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Technical Report - TR-07-02 - Relatório Técnico

April - 07 - April

An Aspect Ratio Upper Bound in 2D Solid Alpha Complexes of Poisson Disc Samplings

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Abstract. We consider 2D Alpha Solid Complexes of Poisson disc samplings of regions of solid objects. In this article we prove that the triangles have aspect ratio less or equal than $4\sqrt{3}$. This fact implies that the mesh has a good quality.

1 Introduction

Almost all problems in computer graphics involve samplings. It is well known that the properties of the sampling distribution can greatly affect the quality of the final result. Poisson disc samplings have excellent blue noise spectra and also mimics the distribution of photoreceptors in a primate eye [3]. They are also a typical example of stochastic sampling also known as blue noise [2].

Generating meshes of point samplings are important steps in numerical simulations, in scientific and engineering. The builded mesh is a partition of a polihedral domain into elements of simple shape and two dimensional triangulations are popular meshes.

The mesh quality is often measured by the shape of the triangles, the edge lengths and the mesh size. A popular shape measure of a triangle τ is the aspect-ratio $\frac{L^2}{area(\tau)}$, where L is the longest edge of τ.

In this paper we consider the aspect ration quality on 2D solid alpha complex of poisson disc samplings. This kind of mesh is a subset of Delaunay triangulation by taking out all triangles which have circuncribed radius greater than a fixed positive real parameter α. This parameter is the same used to build the poisson disc sampling.

1.1 Overview

The outline of this paper is as follows. In the next section we give basic concepts such as Simplicia Complexes, Delaunay Triangulations, Alpha Complexes, Alpha Shapes and Solid Alpha Complexes. We also give some preliminary notations. In secion 3 we define poisson disc samplings and show how to build it. In section 4 we prove the main result of this paper that is the theorem 1. In section 5 we give a conclusion and show some examples.

2 Concepts and Preliminary Notations

2.1 Simplicial Complexes

A k-simplex $\sigma_T = conv(T)$ is the convex combination of an affinelly independent point set $T \subset \mathbb{R}^n$, $\#T = k + 1$; $0 \leq k \leq n$; and $\#$ denotes the cardinality. k is the dimension of the simplex σ_T. A *simplicial complex* K is a finite collection of simplices with the following two properties:

1. if $\sigma_T \in K$ then $\sigma_U \in K, U \subset T$.
2. if $\sigma_U, \sigma_V \in K$, then $\sigma_{T\cap V} = \sigma_U \cap \sigma_V$.

Both properties above imply that $\sigma_{T\cap V} \in K$. The underlying polyhedron of K is $|K| = \cup_{\sigma\in K}\sigma$. A subcomplex L of K is a simplicial complex $L \subset K$.

A *Solid Simplicial Complex* has not isolated simplices,i.e., k-simplices that are not faces of a simplice with greater dimension. Given a simplicial complex K, the collection $\overline{K} \subset K$ is the maximal solid simplicial complex contained in K.

2.2 Delaunay Triangulations

The Delaunay triangulation of a set of points on the plane is a unique set of triangles connecting the points satisfying an "empty circle" property: the circumcircle of each triangle does not contain any other points. It is in some sense the most natural way to triangulate a set of points. We give below a general definition based on simplicial complexes.

Definition 1. *Given a set $S \subset \mathbb{R}^n$ in general position, the Delaunay Triangulation of S is the simplicial complex $\mathrm{DT}(S)$ consisting only of*

1. *all k-simplices,σ_T ($0 \leq k \leq n$) , with $T \subset S$ such that the circumsphere (the smallest sphere such that all points lie on its boundary) of T does not contain any other points of S, and*
2. *all k-simplices which are faces of other simplices in $\mathrm{DT}(S)$.*

2.3 Alpha Complexes

Alpha Complexes are simplicial complexes that describe how a set of points are structurated in clusters. By varying a positive real parameter α we obtain different shapes ranging from fine to crude. The most fine shape is the set of points, which is obtained when $\alpha = 0$. As α increases, the shape grows by adding simplices and develops cavities that may join to form tunnels and voids. The most crude shape is the Delaunay triangulation wich is obtained for large values of α. More precisely, we have for alpha complexes the following definition:

Definition 2. *Let $S \subset \mathbb{R}^n$ be a set of points in general position. For $T \subset S$ with $\#T \leq n$, let b_T and μ_T denote the smallest ball that contains the points of T and its radius, respectively. Given $0 \leq \alpha \leq \infty$, the alpha complex $\mathcal{C}_\alpha(S)$ of S is the following simplicial subcomplex of $\mathrm{DT}(S)$ where a simplex $\sigma_T \in \mathrm{DT}(S)$ is in $\mathcal{C}_\alpha(S)$ if*

1. *$\mu_T < \alpha$ and $b_T \cap S = \emptyset$, or*
2. *σ_T is a face of another simplex in $\mathcal{C}_\alpha(S)$.*

The alpha shapes S_α is defined as the underlying polyhedron of an alpha complex $\mathcal{C}_\alpha(S)$, i. e., $S_\alpha = |\mathcal{C}_\alpha(S)|$. As well in the alpha complexes for large values of parameter α we obtain the Delaunay triangulation likewise in the alpha shapes we obtain precisely the convex hull. Indeed, an alpha shape is a suitable generalization of the convex hull concept that is used in several applications [5, 6].

