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Reconstructing Poisson Disc Samplings of Solid Objects with Topological Guarantees

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Abstract. In this article we show a sampling condition to reconstruct a poisson disc sampling of solid objects by means of 2D Alpha Solid Complexes that is topological equivalent with the original object.

1 Introduction

Almost all problems in computer graphics involve samplings. It is well known that the properties of the sampling distribution can greatly affect the quality of the final result. Poisson disc samplings have excellent blue noise spectra and also mimics the distribution of photoreceptors in a primate eye [4]. They are also a typical example of stochastic sampling also known as blue noise [3].

Generating meshes of point samplings are important steps in numerical simulations, in scientific and engineering. The builded mesh is a partition of a polihedral domain into elements of simple shape and two dimensional triangulations are popular meshes.

Given a solid object R, in this paper we show a sampling condition for a Poisson Disc Sampling of R to reconstruct it by means of a 2D Alpha Solid Complex that is topological equivalent with R. The sampling condition is a positive real number that indicates the radius of the Poisson Discs.

1.1 Overview

The outline of this paper is as follows. In the next section we give basic concepts such as Simplicial Complexes, Delaunay Triangulations, Alpha Complexes, Alpha Shapes and Solid Alpha Complexes. We also give some preliminary notations. In section 3 we define poisson disc samplings and show how to build then. In section 4 we prove the main result of this paper that is the theorem 1. In section 5 we give a conclusion and show some examples.

2 Concepts and Preliminary Notations

2.1 Simplicial Complexes

A k-simplex $\sigma_T = conv(T)$ is the convex combination of an affinelly independent point set $T \subset \mathbb{R}^n$, $\#T = k + 1$; $0 \leq k \leq n$; and $\#$ denotes the cardinality. k is the dimension of the simplex σ_T. A *simplicial complex* K is a finite collection of simplices with the following two properties:

1. if $\sigma_T \in K$ then $\sigma_U \in K, U \subset T$.
2. if $\sigma_U, \sigma_V \in K$, then $\sigma_{T \cap V} = \sigma_U \cap \sigma_V$.

Both properties above imply that $\sigma_{T \cap V} \in K$. The underlying polyhedron of K is $|K| = \cup_{\sigma \in K} \sigma$. A subcomplex L of K is a simplicial complex $L \subset K$.

A *Solid Simplicial Complex* has not isolated simplices,i.e., k-simplices that are not faces of a simplice with greater dimension. Given a simplicial complex K, the collection $\overline{K} \subset K$ is the maximal solid simplicial complex contained in K.

2.2 Delaunay Triangulations

The Delaunay triangulation of a set of points on the plane is a unique set of triangles connecting the points satisfying an "empty circle" property: the circumcircle of each triangle does not contain any other points. It is in some sense the most natural way to triangulate a set of points. We give below a general definition based on simplicial complexes.

Definition 1. *Given a set $S \subset \mathbb{R}^n$ in general position, the Delaunay Triangulation of S is the simplicial complex* $\mathrm{DT}(S)$ *consisting only of*

1. *all k-simplices,σ_T $(0 \leq k \leq n)$, with $T \subset S$ such that the circumsphere (the smallest sphere such that all points lie on its boundary) of T does not contain any other points of S, and*
2. *all k-simplices which are faces of other simplices in* $\mathrm{DT}(S)$.

2.3 Alpha Complexes

Alpha Complexes are simplicial complexes that describe how a set of points are structurated in clusters. By varying a positive real parameter α we obtain different shapes ranging from fine to crude. The most fine shape is the set of points, which is obtained when $\alpha = 0$. As α increases, the shape grows by adding simplices and develops cavities that may join to form tunnels and voids. The most crude shape is the Delaunay triangulation wich is obtained for large values of α. More precisely, we have for alpha complexes the following definition:

Definition 2. *Let $S \subset \mathbb{R}^n$ be a set of points in general position. For $T \subset S$ with $\#T \leq n$, let b_T and μ_T denote the smallest ball that contains the points of T and its radius, respectively. Given $0 \leq \alpha \leq \infty$, the alpha complex $\mathcal{C}_\alpha(S)$ of S is the following simplicial subcomplex of $\mathrm{DT}(S)$ where a simplex $\sigma_T \in \mathrm{DT}(S)$ is in $\mathcal{C}_\alpha(S)$ if*

1. *$\mu_T < \alpha$ and $b_T \cap S = \emptyset$, or*
2. *σ_T is a face of another simplex in $\mathcal{C}_\alpha(S)$.*

The alpha shapes S_α is defined as the underlying polyhedron of an alpha complex $\mathcal{C}_\alpha(S)$, i. e., $S_\alpha = |\mathcal{C}_\alpha(S)|$. As well in the alpha complexes for large values of parameter α we obtain the Delaunay triangulation likewise in the alpha shapes we obtain precisely the convex hull. Indeed, an alpha shape is a suitable generalization of the convex hull concept that is used in several applications [6, 7].

