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Poster

A Primal-to-Primal Discretization of Exterior Calculus on Polygonal Meshes

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Abstract

Discrete exterior calculus (DEC) offers a coordinate-free discretization of exterior calculus especially suited for computations on curved spaces. We present an extended version of DEC on surface meshes formed by general polygons that bypasses the construction of any dual mesh and the need for combinatorial subdivisions. At its core, our approach introduces a polygonal wedge product that is compatible with the discrete exterior derivative in the sense that it obeys the Leibniz rule. Based on this wedge product, we derive a novel primal–primal Hodge star operator, which then leads to a discrete version of the contraction operator. We show preliminary results indicating the numerical convergence of our discretization to each one of these operators.

CCS Concepts
•***Computing methodologies*** → *Mesh geometry models;*

1. Introduction

The discretization of differential operators on surfaces is fundamental for geometry processing tasks. DEC is arguably one of the prevalent numerical frameworks to derive such discrete differential operators. However, the vast majority of work on DEC is restricted to simplicial meshes, and far less attention has been given to meshes formed by arbitrary polygons. We propose a new discretizaton for several operators commonly associated to DEC that operate directly on polygons without involving any subdivision. Our approach offers three main practical benefits. First, by working directly with polygonal meshes, we overcome the ambiguities of subdividing a discrete surface into a triangle mesh. Second, our construction operates solely on primal elements, thus removing any dependency on dual meshes. Finally, our method includes the discretization of new differential operators such as the contraction operator. We examined the accuracy of our numerical scheme by a series of convergence tests on flat and curved surface meshes.

2. Contributions

In the following sections, we discuss our main contributions. Since we provide rather tools and not specific applications, we focus on the main concepts and properties of our operators. Concretely, we define a discrete wedge product on polygonal meshes and show experimental convergence of the product of two discrete forms to the continuous wedge product of respective differential forms (Section 2.1). Then we provide a primal–primal discretization of the Hodge star operator (Section 2.2) that is compatible with our wedge product. And using these two operators we derive a discrete inner product and a discrete contraction operator in Section 2.3.

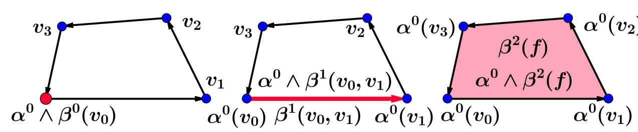


Figure 1: *The wedge product on a quadrilateral: The product of two 0–forms is a 0–form located on vertices (L). The product of a 0–form with a 1–form is a 1–form located on edges (C). The product of a 0–form with a 2–form is a 2–form located on faces (R).*

2.1. The Discrete Wedge Product

Just like the wedge product of differential forms, our discrete wedge product is a product of two discrete forms of arbitrary degree k and l that returns a form of degree $k+l$ located on primal $(k+l)$–dimensional cells (see Figure 1). Inspired by cup products on simplicial and cubical complexes [Arn12], we derive the following formulas for a discrete wedge product on general polygons such that it satisfies the Leibniz rule with discrete exterior derivative (coboundary operator), for details see [Pta17, Section 3.2]).

Definition 2.1 Let K be a surface mesh whose 2–cells (faces) are polygons, possibly non–convex and non–planar. Let $f = (v_0,\dots,v_{p-1})$ be a p–polygonal face, $e = (v_i, v_j)$ an edge, and v a vertex of K. The **polygonal wedge product** of two discrete forms α^k, β^l is a $(k+l)$–form $\alpha^k \wedge \beta^l$ defined on each $(k+l)$–cell of K as:

$$(\alpha^0 \wedge \beta^0)(v) = \alpha(v)\beta(v), \quad (\alpha^0 \wedge \beta^1)(e) = \frac{1}{2}(\alpha(v_i)+\alpha(v_j))\beta(e),$$

$$(\alpha^0 \wedge \beta^2)(f) = \frac{1}{p}\left(\sum_{i=0}^{p-1}\alpha(v_i)\right)\beta(f),$$

$$(\alpha^1 \wedge \beta^1)(f) = \sum_{a=1}^{\lfloor \frac{p-1}{2} \rfloor} \left(\frac{1}{2} - \frac{a}{p}\right) \sum_{i=0}^{p-1} \alpha(i)(\beta(i+a) - \beta(i-a)),$$

where $\alpha(i) := \alpha(e_i)$, $\beta(j) := \beta(e_j)$ and all indices are modulo p.