2.4 Alpha Solid and Solid Alpha Complex

In general, the alpha-complexes and alpha shapes are mixed-dimension complexes and polytopes, respectively. Bernardini et al.[**?**] defined the *solid alpha-shapes* (or alpha-solid) as the alpha-shape without isolated k-simplices. In a similar way we define the *solid alpha complex* as the alpha complex without isolated k-simplices. It is a kind of a "regularized" subcomplex version of the alpha complex $\mathcal{C}_\alpha(S)$. As we discussed before we have that the solid alpha complex is defined as $\overline{\mathcal{C}_\alpha(S)}$. In figure 1 we show a visual difference between alpha-complex and solid alpha complex, in the 2D case.

3 Poisson Disc Samplings

In this section we will define Poisson Disc Samplings (PDS), an important class of stochatic sampling very usefull in Computer Graphics applications. We will define then for regions in the plane. More precisely we restrict then as open, connected and bounded regions in the plane.

Definition 3. *Let $S_\alpha = \{s_1, s_2, ..., s_n\}$ be a samplig of a solid region R ($R = A \cup \partial A$, A bounded, open and connected). We say that S_α is a Poisson disc Sampling (PDS) if $R \subset \cup_{s_i \in S_\alpha} B_\alpha(S_i)$ (Coverage condition) and $S_\alpha \cap B_\alpha(s_i) = \{s_i\}$, $\forall i$ (Poisson condition).*

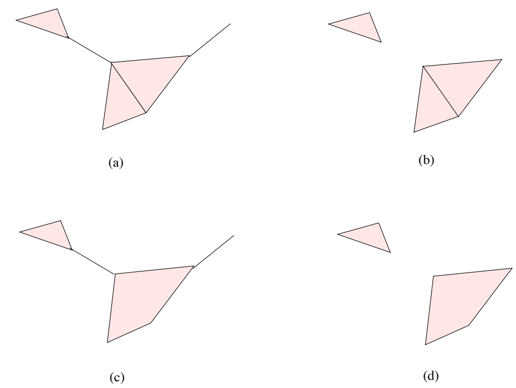

Figure 1: the alpha complex (a) and its solid alpha complex (b). The alpha shape (c) and its alpha solid (d).

Whenever we refer for a region R it will be the union of a bounded, open and connected subset of $\mathbb{R}^2$ with its boundary.

Proposition 1. *There exists a PDS any region R.*

Proof. The sampling can be generated by the algorithmic approach of *dart throwing* [1]. In this approach we have a random sampling generator in the region and a validator that verify if the samplings satisfy the desired geometric criteria. In our case the criteria is the Poisson condition. If a sampling is validated then it is added to the output, otherwise, we descard it. The algorithm stops when all samplings satisfy the covering condition. □

In the figure 2 we have an example of PDS.

4 Aspect Radio Bound

In this section we will prove the following the main result of the paper:

Theorem 1. *Let S_α be a PDS sampling of a region R. The aspect ratio of the triangles of $\overline{C_\alpha(S_\alpha)}$ is bounded by $4\sqrt{3}$.*

Proof. Let τ be a triangle and $b \leq a \leq L$ its edges. Let $S(\tau)$ be the area of the triangle τ and R its circunscribed radius. By the known formula $S(\tau) = \frac{abL}{4R}$, we have that aspect-ratio$(\tau) = \frac{L^2}{S(\tau)} = \frac{4RL}{ab}$.

First lets fix R and L. So we must minimize ab. Looking at the figure 3. We have:

$$\frac{a}{\sin x} = \frac{b}{\sin(\theta - x)} = cte = k$$

$$\Rightarrow ab = k^2 \sin x \sin(\theta - x)$$

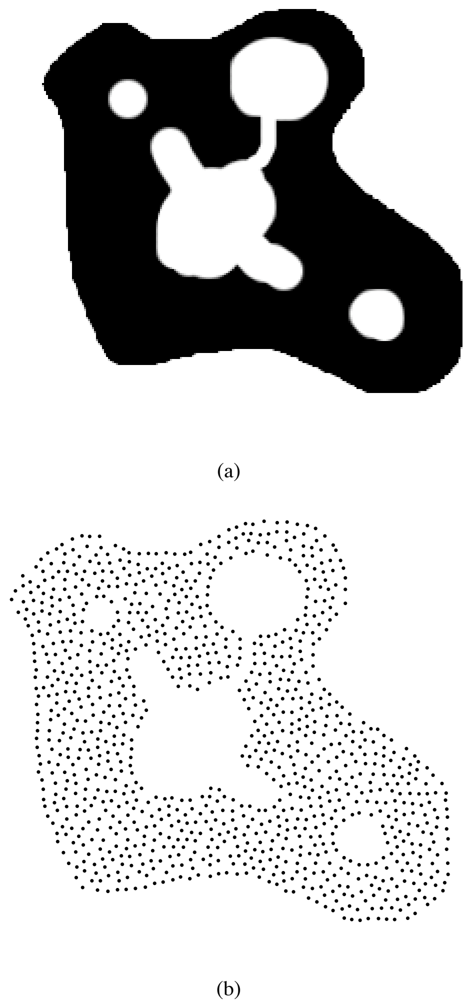

Figure 2: PDS example: (a) solid region, (b) the sampling.

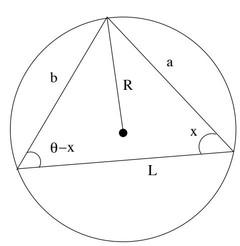

Figure 3: Minimizing ab with R and L fixed.

differentianting $f(x) = \sin x \sin(\theta - x)$ we have:

$$f'(x) = \frac{\partial}{\partial x} \sin x \sin(\theta - x) =$$
$$\cos x \sin(\theta - x) - \cos(\theta - x) \sin x =$$
$$\sin(\theta - 2x)$$

as b is the lowest edge then $\pi/2 > \theta - x > x \Rightarrow \pi/2 > \theta - 2x > 0$ and we conclude that f is growing. But $\alpha \leq b$. So, the lowest value of ab occurs when $b = \alpha$.

Now lets maximize the ratio $\frac{L}{a}$ when we fix R and $b = \alpha$. Looking at the figure 4 we have that:

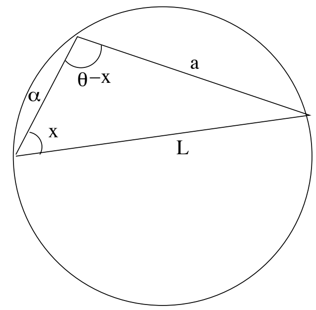

Figure 4: Maximizing $\frac{L}{a}$ with R and $b = \alpha$ fixed.

$$\frac{L}{a} = \frac{\sin(\theta - x)}{\sin x} = \sin\theta \cot x - \cos\theta$$

As we know $0 < x < \pi/2$ the cotangent function is decreasing then the minimal value of the ratio occurs when $a = \alpha$.

Lets now maximize RL fixing $a = b = \alpha$. Looking at the figure 5 we have:

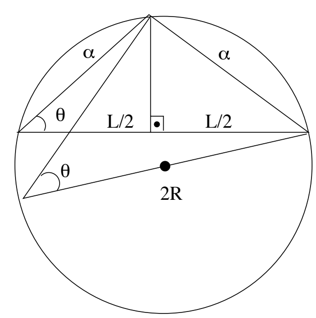

Figure 5: Maximizing RL with $a = b = \alpha$ fixed.

$$\cos\theta = \frac{L}{2\alpha}$$
$$\sin\theta = \frac{\alpha}{2R}$$

$$\frac{L^2}{4\alpha^2} + \frac{\alpha^2}{4R^2} = 1$$
$$16R^2L^2 = 16\alpha^2R^2 - 4\alpha^4$$

It is clear that RL is maximal when $R = \alpha$. This fact define $L = \alpha\sqrt{3}$. In conclusion, the maximal aspect ratio is $4\sqrt{3}$ which is the configuration of the triangle in the figure 6. In this case the greatest angle is $3\pi/2$.

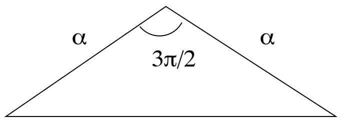

Figure 6: Configuration of the worst case.

□

5 Conclusion

In this paper we showed that the triangles of a 2D Solid Alpha Complex have limited aspect ratio. More precisely we proved that the upper bound is $4\sqrt{3}$ and the equality occurs when the greater angle is $3\pi/2$. In the figure 7 we show three examples with good shaped triangles. In 7.a and 7.b we have a single recangle sample and its reconstruction. In 7.a and 7.b we have a ring example. In 7.a and 7.b we have a more complex region with boundary features and many holes.

