Modeling with Simplicial Diffeomorphisms

Luiz Velho
IMPA – Instituto de Matemática Pura e Aplicada

(*PhD work of Vinicius Mello*)

Outline

- Background / Motivation
- Framework Overview
- Simplicial Diffeomorphisms
- Binary Multi-Triangulations
- Applications
- Open Issues

New Framework

- Combines
  - Parametric Representation
  - Implicit Surfaces

with
  - Spatial Warping

- Also…
  - Subdivision
  - Point Sets

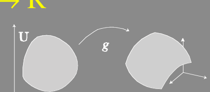
Background

- Surface Descriptions
  - Parametric
  - Implicit
  - Points

- Deformations
  - Structured
  - Unstructured

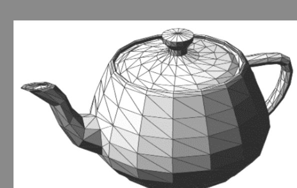
Parametric Surfaces

$$g : U \subset R^2 \rightarrow R^3$$



- Piecewise Descriptions
  - Polygonal Meshes
  - Algebraic Patches
  - Subdivision Surfaces
- Other (Variational, etc…)

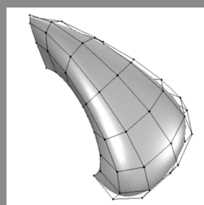
Polygonal Meshes



\+ Simple

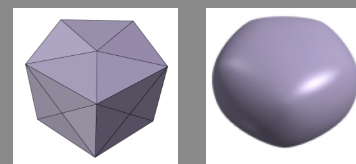
\- Not Smooth

Parametric Patches



+ Smooth
- Regular Mesh Topology

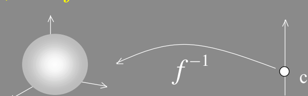
Subdivision Surfaces



+ Arbitrary Base Mesh Topology
- Extraordinary Vertices

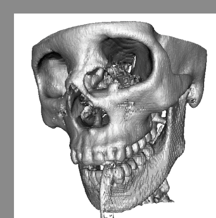
Implicit Surfaces

$f^{-1}(c), \qquad f : \mathrm{R}^3 \to \mathrm{R}$



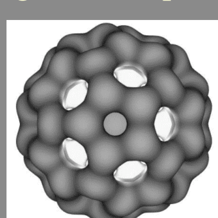
- Primitives
- Compositions
  - Combination of Algebraics
  - Volumetric

Volumetric Data



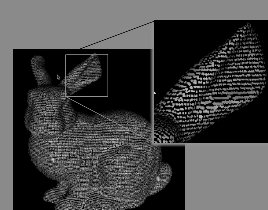
+ Local Control
- Not Smooth

Algebraic Implicits



+ Smooth
- Restricted Control

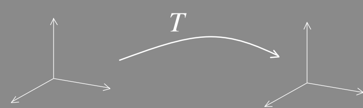
Point Sets



+ Simple
- No Topology
  (*need local approximants*)

Deformations

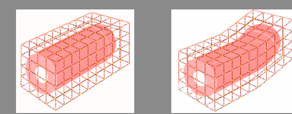
$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$



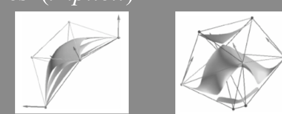
- Structured
  - Grid-Based
- Unstructured
  - Feature-Based

Grid-Based Deformations

- Free-Form Deformations (*parametric* *)



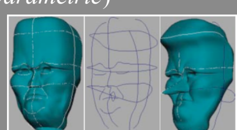
- A-Patches (*implicit*)



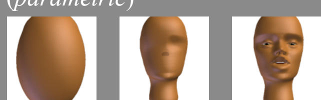
Multiple Sheets

Feature-Based Deformations

- Wires (*parametric*)



- Clay (*parametric*)



