Moebius Transformations and Omnidirectional Images

Luiz Velho
IMPA

Outline

- Moebius Transformations
  - Mathematical Fundamentals
- Omnidirectional Images
  - Basic Concepts
  - 360 Panoramas
- Applications
  - Wide Field of View

Moebius Transformations

Möbius Transformations

• Complex Map

\[ M : \mathbb{C} \mapsto \mathbb{C} \]

• Definition:

\[ M(z) = \frac{az + b}{cz + d} \]

with

\[(ad - bc) \neq 0\]

Anatomy of \( M \)

• Decomposition into Sequence

\[ m_4 \circ m_3 \circ m_2 \circ m_1(z) \]

\[
m_1(z) = z + \frac{d}{c} \quad \text{translation}
\]

\[
m_2(z) = \frac{1}{z} \quad \text{inversion}
\]

\[
m_3(z) = \frac{(be-ad)z}{cz} \quad \text{scaling and rotation}
\]

\[
m_4(z) = z + \frac{a}{c} \quad \text{translation}
\]

Fixing the Inversion

• Point at Infinity \( \infty \)

\[
\frac{1}{\infty} = 0 \quad \frac{1}{0} = \infty
\]

• Extended Complex Plane

\[ \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \]
Riemann Sphere

- Stereographic Projection

\[ \hat{z} = (\theta, \phi) \mapsto z = \cot(\phi/2) e^{i\theta} \]

Complex Projective Space

- Isomorphism

\[ z \mapsto w = M(z) \quad \text{in} \quad \hat{C} \]

induces

\[ \hat{z} \mapsto \hat{w} \quad \text{in} \quad \Sigma \]

- Geometry and Algebra

\[
\begin{align*}
\text{Riemann} & \quad \longleftrightarrow \quad \text{Complex} \\
\text{Sphere} & \quad \text{Plane}
\end{align*}
\]

Properties of \( M \)

- Projective Linear Group (Lie Group) \( PGL(2, \mathbb{C}) \)

- Preservation of:
  - Circles (lines to circles)
  - Angles (conformal)
  - Symmetry (w.r.t. circles)

Defining \( M \)

- Images of 3 points (e.g)

\[
(a/b), \quad (b/c), \quad (c/d)
\]

- Ratios and Uniqueness

\[
\frac{az + b}{cz + d} = M(z) = \frac{ka z + kb}{kcz + kd}
\]

- Normalization

\[(ad - bc) = 1\]

Homogeneous Coordinates

- Ratio of 2 complex numbers

\[
z = \frac{\delta_1}{\delta_2} = [\delta_1, \delta_2] \neq [0,0]
\]

- Two Cases

\[
\begin{align*}
\delta_2 & \neq 0 \\
& \qquad z = \frac{\delta_1}{\delta_2}
\end{align*}
\]

\[
\begin{align*}
\delta_2 & = 0 \\
& \qquad z = \infty
\end{align*}
\]

Cross Ratio

- The unique

\[ z \mapsto w = M(z) \]

sending

\[ q, r, s \mapsto \tilde{q}, \tilde{r}, \tilde{s} \]

\[
\frac{(w - \tilde{q})(\tilde{r} - \tilde{s})}{(w - \tilde{s})(\tilde{r} - \tilde{q})} = [w, \tilde{q}, \tilde{r}, \tilde{s}] = [z, q, r, s] = \frac{(z - q)(r - s)}{(z - s)(r - q)}
\]

- Theorem:

If \( M \) maps 4 points \( p, q, r, s \mapsto \tilde{p}, \tilde{q}, \tilde{r}, \tilde{s} \)

then, the cross-ratio is invariant.
Orientation Properties

• Maps Oriented Circles to Oriented Circles
  s.t. Regions are mapped accordingly

Fixed Points

• Solution of
  \( z = M(z) \)

• \( M \) has at most **two** fixed points
  except for \( \text{Id.} \)

• For \( M \) Normalized
  \( \xi_{\pm} = \frac{(a-d) \pm \sqrt{(a+d)^2-4}}{2c} \)

\[ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

\[ \xi_{\pm} = \frac{(a-d) \pm \sqrt{(a+d)^2-4}}{2c} \]

\[ M(z) = Az + B \]

\[ z \mapsto e^{i\alpha}z \]

\[ z \mapsto \rho e^{i\alpha}z \]

