Geometry independent game encapsulation for non-Euclidean geometries

Filipe Deó Guimarães, Vinícius M. Mello
Instituto de Matemática
Universidade Federal da Bahia - UFBA
Salvador, Brazil
Email: filipedeo@gmail.com, vinicius.mello@ufba.br

Luiz Velho
Vision and Graphics Laboratory
Instituto de Matemática Pura e Aplicada - IMPA
Rio de Janeiro, Brazil
Email: lvelho@impa.br

***Abstract*—This project brings a ludic new approach to the presentation of non-Euclidean geometries by adapting classic games to these geometries. In addition, rethinks the gameplay possibilities enabled by the use of non-Euclidean geometries in a game's design. Finally, it is presented a form of encapsulation that provides a simple adaptation between these geometries.**

***Keywords*-Non-Euclidean; game encapsulation; geometric mechanics; variational integrator.**

I. Introduction

The Euclidean geometry presents itself in virtually all aspects of our lives. From our way of perceiving the world and its distances to the form in which we represent a fictitious video game world. As such, it is consolidated in our society. However, the Euclidean geometry is but a fraction of a much larger set of geometries, with every single one of them as consistent as each other. The presentation of these non-Euclidean geometries is commonly done in a theoretical, hence abstract, manner. As new meanings are given to classic concepts, such as a circle or a straight line, new challenges are exposed, and it becomes vital to find an approach more suitable to introduce these concepts.

Since the conception of the first video game, its visual representation has always been done mainly in the fashion of the Euclidean geometry. Albeit, for a while now, the use of this specific geometry has been occurring not due to technical constraints, but due to a simple "convention". From the most elementary visual representation in a game to the most complex one, they all require a certain degree of abstraction in order to fully comprehend what's represented, [1]. This is true for any given geometry depicted in a game, both Euclidean and non-Euclidean.

Video games are know for their overwhelming capacity of introducing new concepts by engaging its players to immerse in the game's "alternative reality". The engagement generated by games is such that we can perceive phenomena such as the "gamefication", utilized in the most diversified manners. In this context, the employment of video games as a casual and intuitive form of introducing a new, non-Euclidean, geometric model becomes increasingly promising.

Arguably, the best way to present a new concept is to contrast it to a familiar one. Thus, this work aims to use classic and well known games - such as *Atari, Inc.*'s *Asteroids* - to highlight the differences between distinct geometries. By using an interactive environment, not only will the player be able to experience the new reality provided by a different geometry, but also he will be challenged by the new behavior of the game's original challenges, which shall occur in accordance to the new geometry in use.

Contributions: This paper proposes an encapsulation method to dissociate the game development from the geometric space in which it will be represented. This method is able to render the encapsulated game in both Euclidean and non-Euclidean geometries, showcasing all the inner concepts involved in these geometries. Thus, demonstrating some unique gameplay possibilities that emerge through the use of a non-Euclidean space on a "classical Euclidean game dynamic". Finally, the aforementioned encapsulation will be presented as a novel way of converting a given 2D game between distinct geometric models, controlling its geometry through the use of shaders, in the GPU. By developing a simple method to transition between geometric spaces, included in the game's encapsulation process, our approach delivers a straightforward generic way to deal with multiple geometries simultaneously, without any of their specific constrictions.

A. Related work

As far as commercial games go, HyperRogue might be the most popular game to break free from the Euclidean standard. As its website description states, its "strange, non-Euclidean world" uses a hyperbolic plane to represent the space in which the game takes place [2]. HyperRogue's world screen representation happens through the use of a Poincaré disk, although the Minkowski hyperboloid model is used for internal calculations [3].

