Moebius Transformations and Omnidirectional Images

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Outline

- Moebius Tranformations
  - Mathematical Fundamentals
- Omnidirectional Images
  - Basic Concepts
  - 360 Panoramas
- Applications
  - Wide Field of View

Moebius Transformations

Möbius Transformations

- Complex Map

$$M : \mathbb{C} \mapsto \mathbb{C}$$

- Definition:

$$M(z) = \frac{az+b}{cz+d}$$

with

$$(ad - bc) \neq 0$$

Anatomy of *M*

- Decomposition into Sequence

$$m_4 \circ m_3 \circ m_2 \circ m_1(z)$$

$m_1(z) = z + \frac{d}{c}$	*translation*
$m_2(z) = \frac{1}{z}$	*inversion*
$m_3(z) = \frac{(bc-ad)}{c^2} z$	*scaling and rotation*
$m_4(z) = z + \frac{a}{c}$	*translation*

Fixing the Inversion

- Point at Infinity ∞

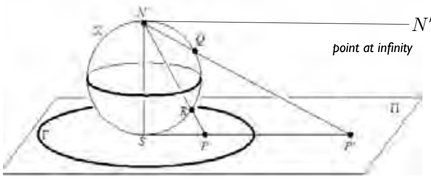
$$\frac{1}{\infty} = 0 \qquad \frac{1}{0} = \infty$$

- Extended Complex Plane

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

Riemann Sphere

- Stereographic Projection



$$\hat{z} = (\theta, \phi) \mapsto z = \cot(\phi/2)e^{i\theta}$$

Complex Projective Space

- Isomorphism

$$z \mapsto w = M(z) \quad \text{in} \quad \hat{\mathbb{C}}$$

induces

$$\hat{z} \mapsto \hat{w} \quad \text{in} \quad \Sigma$$

- Geometry and Algebra

Riemann Sphere ⇔ Complex Plane

Properties of *M*

- Projective Linear Group (Lie Group) $PGL(2, \mathbb{C})$
- Preservation of:
  - Circles (lines to circles)
  - Angles (conformal)
  - Symmetry (w.r.t. circles)

Defining *M*

- Images of 3 points (e.g)

$$(a/b), \quad (b/c), \quad (c/d)$$

- Ratios and Uniqueness

$$\frac{az+b}{cz+d} = M(z) = \frac{kaz+kb}{kcz+kd}$$

- Normalization

$$(ad - bc) = 1$$

Homogeneous Coordinates

- Ratio of 2 complex numbers

$$z = \frac{\delta_1}{\delta_2} = [\delta_1, \delta_2] \neq [0, 0]$$

- Two Cases

$\delta_2 \neq 0$

$$z = \delta_1/\delta_2$$

$\delta_2 = 0$

$$z = \infty$$

Cross Ratio

- The *unique*

$$z \mapsto w = M(z)$$

sending

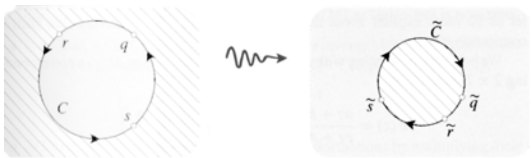
$$q, r, s \mapsto \tilde{q}, \tilde{r}, \tilde{s}$$

$$\frac{(w-\tilde{q})(\tilde{r}-\tilde{s})}{(w-\tilde{s})(\tilde{r}-\tilde{q})} = [w, \tilde{q}, \tilde{r}, \tilde{s}] = [z, q, r, s] = \frac{(z-q)(r-s)}{(z-s)(r-q)}$$

- Theorem:
  If M maps 4 points $p, q, r, s \mapsto \tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}$ *then, the cross-ratio is invariant.*

Orientation Properties

- Maps Oriented Circles to Oriented Circles
  - s.t. Regions are mapped accordingly



Fixed Points

- Solution of

$$z = M(z)$$

- *M* has at most <u>two</u> fixed points
  - except for Id.
- For *M* Normalized

$$\xi_{\pm} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}$$

M - Classification

- Fixed Point at Infinity : $c = 0$

$$M(z) = Az + B$$

- Basic Types
  - Elliptic
  - Hyperbolic
  - Parabolic
  - Loxodromic

Elliptic Transform

- Rotation

$$z \mapsto e^{i\alpha} z$$

- *two fixed points*

$(0, \infty)$

Hyperbolic Transform

- Scaling

$$z \mapsto \rho z$$

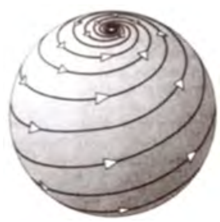
- *two fixed points*

$(0, \infty)$

Loxodromic Transform

- Rotation and Scaling

$$z \mapsto \rho e^{i\alpha} z$$



- *two fixed points*

(combination of elliptic and hyperbolic)