This result can not be generalized for dimension three. We have a counter example of four samples on the vertices of a square with a pertubation. The tetrahedra with vertices in these points can compose a sliver with aspect ratio as far as we desire by moving slightly the points to the plane of the square.

Althoug there is no guarantee of limited aspect ratio for higher dimensions we can measure then by means of probability distribution. Then the question we can make is what is the probability of exists a tetrahedra with aspect ratio grater than a given positive real value?

Our main purpose is to study the relationships between solid alpha complexes, topology and poisson disc samplings of solid objects. The result of this paper gave us strong support for our research. More details can be found in [4].

Acknowledgements

This research has been developed in the VISGRAF Laboratory at IMPA with. VISGRAF is sponsored by CNPq, FAPERJ, FINEP and IBM Brasil.

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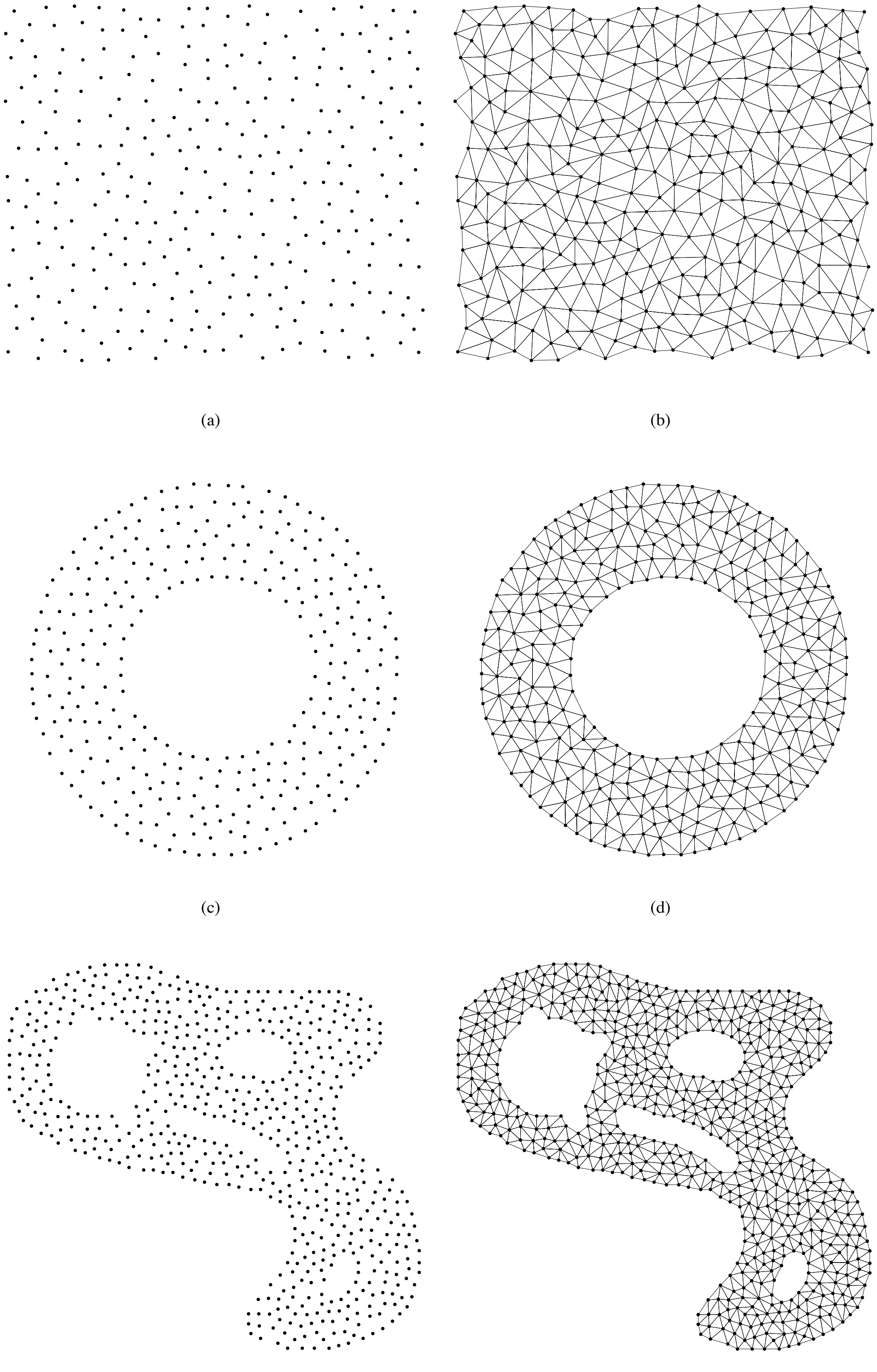

(a) (b) (c) (d) (e) (f)

Figure 7: Examples of poisson disc samples and their solid alpha complexes.