With a distinct approach, Jeff Weeks notable work has more of an educational focus. In [4], it features a selection of hyperbolic games, addressing the hyperbolic geometry in a straightforward manner. While there is a variety of distinct games, all of them using hyperbolic geometry, the final result is almost "too academic". I.e., it's very hard, or even unfeasible, to play some of the games as they are treated more like a brute conversion to the hyperbolic context and less like a game, which should have its playability as a priority. Therefore, the main difference between the work presented in [4] and the work presented in this paper is that our objective goes beyond a mere visualization in an additional geometry. The goal here is to achieve a new representation of the game for each different geometry while remaining engaging, stimulating and challenging to the player; as any relevant game should be [5].

With a comparable but rather distinct work, [6] introduced the concept of *metric neutrality*. This work was based on a projective geometry approach and was able to achieve a "metric-neutral" visualization system, capable of supporting interaction and immersion for Euclidean, Hyperbolic and Elliptic metrics. [6] goes further into explaining why even outdated GPU's are able to handle the necessary isometries of the projective models of the non-Euclidean geometries, and hence how they were capable to integrate real-time rendering into a "metric-neutral" visualization system.

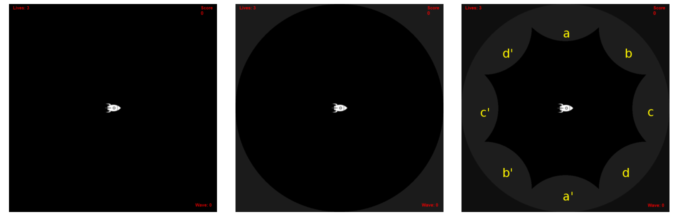
B. Technique overview

By the turn of 19th century, the mathematical community already had a very good understanding of two-dimensional abstract spaces. In particular, the topology and geometry of closed 2-manifolds were completely characterized by the classification theorem of surfaces and the uniformization theorem. The general uniformization theorem states that any connected metrizable surface is a quotient of the: i) elliptical plane; ii) Euclidean plane; iii) hyperbolic plane, by a free action of a discrete isometry group. In other words, there are three fundamental geometries in two-dimensions: Elliptical; flat (Euclidean); and Hyperbolic. These are homogeneous spaces with respectively positive, null (zero), and negative curvatures.

In order to render multiple geometries without directly affecting the game's behavior, the implementation of fundamental features, such as collision detection or movement control, is handled upon the use of a generic geometric model for internal metrics and updates. The definition of which specific geometry is used in the internal calculations and rendering is given by the curvature κ. Hence, every functional aspect of the game is associated with the geometry model in use as the geometry is defined, without ever needing to implement a specific behavior for each geometric model.

Like the internal controls, the game's rendering is geometry independent. This is assured directly through shaders, in a WebGL environment. Once the active geometry is designated, there's a distinct vertex and fragment shader responsible for rendering the game according to the selected geometric model, as seen in Fig. 1.

An interesting feature, present in a few classic games - such as *Asteroids* - is the absence of a limit defined by the screen's edges, i.e., when an object passes through an edge it doesn't collide, but rather reappear in the opposite edge of the screen. In a mathematical perspective, one can say that the edges have been glued together [7]. To achieve this behavior, a 2-dimensional flat torus is used as an orientable surface for the game representation in the Euclidean space, as



(a) Euclidean. (b) Elliptic. (c) Hyperbolic.

Fig. 1. The *Asteroids* game, rendered in each geometric model.

seen in Fig. 3(a). This can also be observed for non-Euclidean geometric spaces, as seen in Fig. 3.

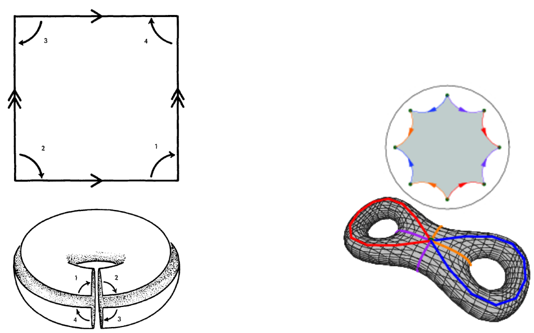

(a) Image reproduced from [7]. (b) Image reproduced from [8].