Pros & Cons

	smooth	*topology*	*control*	**Grid**	**Feature**
Polygonal	-	+	+	✓	✓
Patches	+	-	+	✓	✓
Subdivision	+	-/+	+	✓	✓
Volumetric	-	-/+	-	-	-
Algebraic	+	+	-	- / ✓	-

Motivation

"Combine Parametric and Implicit Surface Descriptions with Spatial Deformations"

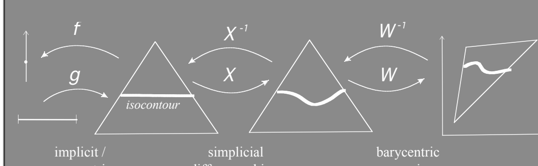
- Exploit:
  - Complementary Aspects
  - Integrated Representation

* Disclaimer:

It does not solve all problems ...

Framework

$F(p) = f \circ X^{-1} \circ W^{-1}(p)$

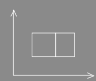


$G(u) = W \circ X \circ g(u)$

Comparison

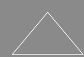
- Deformation + Global Implicit Function

  $F(X^{-1}(p))$



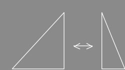
- Deformation of the Implicit Function

  $F(w)$



- Simplicial Deformation

  $F_\sigma(X_\sigma^{-1}(W^{-1}(p)))$



Simplicial Diffeomorphism

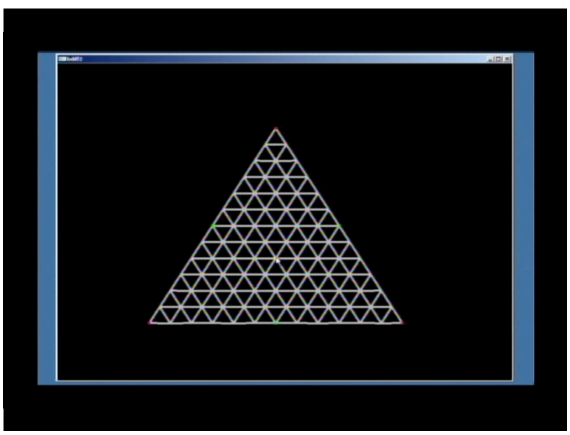
$$\sigma = X(\Delta^n)$$

- *N*-Dimensional Simplex : σ
- Standard *N*-Simplex : Δ^n

* Properties of X

- – Maps Δ to σ, leaving faces invariant.
- – Differentiable, has Differentiable Inverse

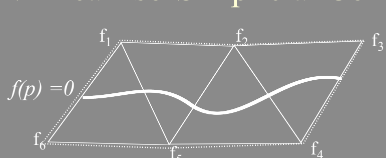
$$\Delta^n = X^{-1}(\sigma)$$



Why such Diffeomorphism is needed?

- Preserves Simplicial Structure
  - – Maps faces to faces of a simplex
- Maintains the Topology of Level Surfaces
  - – Continuous and Bijective
- Induces Smooth Geometry
  - – Differentiable Map

Curvilinear Iso-Simplicial Complex



1. Manifold *M*, triangulated by simplicial complex *K*
2. Function f from the vertices of K to R - {0}
3. Simplicial Diffeomorphisms for $\sigma \in K$

$$X_\sigma : \Delta^n \to \sigma^n$$

Simplicial Model

- Implicit Simplicial Model

$$F(p) = \sum_{i=0}^{n} f(v_i)\, X_{i,\sigma}^{-1}(W_\sigma^{-1}(p))$$

- Parametric Simplicial Model

$$G(u) = \cup\, W_\sigma(X_\sigma(g(u_k)))$$

Main Issues

- *How to guarantee that X is a Simplicial Diffeomorphism ?*
- *How to construct a suitable Space Decomposition K ?*

* Generality (dimension / degree)

Conditions for Diffeomorphism

- A map $X : |K| \rightarrow |K|$ is a diffeomorphism iff. for all $\sigma \in K$