**M - Classification**

• Fixed Point at Infinity : \( c = 0 \)

  \( M(z) = Az + B \)

• Basic Types
  - Elliptic
  - Hyperbolic
  - Parabolic
  - Loxodromic

**Elliptic Transform**

• Rotation

  \( z \mapsto e^{i\alpha}z \)

  • two fixed points
    \((0, \infty)\)

**Hyperbolic Transform**

• Scaling

  \( z \mapsto \rho z \)

  • two fixed points
    \((0, \infty)\)

**Loxodromic Transform**

• Rotation and Scaling

  \( z \mapsto \rho e^{i\alpha}z \)

  • two fixed points
    (combination of elliptic and hyperbolic)
Parabolic Transform
- Translation
  \[ z \mapsto z + b \]
- one fixed point at \( \infty \)

Omnidirectional Images

Basic Concepts
- Plenoptic Function
- Capturing Light Fields
- 360 Panoramas
- Parametrization and Projections

Plenoptic Function
Complete description of Visual Information in a 3D environment
- \( I_\lambda = P(x, y, z, \theta, \phi, t) \)
  Holographic Image
- \( P : \mathbb{R}^3 \times S^2 \times \mathbb{R} \mapsto \mathcal{E} \)
  6D Phase Space

Light Field
A Slice of the Plenoptic Function
- Structured Sampling of \( P \)
  - example: Camera

\[ \begin{array}{c}
\text{x,y,z fixed} \\
\text{Ray Space}
\end{array} \]

Ray Space

Omnidirectional Image
The Set of All Rays incident at a point \( (x,y,z) \)
- Spherical Light Field \( = \) 360 degrees

Representation of Choice
Capturing Point Light Fields

- Omnidirectional Cameras
  - Catadioptric
  - Dioptric
  - Multi-Camera

Catadioptric Cameras

- Mirror-Based (parabolic or hyperbolic)

Dioptric Cameras

- Fish Eye Lenses

Multi-Camera Systems

- Point Grey’s Ladybug (6 Perspective Cameras)

Panoramic Surfaces

Generalized Support for Visual Information

- Data Representation
  - example: Cylindrical Panorama

Parametrizations

Maps 2D Surface to Planar Domain

- Coordinate Systems
  - example: Cylindrical Mapping
360° Image Formats

Omnidirectional Panoramas

- Parametrizations of the Sphere
  - Lat-Long
  - Cube Map
  - Azimuthal
  - Stereographic (*)

Equirectangular Projection

- Latitude-Longitude Mapping  (e.g., Flickr)

Cube Mapping

- 6 Perspective Projections
  suitable for CG rendering

Azimuthal Projection

- Hemispherical Mapping

Applications to 360 Cinema

Exhibition

- Viewing Scenarios
### Field of View
- Reference to Observer
  - 30 to 90 degrees

### Film Language
- Conventional Cinema
  - HD Television
  - Theater Panavision
- 360 Degrees Dome
  - Omnimax
  - Dome Master

### Conventional Cinema
- Camera Moves
  - Track Pan / Tilt Zoom

### 360 Camera
- Camera Moves
  - Track
  - Pan / Tilt
  - Zoom
  - yes, maybe, ?

### Authoring Issues
- OBS: Post-Production
- Passive
  - Movies
- Interactive
  - Google Street View
- Immersive
  - AR Cinema

### 360° Image Transforms
- Moebius Transformations for Manipulation and Visualization of Spherical Panoramas
- Current Research at VISGRAF Lab
- Collaboration with
  - Leonardo Koller Sacht
  - Luis Penaranda
Math of Camera Moves
- Omnidirectional Images and Möbius Transformations
  - Pan / Tilt ⇔ Elliptic Transform
  - Zoom ⇔ Hyperbolic Transform
  - Perspective ⇔ Parabolic Transform

Transformation Pipeline
- Möbius Mapping

Example
- Extreme Zoom

Comparison
- Alternative Projections

Current Work
- Preserving Lines
- Perspective Control
Questions?

Improving Projections of Panoramic Images with Hyperbolic Möbius Transformations

L. Peñaranda  L. Sacht  L. Velho

IMPA