Mathematical Optimization in Graphics and Vision

Luiz Velho
Paulo Cezar Pinto Carvalho

IMPA - Instituto de Matemática Pura e Aplicada

Course Schedule

Module 1 - Computer Graphics and Optimization
Luiz Velho (1:45 minutes)

Module 2 - Continuous and Variational Optimization
Paulo Cezar Pinto Carvalho (1:45 minutes)

Module 3 - Combinatorial Optimization
Luiz Velho (1:45 minutes)

Module 4 - Global Optimization
Paulo Cezar Pinto Carvalho (1:45 minutes)

Questions we will Try to Answer

• Why optimization is important for graphics?
  – Problems and Solutions

• How optimization can be used in graphics?
  – Basic Principles

• What are the main optimization techniques?
  – Mathematical Concepts

• Where optimization has been applied?
  – Examples of Applications

Additional Material

• Website
  – Course Notes
  – Presentation Slides
  – Links
  ⇒ URL
  http://www.visgraf.impa.br/otim-03/

• Course Notes
  – Contents of Presentations
  * OBS: Probability and Optimization

Outline

• Basic Concepts
  – Graphical Objects
  – Operators

• Graphics Problems
  – Direct / Inverse
  – Well-Posed / Ill-Posed

• Optimization Methods
  – Formulating the Problem
  – Classification of Techniques

• Applications
  – Main Criteria
  – Examples in Different Areas
Motivation

- Optimization is a Basic Tool for Graphics
  - Widely Used
  - Flexible

- What Makes Optimization Important?
  - Need to Understand Graphics Problems
  - Conceptual View

Graphics and Vision

- Relation with Physical Universe
  - Photography: (2D representation)
  - Human Visual System: (3D reconstruction)

- Concepts
  - What are the Models?
  - What are the Problems?

Graphical Object

\( O = (U, f), \quad f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \)

- Geometric Support: \( U \)
  (shape)
- Attribute Function: \( f \)
  (properties)

* Dimension
  * Object: \( \text{dim}(U) \)
  * Space: \( n \)

Simple Example

- 2D Drawing

More Examples

- Images:
  - Grayscale, Color
    (simple shape, complex attributes)

- Models (Data)
  - Surfaces, Solids
    (complex shape, simple attributes)

* Graphical Object: Comprehensive Concept

Processing Graphical Objects

- Geometric Modeling
- Image Analysis
- Image Synthesis
- Image Processing
Operator on Graphical Objects

Graphical Problems

- Spaces of Graphical Objects (function spaces)
  
  \[ x \in \mathcal{O} \]

- Operators on Spaces of Graphical Objects

  \[ T : O_1 \rightarrow O_2 \]

  \[ T(x) = y \]

Types of Problems

- Direct Problems
  - Given \( T \) and \( x \), find \( y \)

- Inverse Problems
  - Given \( T \) and \( y \), find \( x \) such that \( T(x) = y \)
  - Given \( x \) and \( y \), find \( T \)

Example: Visualization

- \( x \) is the scene
  - Geometry
  - Illumination
  - Camera

- \( T \) is the \textit{rendering operator}

  \[ T(x) = y \]

- \( y \) is the rendered image

* Direct Problem

\[ \Rightarrow \text{OBS: Vision - Inverse Problem} \]

Problem Characterization

* Hadamard (1902)

- Well-Posed Problem
  - Existence of Solution
  - Uniqueness of Solution
  - Continuous Dependence on Initial Conditions

- Ill-Posed Problem
  (doesn’t satisfy at least one of the above conditions)

Ill-Posed Problems

- Inherent in some Applications
  - Inverse Problems are usually Ill-Posed

- Sources of Ill-Posedness
  - Multiple Solutions
  - Numerical Errors

\[ \Rightarrow \text{Need to get around Ill-Posedness} \]

- Optimization Methods
  - Best Solution (unique)
Example of Ill-Posed Problem

- Linear System: \( T x = y \),
  \[ T = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \], \( x = (u, v) \), \( y = (a, 2a) \), and \( a \in \mathbb{R} \)

- Solution \( y \) must lie on \( 2u - v = 0 \)
  - Ill-Conditioning / Perturbations

- Well-Posed Solution
  \[ y = \arg\min \| y, Tx \| \]

Describing Graphical Objects

- Computer Representation from Mathematical Model

Reconstruction Issues

- Why do we need to reconstruct?
  - Discretization gives incomplete information
  - Working in continuous domain avoid numerical errors
  - Semantics

- Reconstruction
  - Invertibility of the Representation Operator
    - Exact
    - Non-Exact

- Ill-Posed Reconstruction
  - Ambiguous Representation

Ambiguity in Models

- Wireframe Model

- Many Interpretations

Ambiguity in Images

- Two possible interpretations
Optimization in Graphics and Vision

- Techniques for selecting the "best" solution
  - Solve Ill-Posed Problems

- Applications in Graphics
  - Good Representations
    (Unique, Compact and Efficient)
  - Robust Operators
    (Numerical Computation)
  - Automatic Selection of Desired Parameters
    (User Interface)
  - Minimal Energy Solutions
    (Physical Simulations)

Overview of Optimization Techniques

- Optimization:
  Choosing the best among a set of alternatives

\[
\min_{x \in S} f(x)
\]

\(f: S \rightarrow R\) is the objective function

\(S\) is the set of feasible solutions

Classification of Techniques

Three different Criteria:

- Nature of the Solution Set \(S\)
- Description of the Solution Set \(S\)
- Properties of the Objective Function \(f\)

Classification of Techniques I

- Nature of the Set \(S\)
  - Variational
  - Continuous
  - Discrete
  - Combinatorial