2.4 Alpha Solid and Solid Alpha Complex

In general, the alpha-complexes and alpha shapes are mixed-dimension complexes and polytopes, respectively. Bernardini et al.[**?**] defined the *solid alpha-shapes* (or alpha-solid) as the alpha-shape without isolated k-simplices. In a similar way we define the *solid alpha complex* as the alpha complex without isolated k-simplices. It is a kind of a "regularized" subcomplex version of the alpha complex $\mathcal{C}_\alpha(S)$. As we discussed before we have that the solid alpha complex is defined as $\overline{\mathcal{C}_\alpha(S)}$. In figure 1 we show a visual difference between alpha-complex and solid alpha complex, in the 2D case.

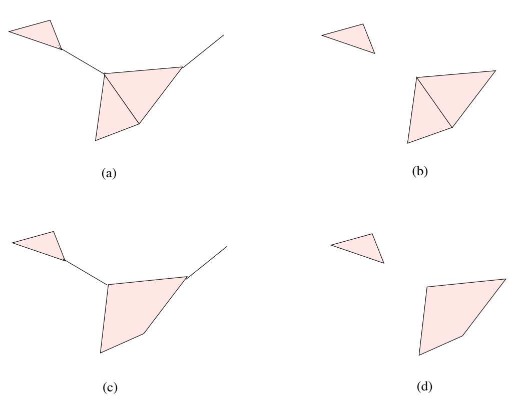

Figure 1: the alpha complex (a) and its solid alpha complex (b). The alpha shape (c) and its alpha solid (d).

3 Poisson Disc Samplings

In this section we will define Poisson Disc Samplings (PDS), an important class of stochatic sampling very usefull in Computer Graphics applications. We will define then for regions in the plane. More precisely we restrict then as open, connected and bounded regions in the plane.

Definition 3. *Let $S_\alpha = \{s_1, s_2, ..., s_n\}$ be a samplig of a solid region R ($R = A \cup \partial A$, A bounded, open and connected). We say that S_α is a Poisson disc Sampling (PDS) if $R \subset \cup_{s_i \in S_\alpha} B_\alpha(S_i)$ (Coverage condition) and $S_\alpha \cap B_\alpha(s_i) = \{s_i\}, \forall i$ (Poisson condition).*

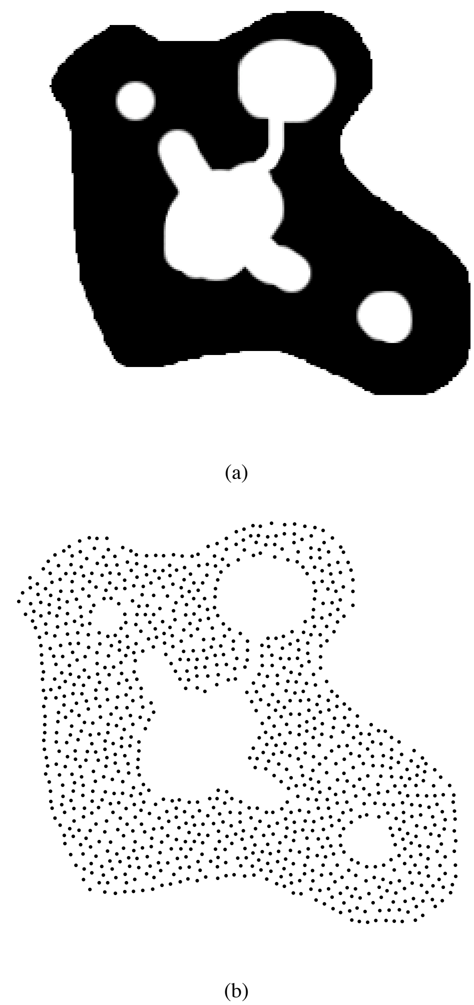

Figure 2: PDS example: (a) solid region, (b) the sampling.

Whenever we refer for a region R it will be the union of a bounded, open and connected subset of $\mathbb{R}^2$ with its boundary.