Fig. 2. Torus and double torus.

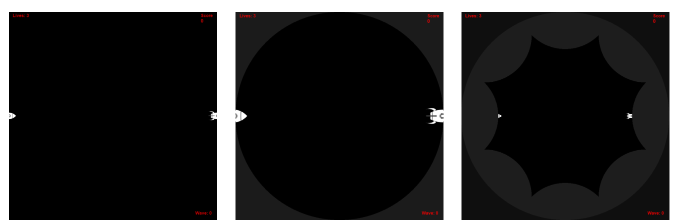

(a) Glued Euclidean. (b) Glued Elliptic. (c) Glued Hyperbolic.

Fig. 3. The space continuity is demonstrated for each geometry in use.

Regarding the representation of the hyperbolic space, the connected sum of two tori is used (lower part of Fig. 2(b)). Still, before the surface can be reproduced in the screen, it must be flattened. Thus, a hyperbolic octagon was used to represent the game in the hyperbolic space (upper part of Fig. 2(b)). The hyperbolic octagon is exhibited inside a Poincaré disk, a two-dimensional space defined as the disk $x \in \mathbb{R}^2 : |x| < 1$. It's worth to note that the octagon's limits, generated by a 2-torus, were manipulated so that its edges stayed diametrically opposed to their pairs, as shown in Fig. 1(c), and not in its usual form, upper part of Fig. 2(b).

Finally, the elliptic space uses a representation of the elliptic plane, a disk visually similar to the Poincaré disk, but with a distinct behavior, Fig. 1(b).

II. Technical Background

To any given surface, there's an unique *Euler number*, χ, containing essential information about the surface's global topology [7]. The Gauss-Bonnet formula presents the relation between the surface's Euler number, its area and its curvature. A surface can admit only a single homogeneous geometry. So, a curvature κ, of a surface's homogeneous geometry, must have the same sign as the Euler number assigned to that surface [7]. Thus, a surface of area A, Euler number χ and constant Gaussian curvature κ, as demonstrated in [7], has the Gauss-Bonnet formula defined as $\kappa A = 2\pi\chi$; where $\kappa = -1$, 0 or $+1$, according to the following classification:

- Hyperbolic space: negative curvature.
- Euclidean space: zero curvature.
- Elliptic space: positive curvature. The spherical space is a particular case of this space.

The geometric approach presented by [9] to the time integration problem served as a starting point to the development of the mechanical model employed in this project, which is capable of correctly representing distinct geometries. In [9], the author specifies a method that ensures a good statistical predictability while working with time integrators.

Geometric mechanics, such as the Lagrangian or the Hamiltonian, consider the mechanics from a variational standpoint, going beyond the Newtonian notion of composition of forces over a body, $F = ma$. The Lagragian mechanics considers the state variable q as the position and $\dot{q}$ as the velocity to establish the Lagrangian function $L(q, \dot{q}) = K(\dot{q}) - U(q)$; which is defined as the kinetic energy K minus the potential energy U.

The main formulation difference between the Hamiltonian and the Lagragian mechanics is the utilization of the *phase space* for describing dynamics [9]. Hamiltonian mechanics can determine the state of a dynamical system through a pair (q, p), where q is the state variable and p is the momentum. In an one-dimensional system, the phase plane is formed by the position q in one axis and the momentum $p = m\dot{q}$, or the velocity $\dot{q}$, in the other axis. Obviously, higher dimensional systems will require an additional axis corresponding to each additional position component q^i and its matching velocity $\dot{q}^i$, or momentum p_i. The work of [9] goes further to explain all the process of obtaining the following discrete Euler-Lagrange equation:

$$D_1 L_d(q_j, q_{j+1}) + D_2 L_d(q_{j-1}, q_j) = 0, \tag{1}$$

but, to this paper understanding, it suffices to acknowledge that for two consecutive positions, q_j and q_{j+1}, the equation 1 defines the next position, q_{j+2}. This is the variational integrator proposed by [9].

