Problem Statement

Surface Reconstruction and Denoising

\[ Q(r) = r + t \cdot n_i, \]

- Projection Operator
  \[ \{ x \mid Q(x) = x \} \]
- Smoothing Operator
  \[ x' \leftarrow Q(x) \]

Surface Estimation

Minimize Error Norm: \( E(x) \)

- Non-Robust Statistics
  - Least Squares
    \[ g(x) = x^2 \]
  - W-Estimator
    \[ \rho_w(x) = \sigma^2 \left(1 - e^{-r^2/2\sigma^2} \right) \]
  - M-Estimator
    \[ \theta(x) = \arg \min_{\theta} \sum \rho(x, \theta) \]

Related Work I

- Moving Least Squares (Levin et al.)
  \[ S = \{ x \in \mathbb{R}^3 \mid \psi(x) = x \} \]
  \[ \psi(x) = r + (t + \gamma(0,0))n \]
  - Reference Plane: \( H_r = (n,t) \)
    \[ (n,t) = \arg \min_{n,t} \sum_{p \in \mathbb{R}^3} (n \cdot (p-r-tn))^2 W_p(\| p-r-tn \|) \]
  - Height Polynomial: \( \gamma(u) = \sum_{i=0}^{k} a_i u^i \)
    \[ \min_{a_i, u} \sum_{p \in \mathbb{R}^3} (\gamma(u_p) - h_p)^2 W_p(\| p-r-tn \|) \]

Proposed Method

Given a Point Cloud

\[ C = \{ p \in \mathbb{R}^3 \mid p = q + \epsilon, \ q \in S, \ \epsilon \sim N(0, \sigma) \} \]

Compute Operator

\[ r' = Q(r) = r + t, n_i \quad \text{with} \quad \| n \| = 1 \]

by Solving Equation (1)

\[ \{ n', t' \} = \arg \min_{n, t} \sum_{q \in S} g(n' \cdot (q-p) - t) W(\| q-p \|) \]
Adaptive, Feature Preserving Reconstruction

- Robust Error Norm
  \[ g(x) = \sigma_x \left( 1 - \exp \left( -\frac{x^2}{2\sigma_x^2} \right) \right) \]

- Smoothing Weight
  \[ w(x) = \exp \left( -\frac{x^2}{2\sigma_w^2} \right) \]

- Local Neighborhood (progressive propagation)
  \[ N_g(p) = \{ s; \forall s \in K_g(s) \land \| s - p \| < \sigma_g \} \]

Main Algorithm

- Iterative Approach
  – Alternate Minimization with Respect to \( n \) and \( t \)

\[ t_0 = 0 \]
\[ k = 0 \]

repeat

(i) \( n_{k+1} = \text{minimum of Eq.}(1) \) with respect to \( n \) with fixed \( t = t_k \)
(ii) \( t_{k+1} = \text{minimum of Eq.}(1) \) with respect to \( t \) with fixed \( n = n_{k+1} \)

\[ k = k + 1 \]

until (close to the minimum)

\[ p' = p + t_k n_k \]

Stage (i)

- Constrained Optimization Problem
  \[ n^* = \arg \min_n \sum_{q \in \mathcal{N}_p} g(n^T (q - p) - t) w(\| q - p \|) \]
  subject to \( \| n \|^2 = 1 \)

  * Solve with Newton Method in \( S^2 \)
  * Geodesic Path
    \[ \gamma(0) = n_i, \quad \gamma(1) = n_{i+1}, \quad \gamma'(0) = H_i \]

Stage (ii)

- Unconstrained Optimization Problem
  – First Order Condition
    \[ \sum_{q \in \mathcal{N}_p} \nabla \Psi_q (\| q - p \|) \Psi_q (h_q) (n^T (q - p) - t) = 0 \]
  with \( \Psi_q (x) = \frac{1}{x \sigma_q^2} g'(x) \), \( \Psi_q (w(x)) = w(x) \), and \( h_q = n^T (q - p) - t \)

  – Solving for \( t \) (recurrence relation)
    \[ t_{i+1} = \frac{1}{2} \sum_{q \in \mathcal{N}_p} \nabla \Psi_q (\| q - p \|) \Psi_q (n^T (q - p - t_i n)) n^T (q - p) \]