Proposition 1. *There exists a PDS any region R.*

Proof. The sampling can be generated by the algorithmic approach of *dart throwing* [2]. In this approach we have a random sampling generator in the region and a validator that verify if the samplings satisfy the desired geometric criteria. In our case the criteria is the Poisson condition. If a sampling is validated then it is added to the output, otherwise, we descard it. The algorithm stops when all samplings satisfy the covering condition. □

In the figure 2 we have an example of PDS.

4 Sampling Condition

In this section we will prove the following main result of the paper that is the theorem 1. First lets review the definition of *medial axis*.

Definition 4. *The Medial Axis of a boundary ∂R is closure of points in the plane which have two or more closest points in ∂R.*

Our condition is based on *Local Feature Size* function, which in some sense quantifies the local level of detail at a point on smooth curve.

Definition 5. *The Local Feature Size, $LFS(p)$, of a point $p \in \partial R$ is the Euclidian distance from p to the closest point m on the medial axis.*

Notice that, because it uses the medial axis, this definition of Local Feature size depends on both the curvature at p and the proximity of nearby features.

The next defintion will be usufull to assign topology information to sampling points.

Definition 6. *Let S_α be a PDS of a region R and a $p \in S_\alpha$. We say that p is a boundary point if $B_\alpha(p) \bigcap \partial R \neq \emptyset$. On the contrary we say that p is a interior point.*

Theorem 1. *Let R be a region and S_α a PDS such that $\alpha \leq \frac{1}{2} inf_{p \in \partial R} LFS(p)$. Then $|C_\alpha(S_\alpha)| \sim R$. The symbol $\sim$ means that the two spaces are topologicaly equivalent.*

Proof.

Lemma 1. $C_\alpha(S_\alpha) = \overline{C_\alpha(S_\alpha)}$.

Proof. First lets proof that $C_\alpha(S_\alpha)$ does not have isolated points. Suppose that p is an isolated point. Then we have that

$$B_\alpha(p) \cap (\cup_{q \in S_\alpha q \neq p} B_\alpha(q)) = \emptyset$$

Let $w \in S_\alpha - \{p\}$. As R is connected there exists a path pw contained in R. But this is an absurd because there is a ϵ ring neighborhood of $B_\alpha(p)$ that does not cointain any point of R and pw must pass through this ring.

Note that we proved this first result by using only connectivity arguments which means that $C_\alpha(S_\alpha)$ is always a connected graph independently of α.

Now lets proof that $C_\alpha(S_\alpha)$ does not have isolated edges. Suppose that e is an isolated edge. Looking at the figure 3 we consider p and q the vertices of e. Let $\{x, y\} = \partial B_\alpha(p) \cap \partial B_\alpha(q)$. If $x \in R$ then there exists a point $w \in S_\alpha$ such that $x \in B_\alpha(w)$. Then we have that $\{w, p, q\} \in B_\alpha(x)$ and we conclude that the circunscruibed circle of the triangle wpq has radius less or equal than α and it does belong to $C_\alpha(S_\alpha)$. This is an absurd because we have supposed that e is isolated. Hence $x \notin R$. Using the same arguments we can prove that $y \notin R$. As a consequence of the intermediate value theorem the line segment px intersects ∂R in some point x'. Analogously for the line segments qx, py and qy we have the boundary points x'', y' and y''.

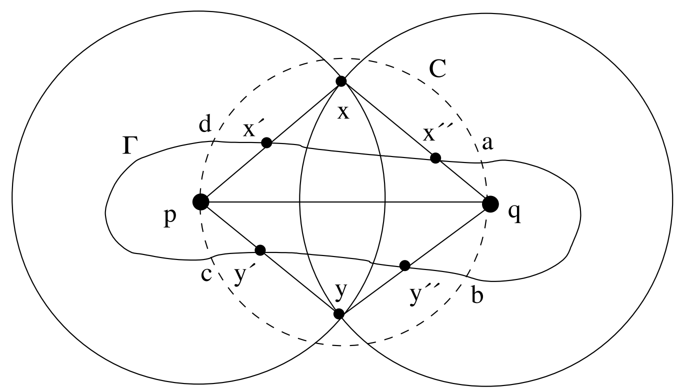

Figure 3: The isolated edge pq and the boundary configuration.