Parabolic Transform

- Translation

$$z \mapsto z + b$$

- *<u>one</u> fixed point at* ∞

Omnidirectional Images

Basic Concepts

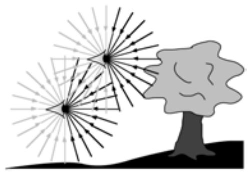
- Plenoptic Function
- Capturing Light Fields
- 360 Panoramas
- Parametrization and Projections

Plenoptic Function

Complete description of Visual Information in a 3D environment

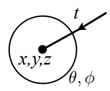
- $I_\lambda = P(x, y, z, \theta, \phi, t)$

  Holographic Image



- $P : \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \mapsto \mathcal{E}$

  6D Phase Space

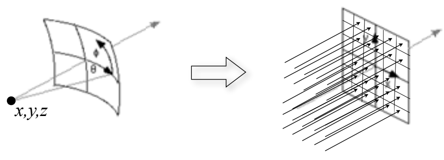


Light Field

A Slice of the Plenoptic Function

- Structured Sampling of P

  - example: *Camera* *Image*



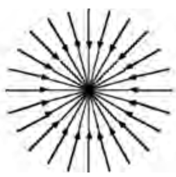
x,y,z fixed

Ray Space

Omnindirectional Image

The Set of <u>All Rays</u> incident at a point (x,y,z)

- Spherical Light Field = 360 degrees



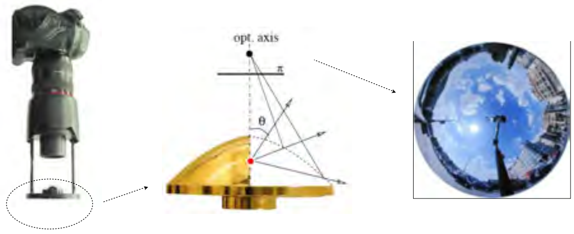
Representation of Choice

Capturing Point Light Fields

- Omnidirectional Cameras
  - Catadioptric
  - Dioptric
  - Multi-Camera

Catadioptric Cameras

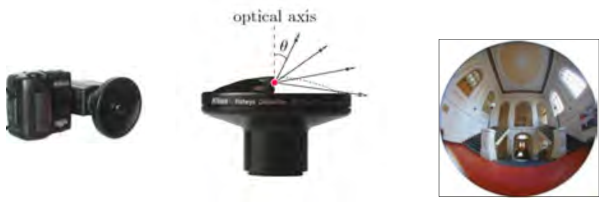
- Mirror-Based (parabolic or hyperbolic)



camera *mirror* *image*

Dioptric Cameras

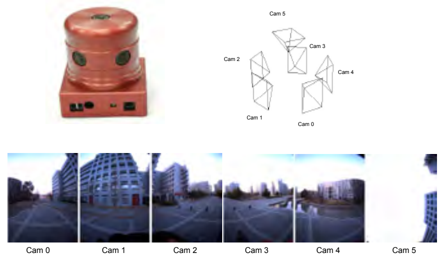
- Fish Eye Lenses



camera *lens* *image*

Multi-Camera Systems

- Point Grey's Ladybug (6 Perspective Cameras)

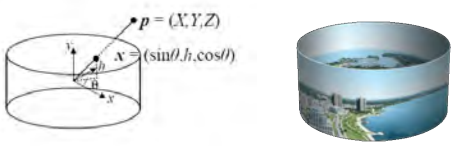


Practical Option

Panoramic Surfaces

Generalized Support for Visual Information

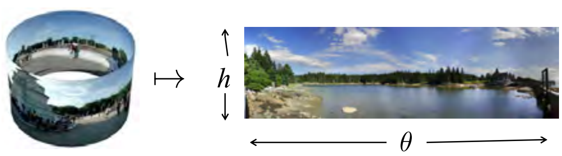
- Data Representation
  - example: *Cilindrical Panorama*



Parametrizations

Maps 2D Surface to Planar Domain

- Coordinate Systems
  - example: *Cilindrical Mapping*



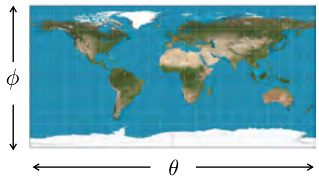
360° Image Formats

Omnidirectional Panoramas

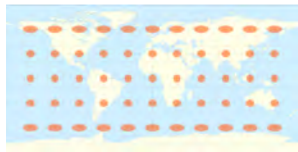
- Parametrizations of the Sphere
  - Lat-Long
  - Cube Map
  - Azimuthal
  - Stereographic (*)