The polygonal wedge product is an anticommutative bilinear operation, but it is not associative in general (see [Pta17, Proposition 3.2.3]). Numerical tests on several forms and surfaces suggest at least linear convergence of our approximation to analytical solutions, in Figure 2 left we show experimental convergence on torus. L^2 errors were computed using inner product matrices of [AW11].

2.2. The Hodge Star Operator

We define a discrete Hodge star as a homomorphism (linear operator) from the group of k–forms to $(2-k)$–forms, $0 \le k \le 2$, both located on primal elements. Since there is no isomorphism between the groups of k– and $(2-k)$–dimensional cells, our Hodge star is not an isomorphism (invertible operator), unlike its continuous counterpart and diagonal approximations. On the other hand, as the dual forms are located on elements of our "primal" mesh, we can compute discrete wedge products of primal and dual forms and thus define discrete inner product and contraction operator later on.

In order to maintain the discrete Hodge star operator exact on constant forms, we define it on 0– and 2–forms as

$$\star_0 = \mathsf{W}_F\, \mathsf{fv}, \quad \star_2 = \mathsf{W}_V^{-1}\, \mathsf{fv}^\top,$$

where $\mathsf{W}_F \in \mathbb{R}^{|F|\times|F|}$, $\mathsf{W}_V \in \mathbb{R}^{|V|\times|V|}$, $\mathsf{fv} \in \mathbb{R}^{|F|\times|V|}$ are such that

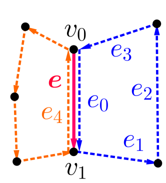
$$\mathsf{W}_F[i,i] = |f_i|, \quad \mathsf{W}_V[i,i] = \sum_{f_k \succ v_i} \frac{|f_k|}{p_k}, \quad \mathsf{fv}[i,j] = \begin{cases} \frac{1}{p_i} & \text{if } v_j \prec f_i, \\ 0 & \text{otherwise.} \end{cases}$$

On 1–forms it is first defined per a p–polygonal face f as:

$$\star_1 = \mathsf{W}_1\, \mathsf{R}^\top, \quad \mathsf{W}_1[i,j] = \frac{\langle e_i, e_j \rangle}{|f|}, \quad \mathsf{R} = \sum_{a=1}^{\lfloor \frac{p-1}{2} \rfloor} \left(\frac{1}{2} - \frac{a}{p}\right) \mathsf{R}_a,$$

$$\mathsf{R}_a[k,j] = \begin{cases} 1 & \text{if } e_j \text{ is the } (k+a)\text{–th halfedge of } f, \\ -1 & \text{if } e_j \text{ is the } (k-a)\text{–th halfedge of } f, \\ 0 & \text{else.} \end{cases}$$

If e is not a boundary edge, it has two adjacent faces, we compute the values of a given 1–form $\star\beta$ on corresponding halfedges, sum their values with appropriate orientation sign and divide the result by 2. E. g. in the inset, $e = (v_0, v_1)$ has halfedges e_0, e_4, thus $\star\beta(e) = \frac{\star\beta(e_0) - \star\beta(e_4)}{2}$, where $\star\beta(e_0)$ is a linear combination of values of β on blue edges and $\star\beta(e_4)$ is a linear combination of values of β on yellow edges. Although our discrete Hodge stars are not diagonal matrices, they are highly sparse. Numerically, our approximation of the Hodge star exhibit at least linear convergence rate on all tested cases, an example is shown in Figure 2 center.