Lemma 2. *A disk containing a point $p \in \partial R$, with diameter at most $LFS(p)$, intersects ∂R in a topological disk.*

Proof. See Amenta et. al[1]. □

Take a circle with diameter $x'x''$ with size less than α. Using the fact that $\alpha < \frac{1}{2} LFS(z)$ for all z in ∂R and using the lemma 2 this circle intersects ∂R in a topological disk. Lets denote this interval as $[x', x'']$. Analogously we have the intervals $[x'', y'']$, $[y'', x']$ and $[y', x']$. All these interval joined compose a the closed boundary Γ. Since the interval $[x'', y'']$ is also part of the interval of the disk $B_\alpha(q)$ it separates this disk into two regions, one in the interior of R and other exterior of R. As p is a interior point then the interval $[x'', y'']$ must intersect the circle C with diameter pq at least in two distinct points a and b in order to contain the point q. Analogously $[x', y']$ intersects C in more two points b and c. We have that $pq < \alpha$ and, again, using the lemma 2, C intersects ∂R in an interval. This is an absurd because all interval has only two extremes and we found at least four, that is a, b, c and d.

This lemma show us that $C_\alpha(S_\alpha)$ is a solid object.

□

Lemma 3. *If p and q are two boundary points such that $B_\alpha(p) \cap B_\alpha(q) \neq \emptyset$ then $(B_\alpha(p) \cup B_\alpha(q)) \cap \partial R$ is a topological disk.*

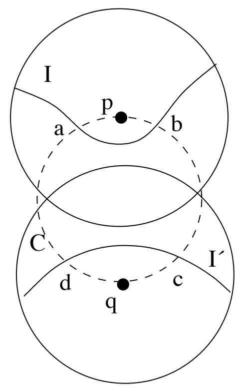

Figure 4: Configuration of two boundary disks with no empty intersection and disjoint intervals.

Proof. Suppose that $B_\alpha(p) \cap B_\alpha(q) \cap \partial R$ is not a topological disk. Then we have the configuration of the figure 4. We will take an argument similar to the one used in the preceding lemma. The interval I splits the disk $B_\alpha(p)$ into two regions, one interior of R and other exterior. As p is inside R then the boundary of the dashed circle C with diameter pq intersects I in two points a and b. In the same way the interval I' intersects the boundary of C in two points c and d. The circle C has diameter less than α and it intersects ∂R in a topological disk. This is an absurd because ∂C has four intersection points with ∂R, that is a, b, c and d.

In conclusion, given B_i a boundary component of R then it is covered by a closed chain of circles, that is, its dual graph is homeomorphic to B_i. □

Lemma 4. *Given $p \in S_\alpha$, if p is a interior point then p is in the interior of $C_\alpha(S_\alpha)$.*

Proof. As p is a interior point then the disk $B_\alpha(p)$ is all contained in R. Let $A \subset S_\alpha$ be a set of points such that $\partial B_\alpha(p) \subset \cup_{q \in A} B_\alpha(q)$. The points in A falls exactly in the link of p and they are conectected. Then p falls in the interior of $C_\alpha(S_\alpha)$. See figure 5.

□

Lemma 5. *The region R and $|C_\alpha(S_\alpha)|$ have the same number of boundary components.*

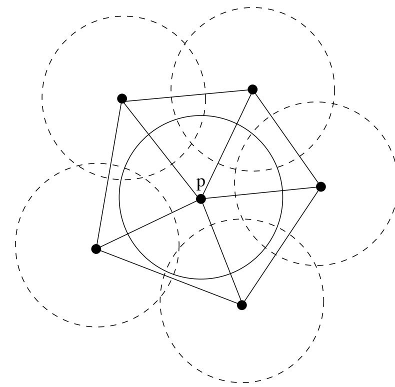

Figure 5: Interior point p and its neighbor sampling points.

Proof. We will establish a bijection between the boundary components of R and $|C_\alpha(S_\alpha)|$.

Consider B_i a boundary component of ∂R. By lemma 3 there exists a chain of circles covering this component such that its dual graph C_i is homeomorphic to B_i.

Let C_i be a boundary component of $|C_\alpha(S_\alpha)|$. By lemma 4 a interior point $p \in S_\alpha$ is in the interior of $|C_\alpha(S_\alpha)|$ then we conclude that C_i has only boundary points. C_i is associated directly with the boundary such that its points belong. □

As R and $|C_\alpha(S_\alpha)|$ are solid objects and they have the same number of connected components. It is well known that this result implies they are topologicaly equivalents.

□

5 Conclusion

In this paper we showed a sampling condition to reconstruct a poisson disc sampling of solid objects by means of 2D Alpha Solid Complexes that is topological equivalent with the original object. In the figure 6 we show three examples of samplings of solid objects and their reconstructions topologicaly equivalent. In 6.a and 6.b we have a single rectangle sample and its reconstruction. In 6.a and 6.b we have a ring example. In 6.a and 6.b we have a more complex region with five boundary curves(including the outer one) and many features.