Equirectangular Projection

- Latitude-Longitude Mapping (e.g., *Flickr)*



natural coordinate system

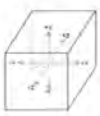


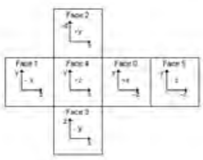
distortion toward poles

Most Convenient Format

Cube Mapping

- 6 Perspective Projections

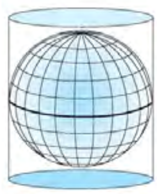




suitable for CG rendering

Azimuthal Projection

- Hemispherical Mapping



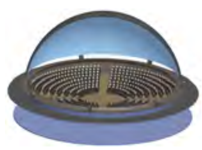
Dome Master standard

Applications to 360 Cinema

Exhibition

- Viewing Scenarios

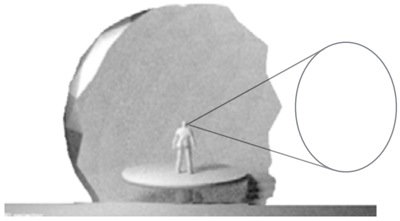
conventional theater



full dome

Field of View

- Reference to Observer
  - 30 to 90 degrees



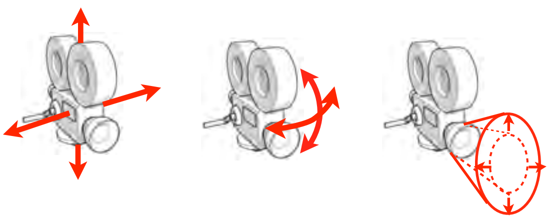
Film Language

- Conventional Cinema
  - HD Television
  - Theater Panavision
- 360 Degrees Dome
  - Omnimax
  - Dome Master

Conventional Cinema

- Camera Moves

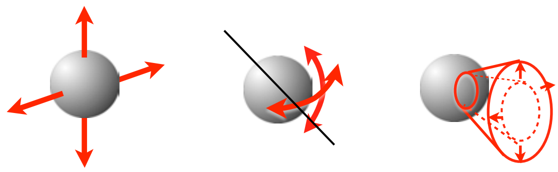
Track   Pan / Tilt   Zoom



360 Camera

- Camera Moves

Track   Pan / Tilt   Zoom



yes   *maybe*   ?

Authoring Issues

OBS: Post-Production

- Passive
  - Movies
- Interactive
  - Google Street View
- Immersive
  - AR Cinema

Emerging Technologies

360° Image Transforms

Moebius Transformations for Manipulation and Visualization of Spherical Panoramas

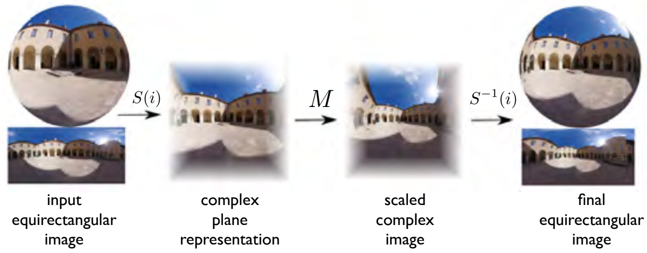
- Current Research at VISGRAF Lab
- Collaboration with
  - Leonardo Koller Sacht
  - Luis Penaranda

Math of Camera Moves

- Omnidirectional Images and Moebius Transformations
  - Pan / Tilt ⇔ Elliptic Transform
  - Zoom ⇔ Hyperbolic Transform
  - Perspective ⇔ Parabolic Transform ?

Transformation Pipeline

- Möbius Mapping

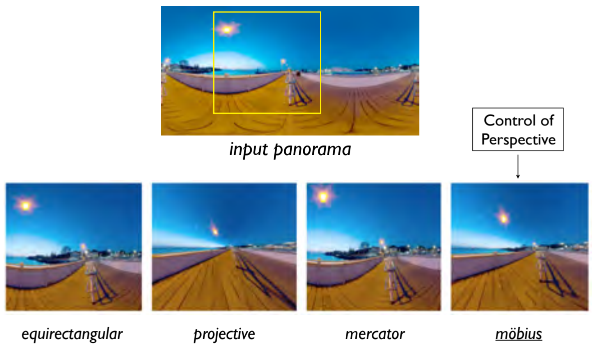


Example

- Extreme Zoom

Comparison

- Alternative Projections



Video 1

Different scales applied to an equi-rectangular image

Current Work

- Preserving Lines
- Perspective Control

Questions?

Improving Projections of Panoramic Images with Hyperbolic Möbius Transformations

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