1. $X(\sigma) = \sigma$
2. $X|_{\text{int}(\sigma)}$ is differentiable
3. $D(X|_{\text{int}(\sigma)})(p)$ is injective for all $p \in \text{int}(\sigma)$

Simplicial-Invariant Function

* Condition (1)

Definition:

Let K be a simplicial complex and $X : |K| \rightarrow |K|$ *a continuous function.*
X is simplicial-invariant w.r.t. to K, or K-invariant, if for all $\sigma \in K$, $X(\sigma) = \sigma$

- Properties
  - Maps faces of *K* to themselves
  - Vertices remain fixed

Differentiability and Injectivity

* Combines conditions (2) and (3)

General Results: (Meisters-Olech)

Let X be a differentiable function in Ω .
If X is injective in $\partial\Omega$ *and locally injective in* $\text{int}(\Omega)$
then X is globally injective.

Characterization

- Theorem:

A Simplicial Diffeomorphism X is:

1. A *K-invariant* function $X : |K| \rightarrow |K|$
2. $\det J_X > 0$ in $\sigma \in K$

Polynomial Simplicial Diffeomorphisms

- Barycentric Representation

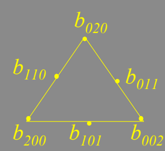
$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ p_0^1 & p_1^1 & \cdots & p_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ p_0^n & p_1^n & \cdots & p_n^n \end{pmatrix} \begin{pmatrix} H^0(w) \\ H^1(w) \\ \vdots \\ H^n(w) \end{pmatrix} = \begin{pmatrix} 1 \\ X^1(x) \\ \vdots \\ X^n(x) \end{pmatrix}$$

- $H(w)=(H^0(w), H^1(w), \dots, H^n(w))$

where H^k is a Homogeneous Polynomial of degree m

Bernstein-Bézier Diffeomorphisms

$$H^k = \sum_{|I|=m} b_I \; B_I$$



- Control Points

  $b_I \in A^n = \{ w \in R^{n+1} \mid \sum w_i = 1 \}$

- Bernstein-Bézier Polynomials

  $B_I = \binom{m}{I} W^I$

Revisiting the Conditions

- B.B. Polynomials $\Longrightarrow$ Conditions on b_I
  - Simple
  - Easy to Compute

- Looking Ahead…
  - Condition (1): *Restrict Position*
  - Conditions (2,3): *Several Special Cases*

K-Invariance for B.B. Polynomials

- Definition:

  Let $\chi(a_0,\ldots,a_m) = (e_0,\ldots,e_m)$ where $e_i = 1$ if $a_i > 0$, $e_i = -1$ if $a_i < 0$ and $e_i = 0$ if $a_i = 0$

- Theorem (*Adjusted Mapping*):

  H is K-invariant if $\chi(b_I) = \chi(I)$

* *Intuition*

- Control Points b_I are restricted to their faces

Injectivity for B.B. Polynomials

- Theorem (*Pólya*):

  Let $F \in R[W]$ a homogeneous polynomial of degree m.
  Then $F(w) > 0$ for all $w \in \Delta^n$ iff $F \cdot (W^0 + \ldots + W^n)^M$ has for some $M \in \aleph$ the form $\sum_{|K|=m+M} a_K W^K$ with $a_K > 0$.

- Apply *Pólya* with $F = H'$

* Particular Cases
  - Degree 2 Polynomials
  - Central Control Point

➢ Composition / Stratification

Quadratic Case $(m = 2)$

- Theorem:

  Let $H = \sum_{|I|=2} b_I B_I$ be a degree 2 adjusted polynomial.
  Then $H|_{\Delta^m}$ is a Δ^n-invariant diffeomorphism.

* *No additional restrictions on* b_I

Central Control Point

➢ *Degree m* = dim(Δ) *+1*

- Theorem:

  Let $H = \sum_{|I|=m} b_I B_I$ be an adjusted degree m polynomial and $J = (J^0,\ldots,J^n) \in \aleph^{m+1}$ with $|J| = m$ and $J^i \in \{0,1\}$.
  If $b_I = I/m$ for $I \neq J$, then $H|_{\Delta^m}$ is a Δ^n-invariant diffeo.