We hope that this result can be generalized for dimension three ond others. The inf constraint for local feature size of the boundary is very strong it turns the sampling condition as a global parameter. Indeed, the Poisson Disc Sampling is also a global parameter. In a future work we plan to create a less constraint condition based on a formalization of adaptive poisson disc samplings. This adaptiveness can also be obtained by wtih others metrics assigned to the plane. It will give us a local sampling condition instead of a global in uniform poisson disc samplings.

Our main purpose is to study the relationships between solid alpha complexes, topology and poisson disc samplings of solid objects. The result of this paper gave us strong support for our research. More details can be found in [5].

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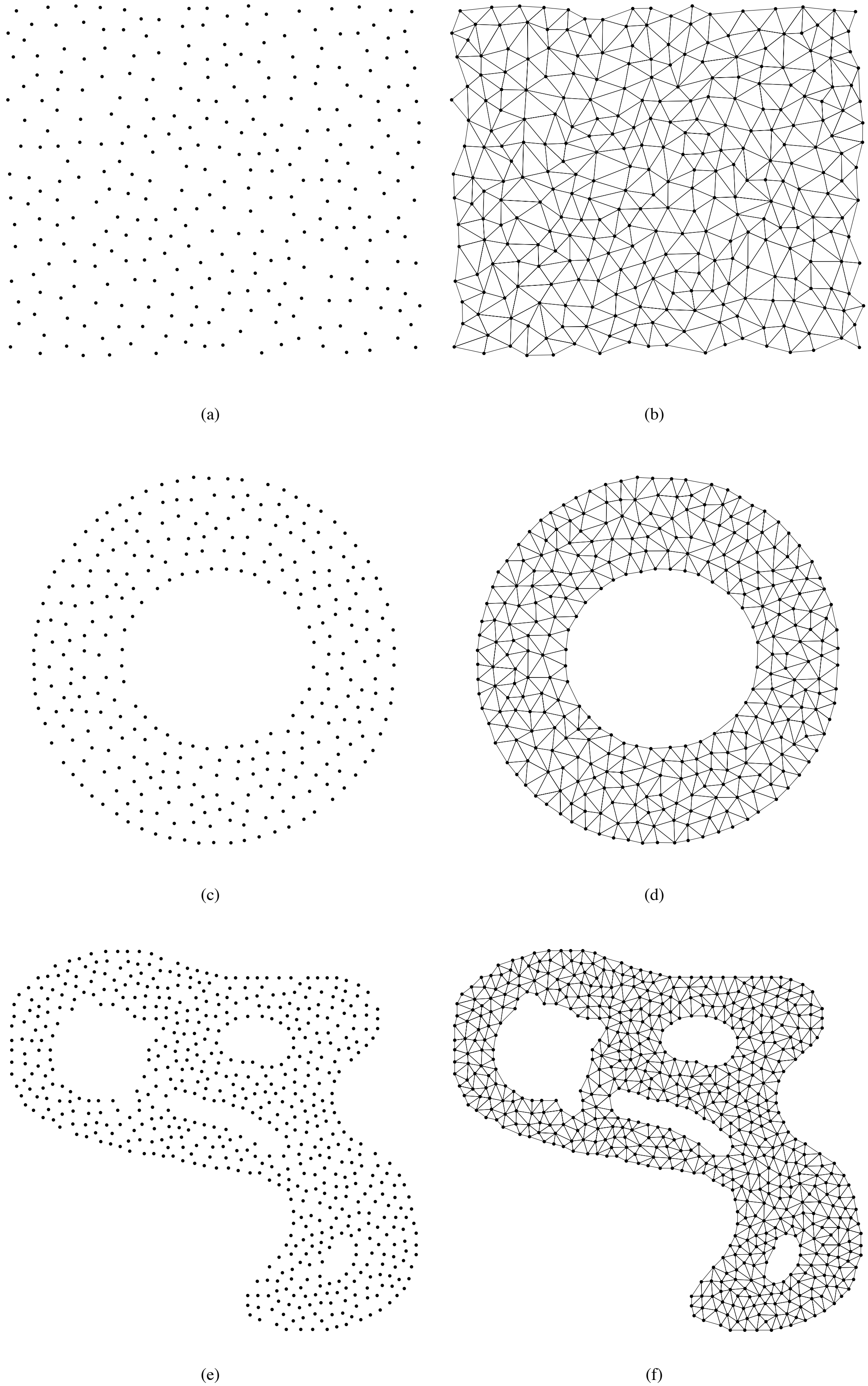

Figure 6: Examples of poisson disc samples and their solid alpha complexes.