  ✓ Convergence

Newton Method on Riemannian Manifolds

- (Smith, 1994)
  Select initial point \( s_1 \in M \) such that \( (\nabla^2 f)_n \) is non-degenerate
  \[ i = 1 \]
  repeat
    (i) \( n_{i+1} = \text{minimum of Eq.}(1) \) with respect to \( n \) with fixed \( t = t_i \)
    (ii) \( t_{i+1} = \text{minimum of Eq.}(1) \) with respect to \( t \) with fixed \( n = n_{i+1} \)
    \[ i = i + 1 \]
  until (close to the minimum)

  With \( f : M \rightarrow \mathbb{R} \)

✓ Quadratic Convergence
✓ Need a Good Starting Point

Convergence Analysis

Lemma: The sequence \( t_i \) is strictly monotone decreasing \( \delta_i > t_{i+1} \) and converges to the minimum of the function:

\[ F_i(t) = \sum_{q \in \mathcal{N}_p} g(h_q - t) \| q - p \| \]

Proof: This sequence \( t_i \) is a particular case of an M-estimator

✓ Note: Fleishman et al. method corresponds to the first iteration of the above algorithm, starting at \( t_0 = \theta \).

\[ t_i = \frac{1}{2} \sum_{q \in \mathcal{N}_p} W_q (h_q) W (\| q - p \|) h_q \]

\[ t_i = \sum_{q \in \mathcal{N}_p} W_q (h_q) W (\| q - p \|) h_q \]
Initialization

- Need Good Estimate of the Normal for Stage (i)
- Robust Method to Compute Normals of C
  - Make $t = 0$ in Eq. (1) and Compose the Lagrange Equation
    
    $L(n, \lambda) = \sum_{p \in N_C} w(\|q - p\|) g(n^T (q - p)) + \lambda \left( n^T n - 1 \right)$

  - Compute the derivative w.r.t. $n$, and set it to zero
    
    $L_n(n, \lambda) = \sum_{p \in N_C} \psi_{\|q - p\|} \psi_{s} (s) (q - p) (q - p)^T n - \lambda n = 0$

Matrix Solution

- Rewrite previous Equation as:

  
  $M(n) n = \lambda n$

  
  where

  
  $M(n) = \sum_{p \in N_C} \psi_{\|q - p\|} \psi_{s} (s) (q - p) (q - p)^T$

- Iterative Scheme

  
  $M(n_k) n_{k+1} = \lambda_{k+1} n_{k+1}$

  
  where

  
  $\lambda_{k+1}, n_{k+1}$ smallest eigenvalue, eigenvector of $M(n)$

  
  ✓ Good results in few iterations

Experiments

- Tested on Several Datasets
  - Polygonal models, discarding connectivity (for comparisons)
- Corrupted Models
  - Gaussian noise (with variance 0.02) in the normal direction
- Parameters
  - $\sigma_c = 1.5h, 3h$
  - $\sigma_w = 2h, 4h$
  - $h$: mean spacing between sample points

Bunny

Igea

Dragon
**Performance**

- Convergence to $10^{-10}$
  - Stage (i): average of 6 iterations
  - Stage (ii): average of 9 iterations

- Timings for Bunny model
  - Pentium IV, 1.6 GHz, 512 Mb RAM
  - $\sigma_w = 2h_r$: 9.91 sec
  - $\sigma_w = 4h$: 19.67 sec

**Comparison**

<table>
<thead>
<tr>
<th>Proposed Method</th>
<th>Bilateral Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iterative</td>
<td>Non Iterative</td>
</tr>
<tr>
<td>$M$-estimator and $W$-estimator</td>
<td>$W$-estimator</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Proposed Method</th>
<th>Moving Least Squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robust Statistics</td>
<td>Non-Robust Statistics</td>
</tr>
<tr>
<td>Two parameters</td>
<td>One parameter</td>
</tr>
<tr>
<td>Local Neighborhood</td>
<td>Global Neighborhood</td>
</tr>
<tr>
<td>One step</td>
<td>Two steps</td>
</tr>
</tbody>
</table>

**Conclusions**

- Modified MLS to be Feature Preserving
- Extended Bilateral Filter to Point Sets
- Robust Surface (and Normal) Estimation
- Effective Numerical Optimization Methods