* *Restriction to move only the* <u>*Central*</u> *Control Point*

Obs: Nice Handle

Composition

* *The Composition of Simplicial Diffeomorphisms is also a Simplicial Diffeomorphism*

- Definition:
  Let $F = \{F_\sigma\}_{\sigma \in K}$ and $G = \{G_\sigma\}_{\sigma \in K}$ be simplicial diffeomorphisms.
  The composition $FG = \{F_\sigma G_\sigma\}_{\sigma \in K}$ of F and G is obtained by applying $F \circ G$ in each simplex $\sigma \in K$.

Obs: Raises degree

Composition is not commutative in general

Stratified Scheme

- Composition $G = G_1 \ldots G_n$ of n mappings G_j which correspond to actions in each topological dimension

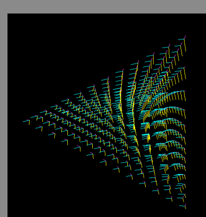
$$G^i_j = W_i + \sum_{|J|=j+1, J^k \in \{0,1\}} (b^i_J - J^i / |J|) B_J$$

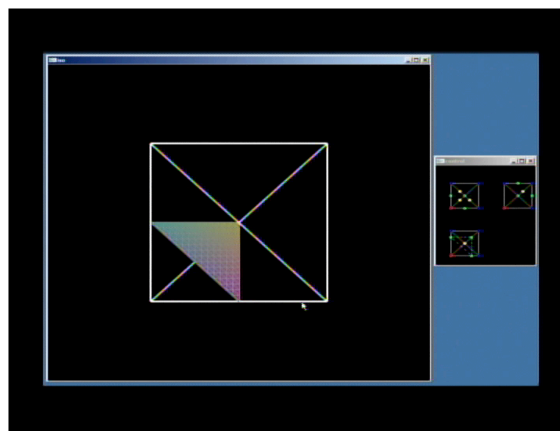
* *Preserves Symmetry of* b_I

➢ Proposition:
G_1, G_2 and G_3 are Δ^3-invariant diffeomorphisms.

Wrap-Up

- Summary
  - Powerful Scheme
  - Flexible and General
  - Arbitrary Dimension
- Examples
  - 3D





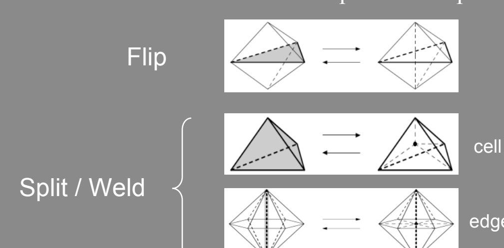
Space Decompositions

* *Need Decomposition of Ambient Space to Construct Iso-Simplicial Complex*

- Desired Properties
  - Semi-Regular Tiling
  - Adaptation Power
  - Gradual Transitions between Cells
  - Natural Multi-Resolution Structure

Stellar Operators

- Basic Transformations on a Simplicial Complex



Some Results of Stellar Theory

- Theorem (Newman, 1931):

  Two n-*dimensional simplicial complexes are piecewise homeomorphic iff they are related by a finite sequence of elementary stellar operations.*

- Proposition:

  Any stellar operation can be decomposed by a sequence of stellar operations on edges.