Classification of Techniques II

- Constraints on the Set \(S\)
  \(x \in S\) subject to
  - Equality Constraints
    \(h_i(x) = 0, \ i = 1, \ldots, m\)
  - Inequality Constraints
    \(g_j(x) \leq 0, \ j = 1, \ldots, n\)

Classification of Techniques III

- Properties of the function \(f\)
  - Linear
  - Quadratic
  - Convex
  - Sparse
Classification of Techniques IV

Other Issues

- Local versus Global Solutions
- NP-complete Problems
- Probability and Optimization

Studying Optimization

- Approach:
  - Analyze from 3 different points of view

Optimization Techniques x Graphics Problems

Optimization in Graphics Applications

* Typical applications from different areas

- Selected Applications
  - Modeling: Variational Curves
  - Visualization: Camera Control
  - Vision: Edge Detection
  - Image Processing: Contour Following

Variational Modeling of Curves

- Energy Minimization Model
  \[ \gamma(t) = \arg \min \ E_{\text{total}}(\gamma) \]

- Energy Functional
  - Shape Quality
  - Model Control
  \[ E_{\text{total}}(\gamma) = E_{\text{internal}}(\gamma) + E_{\text{external}}(\gamma) \]

Internal Energy

- Fair Shape (Physical Model)
  \[ E_{\text{internal}}(\gamma) = \lambda E_{\text{bend}} + (1 - \lambda) E_{\text{stretch}} \]

- Thin Plate
  \[ E_{\text{bend}}(\gamma) = \int k^2(t) dt \]

- Membrane
  \[ E_{\text{stretch}}(\gamma) = \int \| \gamma'(t) \| dt \]
**External Energy**

- Modeling Controls
  - Attraction / Repulsion Forces
- Punctual
  \[ E_{\text{punctual}}(\gamma) = \min_t \| \gamma(t) - p \|^2 \]
- Directional
  \[ E_{\text{directional}}(\gamma) = \min_t \| \gamma(t) \times p \| \]

**Applications**

- Reconstruction from Points
- Interactive Modeling

**Camera Control**

- Projective Transformation
  \[ T(x) = p \]
- Inverse Problems
  - Compute Camera Transformation \( T \)
  - Compute Scene Points \( x \)
- Best Parameter Estimation
  - Camera Paths

**Differential Camera Control**

- Camera Motion with Constrains
  \[ \text{minimize} \quad E = \frac{1}{2} \left\| \frac{dc}{dt} - \frac{dc}{dt} \right\| \]
  \[ \text{subject to} \quad \frac{dp}{dt} - \frac{dp}{dt} = 0 \]

**Image Analysis**

- Primal Sketch
  - Edges
- David Marr’s Conjecture
  - Image is Characterized by its Edges
- Boundary Operator
  \( \text{(Computation of Edges)} \)
  - Geometric Methods
    * snakes

**Snakes**

- Geometric Computation of the Boundary
- Energy Minimization Approach
  - Boundary function (e.g. \( \text{grad } I(u,v) \))
Classification of Techniques (again)...

• Variational optimization
  $S$ has infinite dimension
  (elements of $S$ are functions)

• Continuous optimization
  $S$ has finite dimension, i.e., $S \subset \mathbb{R}^n$
  (elements of $S$ are vectors)

• Discrete optimization
  - usually, $S = \mathbb{Z}^n$

• Combinatorial optimization
  - concise description by exploiting combinatorial structure

What type of optimization technique?

• Often, the type of optimization technique to adopt is a matter of the user’s choice.

• Basic question:
  At which level to discretize?

Concrete Example (Image Processing)

• Contour extraction
  Find $x: [0, 1] \to \mathbb{R}^2$
  with $x(0) = P_0$ and $x(1) = P_1$
  as to minimize

$$\int_0^1 |x'(t)| \omega(x(t)) \, dt,$$

where

$$\omega(x) = \frac{1}{k + \|\nabla I(x)\|}$$

Possible approaches ... (1)

• Tackling directly the variational problem
  - infinitely-dimensional space
  - Euler-Lagrange formulation
    (partial differential equations)
  - solution process requires discretization

Example (Variational formulation)

Euler-Lagrange Equations

$$f(t, x, x') = \|x'(t)\| \omega(x(t))$$

$$\min \int f(t, x', x) \, dt$$

$$\frac{\partial f}{\partial x_i} = \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial x'} \right), \quad i = 1, \ldots, n$$

Numerical Discretization

Possible approaches... (2)

• Approximating by an optimization problem in finite dimension
  - for instance, restrict attention to polynomials
    (characterized by a finite set of parameters)
  - use methods of continuous optimization in finite dimensions
Example (Continuous formulation)

Parametric Cubic Curves

\[ x(t) = (x_1(t), x_2(t)) \]
\[ x_1(t) = c_{10} + c_{11}t + c_{12}t^2 + c_{13}t^3 \]
\[ x_2(t) = c_{20} + c_{21}t + c_{22}t^2 + c_{23}t^3 \]

8-dimensional space

Possible approaches... (3)

- Approximating by a discrete optimization problem
  - discretization occurs at the beginning of the process
  - for instance, consider a regular grid and solve a shortest path problem

Example (Discrete formulation)

\[ \text{cost} = \text{length} \cdot \frac{1}{k + \|N\|} \]

The need for discretization

- Discretization is always needed...
- ... but can occur at different points of the solution process...
- ... leading to different computational schemes

Conclusions

* Optimization has many uses in Graphics

- Representation
  - Solves Ambiguity
  - Incorporates User-defined Criteria

- Algorithms
  - Efficient
  - Robust

- User Interface
  - Intuitive Controls

- Optimization Techniques
  - Main Issue: discretization