The equation 1 can be rewritten in a position-momentum form:

$$p = -D_1 L_d(q, q'), \qquad p' = D_2 L_d(q, q'), \tag{2}$$

where q' and p' represent the following position and momentum relative to q and p. Thus, knowing (q, p), q' can be obtained by the equation 2(left). By substituting the obtained values in the equation 2(right), the momentum p' is found; and this specifies an update rule in phase space.

It's worth mentioning that the aforesaid variational integrator preserves the underlying geometry of the physical system as well as it is guaranteed to preserve the system's discrete momenta [9]. Therefore, by using this integrator in a game, a great physical and visual fidelity, with low computational cost, can be achieved.

III. Regarding a Geometry Independent Model

Non-Euclidean geometries utilize a set of metrics that go beyond the Euclidean notion of distance by straight lines. Hereafter, a set of geometry independent metrics will be exhibited alongside with all the necessary rules to fully work with a geometry independent model, as we desire.

A. Metrics

The concept of metric is delineated by the function of the distance between two points, $d(p_1, p_2)$. Such that, geometrically, d is the shortest geodesic between the points p_1 and p_2. The hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}, \tag{3}$$

serves only to hyperbolic geometry representations. Nevertheless, it represents an interesting starting point to our goal of a generic metric system. Hence, by considering the surface's curvature κ, it's possible to rearrange equation 3 as:

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 + \kappa(x^2 + y^2))^2} = \frac{4dz\overline{dz}}{(1 + \kappa\,|z|^2)^2}. \tag{4}$$

Given the equation:

$$\operatorname{arctan}_\kappa(z) = \sum_{n=0}^{\infty} \frac{\kappa^n z^{2n+1}}{2n+1}, \tag{5}$$

it's fundamental to specify its behavior for each possible geometric model, indicated by κ. Thus, by taking the hyperbolic, Euclidean and elliptic space into account, the following behavior is defined:

$$= \begin{cases} \operatorname{arctanh}(z), & \text{if } \kappa = -1 \\ z, & \text{if } \kappa = 0 \\ \arctan(z), & \text{if } \kappa = 1 \end{cases}. \tag{6}$$

Thusly, the distance between two points, (z_1, z_2), is described by the equation:

$$d(z_1, z_2) = 2\operatorname{arctan}_\kappa \left| \frac{z_1 - z_2}{1 + \kappa z_1 \overline{z_2}} \right|. \tag{7}$$

B. Isometries

Basic geometric transformations too must be adapted to a new form, compatible with the three previously considered geometric models. Thus, as expected, the space curvature is taken into account in some of the following isometries.

The translation $(a \to 0)$, of a point z, is designated by the function $T_a(z) = \frac{z-a}{1+\kappa z\bar{a}}$.

The rotation makes use of the Euler's formula - $e^{i\theta}$ - and is characterized by the function $R_\theta(z) = e^{i\theta}z$, in which e represents the base of the natural logarithm; i, the imaginary unit; θ, the rotation angle, and; z, the rotated point.

The reflection is not a direct isometry, but rather an inverse isometry. It's expressed in complex coordinates and its function is defined as $R(z) = -z$, reflecting through the imaginary axis.

The mirroring $(a \leftrightarrow 0)$ behaves differently from a simple reflection as it isn't limited to the reflection of a single point a when mirroring it. Given a mirroring axis, the element positioned in the location where a is being mirrored to will also be mirrored to a's initial position. Thus, the mirroring equation is defined as $M(z) = \frac{a^2}{|a|^2}\frac{\overline{a-z}}{1+\kappa a\bar{z}}$.

The perpendicular bisector $(\overline{0a})$ can be found through the following equation:

$$\kappa\,|a|^2\,(x^2+y^2) + 2a_x x + 2a_y y = |a|^2\,. \tag{8}$$

However, in specific cases where $\kappa \neq 0$, the equation 8 can be reformulated as $(x+\kappa\frac{a_x}{|a|^2})^2 + (y+\kappa\frac{a_y}{|a|^2})^2 = \frac{1}{|a|^2} + \kappa$.