Binary Multi-Triangulations

- Definition:

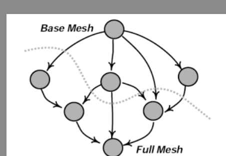
  A binary multi-triangulation is a poset (T, >), *where* T={M_0, M_1, ..., M_k} *is a finite set of simplicial* n-*complexes and the order relation > satisfies:*

  1. *There is a* maximum, $\overline{M}$ *and a* minimum $\underline{M}$ *in* T;
  2. $M' \succ M$ *if and only if,* $M \xrightarrow{\xi(A)} M'$ *for a stellar operation* ξ *on some edge* A

* Best Properties (Puppo & De Floriani, 1998)

Graph Representation of BMT

- BMT is a *Lattice*



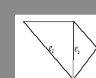
- Nodes are sub-meshes, arrows are stellar operations
- Any separating cut in the graph is a valid complex

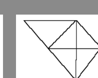
Adaptive Simplicial Decomposition

* *Every n-simplex,* $t \notin \overline{M} \cup \underline{M}$ *, has a* **refining element** *(split edge or flip edge) and an* **unrefining element** *(weld vertex or flip edge).*

- Algorithm:

```
adapt(e)
  for all t_i incident in e do
    if element(t_i) != e then
      adapt(element(t_i))
  apply(stellar, e)
```





➢ Enforces Transition (*Restricted Structure*)

➢ Same Algorithm for Local Refining and Unrefining

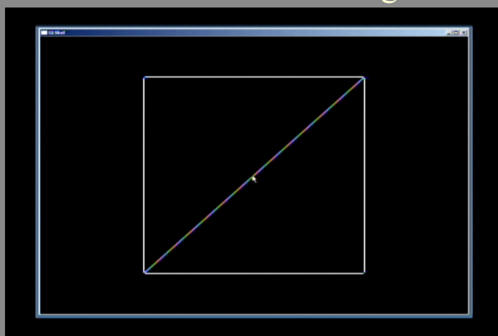
Construction Methods

- Base Complex
  - Semi-Regular Structure
- Subdivision
  - Refinement
- Simplification
  - Coarsening
- Dynamic Adaptation
  - Refinement *and* Coarsening

Applications

- Free-Form Modeling
- Surface Approximation
- Reconstruction from Points

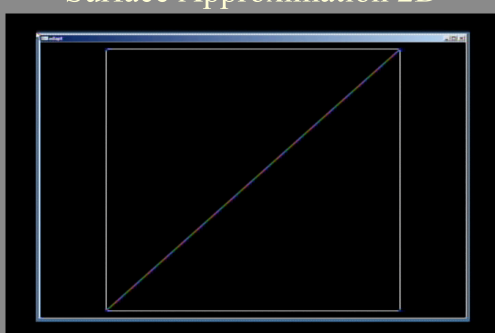
Free-Form Modeling



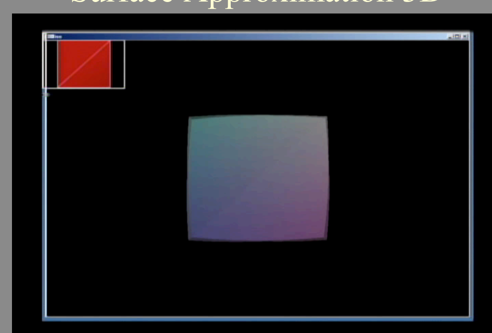
Free-Form Modeling

- Current Implementation
  – One Control Point per *k*-Face
  – Iterate Mappings Two Times
- Future Plans
  – Hermite Handles (Normal + Point)
  – Natural Continuity Constraints (*same* Normal)

Surface Approximation 2D



Surface Approximation 3D



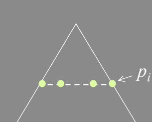
Surface Approximation

- Fitting Based on Implicit Function

$$\min \sum_{\sigma \in T} \left\| f(X_{\sigma(b_I)}(p_i)) \right\|^2$$

s.t.

$$b_I \in \Delta^{|I|-1}$$



- Optimization Method
  – **L-BFGS-B**
  (*Zhu, Byrd and Nocedal*)

Reconstruction from Points

* *Under Development*

- Similar to Implicit Surface Fitting
  – Optimization Based on Distance to Samples
- Need to Compute Distance from Surface
  – Project Points onto Current Approximation

Open Issues

- Continuity
- Rendering
- Polynomial Approximation
- Rational Diffeomorphisms

Thanks!