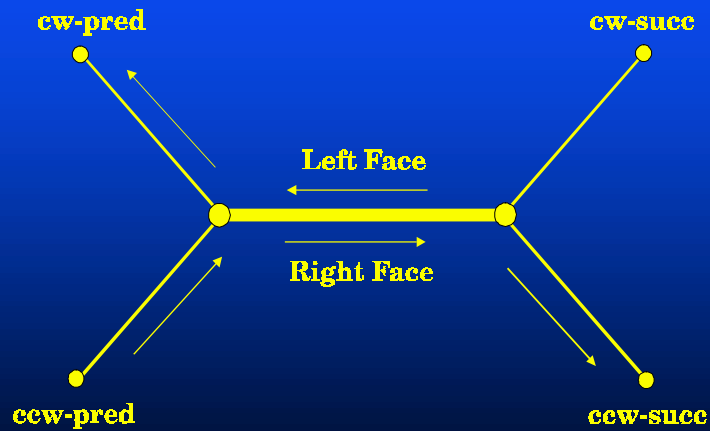
Manifold Decomposition

- **Boundary Representation**
- **Faces, Edges, Vertices**
 - **Nine adjacency relationships**
 - **V-V V-E V-F**
 - **E-V E-E E-F**
 - **F-V F-E F-F**

Adjacency Relations



Winged Edge



Trees

- **Nested subdivisions of Space**
- ***N-Trees*** (Quadtrees, Octrees)
- ***K-d Trees***
- ***BSP-Trees***
- **Restricted trees**
 - ***Adaptive decompositions***

Modeling Techniques

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Modeling Techniques

- **Constructing the Mathematical Universe**
- **Construction**
- **Check for consistency**
 - **Belong to the Math Universe?**

Modeling Techniques

- **Modeling with primitives**
- **Curve and surface construction**
- **Sweep Techniques**
- **Deformation**
- **Free form**
- **Data set**
- **Algorithmic techniques**

Modeling with Primitives

- **Constructive representations**
- **Pure primitive representations**
- **Building blocks paradigm**

Curve and Surface Construction

- **Surfacing systems**
 - **Coons, Bezier, De Casteljau**
- **Parametric**
 - **Spline carpentry**
- **Implicit**
 - **Marching cubes**

Sweep Techniques

Dimension N



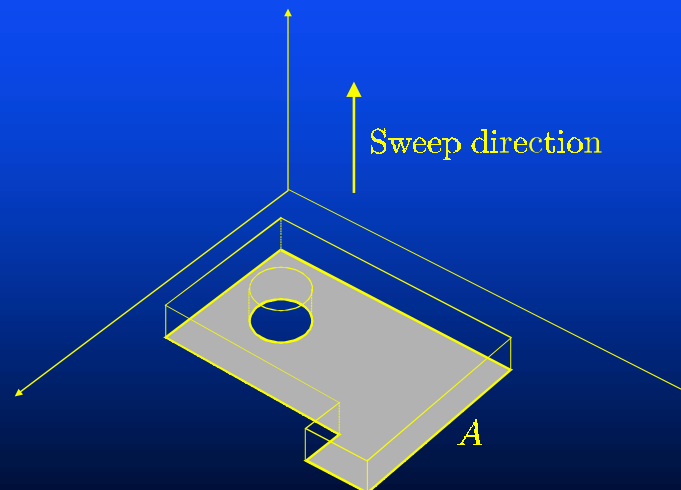
Sweep

Dimension N+1

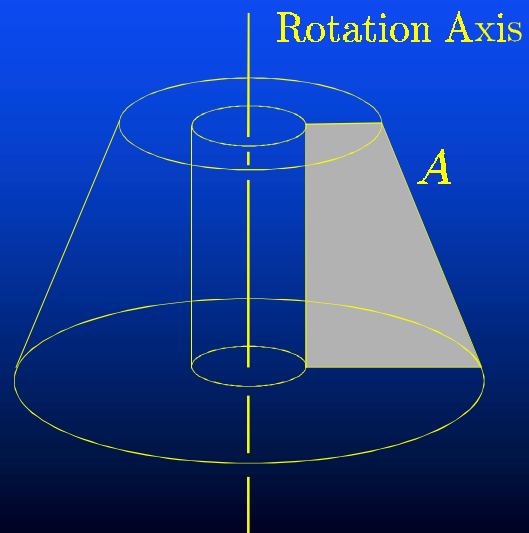
Sweeping Techniques

- Translational
- Rotational
- General
- Generative Modeling

Translational Sweep

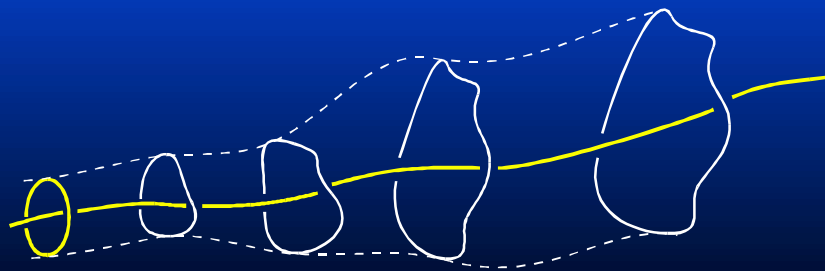


Rotational Sweep



General Sweep

- **Powerful Technique**
- **Consistency checking difficult**



Model Deformation

- **Space transformations**
 - **Projective transformations**
 - **Affine**
 - **Rigid motion**
- **Non-linear deformations**

Data Set

- **Sample to model**
- **Reconstruction Problem**
 - **Scattered data interpolation**
 - **Cross sections**
 - **Inverse problems**

Algorithmic Techniques

- **Non-linear deformations**
- **Inverse problems**
- **Physically based modeling**
- **Variational modeling**

“It is necessary to have a better understanding of algorithmic modeling and its relation with geometric and solid modeling”