At last, the midpoint m relative to the perpendicular bisector $(\overline{0a})$ is defined as:

$$m = \begin{cases} \frac{1}{2}a, & \text{if } \kappa = 0 \\ \kappa\frac{\sqrt{1+\kappa|a|^2}-1}{|a|^2}a \end{cases}. \tag{9}$$

C. Geodesic Circle (O, R)

The classical definition of a circle is restricted to the Euclidean plane. While considering nonzero space curvatures, its crucial to revise a few definitions so that all space curvatures can be contemplated. Therefore, the geodesic circle is defined as $(x - \frac{O_x(1+\kappa p^2)}{1-p^2\kappa^2|O|^2})^2 + (y - \frac{O_y(1+\kappa p^2)}{1-p^2\kappa^2|O|^2})^2 = (\frac{p(1+\kappa|O|^2)}{1-p^2\kappa^2|O|^2})^2$, with $p = \tan_\kappa(\frac{R}{2})$.

D. Geodesic update rule

As the update rule in phase space, mentioned in Section II, was adapted to this work's context, some of its equations were reevaluated. In addition, considering the 2-dimensional aspect related to 2D games, the Lagrangian function was reformulated by taking the position and momentum into consideration for each one of the axis: $L(q_x, q_y, p_x, p_y) = \frac{4(p_x^2+p_y^2)}{(1+\kappa(q_x^2+q_y^2))^2}$.

The discrete Lagrangian, considers both current and consecutive position, q and q', in addition to h, the time interval between two samples. In this equation, every position is dismembered according to its respective axis, $q'_x = q_x + \Delta q_x$ and $q'_y = q_y + \Delta q_y$. Henceforth, the final discrete Lagrangian becomes: $L_d(q_x, q_y, q'_x, q'_y, h) = hL(q_x, q_y, \frac{q'_x - q_x}{h}, \frac{q'_y - q_y}{h})$.

As the equation 2 was redesigned to determine the momentum p in this new context, the displacement Δq, the time interval h, and the space curvature κ were taken into account. So, the momentum relative to each axis is portrayed as:

$$p_x = \frac{8}{h(1+\kappa\,|q|^2)^2}(\Delta q_x + 2\kappa q_x((\Delta q_x)^2 + (\Delta q_y)^2)); \tag{10}$$

$$p_y = \frac{8}{h(1+\kappa\,|q|^2)^2}(\Delta q_y + 2\kappa q_y((\Delta q_x)^2 + (\Delta q_y)^2)). \tag{11}$$

Finally, the successive momentum p', for each axis, is presented in the following equations: $p'_x = \frac{8\Delta q_x}{h(1+\kappa|q|^2)^2}$; and $p'_y = \frac{8\Delta q_y}{h(1+\kappa|q|^2)^2}$.

IV. Results and Discussion

Our encapsulation method and its geometry independent essence are validated through the implementation of the classic *Atari, Inc.* game *Asteroids*. This specific game was chosen as our main case study because of its vast popularity. As mentioned in Section I, it represents a great proof of concept to showcase the differences and particularities between its original Euclidean geometry and the hyperbolic and elliptic ones. This game has been implemented and is fully functional in all three geometric spaces, as seen in [10].

As future work, a second experiment is intended and it shall further demonstrate the potential that our encapsulation method presents. It consists in developing a simple and novel game with a non-Euclidean geometry gameplay experience in mind.

V. Conclusion

In this paper, we introduced an encapsulation method focused on making a game's internal controls development independent from its final/underlying geometry, utilized in its screen representation. This has the potential to impact the introduction of non-Euclidean geometries to a wide range of students and enthusiasts, as well as demonstrate, in a interactive way, the particularities of each specific geometric space. Alongside, the presented method is also a efficient and simple form of developing games to non-Euclidean geometries.

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