2.3. The Inner Product and the Contraction Operator

Similarly to the L^2 inner product on Riemannian manifolds, we define a **discrete L^2–Hodge inner product** of two l–forms α, β by $(\alpha, \beta) := \sum_f (\alpha \wedge \star\beta)(f)$. The discrete inner product thus read:

$$\mathsf{M}_0 = \mathsf{fv}^\top \mathsf{W}_F\, \mathsf{fv}, \quad \mathsf{M}_1|_f = \mathsf{R}\mathsf{W}_1 \mathsf{R}^\top|_f, \quad \mathsf{M}_2 = \mathsf{fv}\,\mathsf{W}_V^{-1}\, \mathsf{fv}^\top,$$

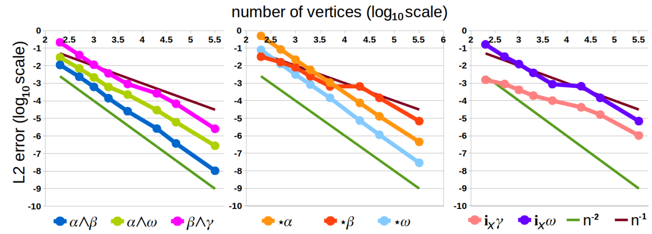


Figure 2: *L^2 errors of approximation of the wedge product (L), Hodge star (C), and contraction (R) on a set of irregular polygonal meshes on torus. Here $\alpha = x^2 + y^2$, $\beta = -ydx + xdy$, $\gamma = -2xzdx - 2yzdy + 2(x^2 + y^2 - \sqrt{x^2+y^2})dz$, $\omega = \frac{\beta\wedge\gamma}{x^2+y^2}$, and $X = (-y, x, 0)$.*

where $\mathsf{M}_1|_f$ is the matrix of product of two 1–forms restricted to a face f. It can be shown that our inner product of 1–forms is identical to the one of [AW11, Lemma 3], i. e., $\mathsf{R}\mathsf{W}_1 \mathsf{R}^\top|_f = \frac{1}{|f|} B_f B_f^\top$, where B_f is a matrix with edge midpoint vectors as rows.

We define a **discrete contraction operator** using the following property that holds on Riemannian surfaces [Hir03, Lemma8.2.1]:

$$\mathbf{i}_X\,\alpha = (-1)^{k(2-k)} \star (\star\alpha \wedge X^\flat), \ \alpha \text{ is a } k\text{–form}, X^\flat(e) = \int_e \langle X(\mathbf{x}), e'(\mathbf{x})\rangle d(\mathbf{x}).$$

Just like its continuous version, our $\mathbf{i}_X$ is a linear map sending k–forms to $(k-1)$–forms s. t. $\mathbf{i}_X\,\mathbf{i}_X = 0$. An example of experimental convergence of our approximation is shown in Figure 2 right.

3. Ongoing Work

Currently, we are numerically evaluating a **discrete Lie derivative** given by the Cartan's magic formula $\pounds_X \alpha = \mathbf{i}_X\, d + d\, \mathbf{i}_X$. We are also starting to study **discrete Hodge decomposition** using a discrete codifferential $\delta(\alpha^k) = (-1)^k \star d \star \alpha^k$, where $\star$ is our discrete Hodge star, and a discrete Laplacian given by $\Delta := \delta d + d\delta$.

4. Conclusion

Geometry processing with polygonal meshes is a new developing area, maybe one of the first steps and also the most influential ones has been the definition of discrete Laplacians on general polygonal meshes by [AW11]. Our objective was to continue in this venue by presenting a novel discretization of several operators and operations that act directly on general polygonal meshes, are compatible with each other and easy to implement. We believe that the generality of our framework will make it a useful tool in many geometry processing tasks and will inspire further research in the area.

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