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Discrete Exterior Calculus: History, Theory and Applications

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Introduction

The *Exterior Calculus of Differential Forms*, first introduced by E. J. Cartan [Cartan 1945], is a geometry-based calculus that has become the foundation of modern differential geometry. It is independent of a coordinate system and thus allows for expressing the geometric nature of many current physical theories. The *Discrete Exterior Calculus* (DEC) aims to offer a discrete counterpart of the exterior calculus on differentiable manifolds, now on simplicial and cellular chains.

In this report, we first give a brief introduction to the DEC, with the main goal to set the discipline into a broader perspective by drawing its connections to other areas of mathematics and different computational methods. Next, we look at the development of DEC in the last few decades, we talk about research groups and people involved in the area (Chapter 1). We proceed with basic definitions and the theory of DEC (Chapter 2). Finally some applications of DEC in computer graphics are presented and open problems are pointed to (Chapter 3).

The Objective of Discrete Exterior Calculus

As said above, exterior calculus explores geometric meaning of quantities in the continuous setting and uses geometric insight to model theories such as electromagnetism or fluid mechanics. DEC translates the tools of exterior calculus into the discrete world of simplicial or more general CW complexes (Definition 2.1.3), maintaining the consistency with the continuous setting. The choice to focus on CW complexes is very reasonable, because it can be shown (using Morse theory) that every differentiable manifold has the *homotopy type* of a CW complex, which informally means that one can be "continuously deformed" into the other. And every compact manifold has the homotopy type of a finite CW complex.

Preserving the essential structures at the discrete level leads to faster, simpler and more exact computations. To put it short, DEC is an extension of the exterior calculus to discrete spaces including graphs and simplicial complexes.

DEC vs. Other Computational Techniques

There have been many computational techniques offering discretization of differential equations, but they often fail to preserve the geometric structures they are simulating.

Finite difference or particle methods focus on accurate discretization of local laws, which often leads to loss of the global structures and invariants.

Finite element methods remedy this inadequacy to some extent by satisfying local conservation laws on average and preserve some important invariants. But there is some loss of fidelity following from a discretization process that does not preserve fundamental geometric and topological structures of the underlying continuous models, according to [Desbrun et al. 2008].

Discrete exterior calculus, unlike the other methods, maintains the separation of the topological (metric-independent) and geometric (metric-dependent) components of quantities at play. Moreover, it stores and manipulate quantities at their geometrically meaningful locations. Pointwise

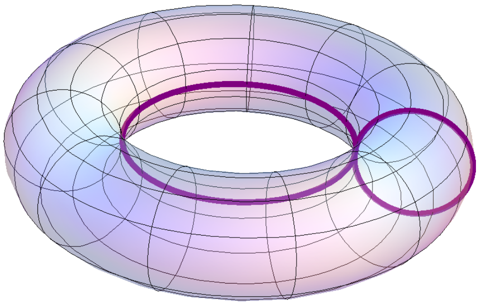

Figure 1: Simple torus.

evaluations of an approximation are not appropriate discrete analogs of n-dimensional volume integrals. Instead, we consider values on vertices, edges, faces as proper discrete versions of pointwise functions, line and surface integrals, respectively.

Related Mathematical Disciplines

Not surprisingly, DEC is closely related to other mathematical disciplines. *Differential geometry* studies problems in geometry using techniques of differential and integral calculus and algebra. *Exterior calculus* is a geometry based calculus that has become the modern language of differential geometry, where it is used to define differential forms. The exterior algebra of differential forms, together with the exterior derivative plays a vital role in the algebraic topology of differentiable manifolds.

Algebraic topology deals with topological spaces and aims to find algebraic invariants that classify topological spaces up to homeomorphism. Algebraic topology of simplicial and CW complexes (duals of simplicial complexes are not simplices in general, but the most reasonable duals of simplicial meshes are the so called CW complexes) then studies the topological invariants of the spaces of our interest.

Example. In algebraic topology, **Betti numbers** are used to classify topological spaces based on the connectivity of n-dimensional simplicial complexes. The first Betti number of an orientable closed finite surface fully characterizes its topology [Stillwell, 1993, p. 182-183].

The k-th Betti number β_k of a 3D simplicial complex has the following intuitive interpretation: β_0 is the number of connected components, β_1 is the number of one-dimensional holes or tunnels (non-contractible circles), and β_2 is the number of two-dimensional voids or cavities. A simple torus has one connected component, thus $\beta_0 = 1$, it has two one-dimensional holes (see the circles in Figure 1), hence $\beta_1 = 2$. And finally, it has one void (the connected empty space inside the torus), therefore $\beta_2 = 1$.

As stated in the abstract of [Harrison 2006], the full calculus on Euclidean spaces, cell complexes, classical Newtonian an the Cartan exterior calculus on smooth manifolds can be obtained as special cases of chainlet operators, products and integrals on the chainlet complex. The reader might be interested to see also other work by Janny Harrison (UC Berkeley).

Chapter 1

History and Research in the Area

At this point we will not give a complete history of the area, but we refer the interested reader to [Desbrun et al. 2005a, Paragraph 2]. We only comment that the papers published could be divided into two main groups depending on their objective: theory and applications. Some authors have been interested in developing a theoretical ground for the area, whereas others have been applying the theory to solve problems in computer graphics in a novel way.

The whole report pretty much shows the state of the art in the theory of DEC. On the other hand, DEC has been applied in many tasks such as mesh smoothing (see [Crane et al. 2013a] and references therein), improving the quality of meshes (e.g., [Mullen et al. 2011]), computing self–supporting masonry structures [de Goes et al. 2013a], or computing distances on general meshes [Crane et al. 2013b]. The reader can get a broader view by looking at Bibliography.

Research Groups and People

There are several institutions involved in the area of Discrete Differential Geometry, that is closely related to DEC, some of these institutions are listed at the website http://ddg.cs.columbia.edu/ of the Columbia University.

We have been studying mainly from materials published by researchers and alumni of the Applied Geometry Lab at Caltech, such as Mathieu Desbrun, Fernando de Goes, and Amil Hirani. All these publications are included in Bibliography.

Chapter 2

Theory

We search for discrete versions of forms and their domains that would be formally identical to its continuous counterparts. Thus domains are represented as chains of simplicial or finite regular CW complexes (Definition 2.1.3) and forms as cochains.

2.1 Primal and Dual Complexes

In the following, we give some basic definitions, such as of simplicial and cell complexes, which are widely studied in algebraic topology and we encourage the interested reader to see [Munkres 1984], [Massey 1991], or [Hatcher 2001]. We introduce a notion of complex more general than that of simplicial complex, called CW complex and in Section 2.2 we assign to it a chain complex, called cellular chain complex.

During the discretization of a smooth manifold we usually create first a simplicial complex and consider it as our primal discretization. And later we define its dual, which is not a simplicial complex in general, but a finite regular CW complex (hereinafter also simply called as cell complex).

In computer graphics, we might want to work with quadrilateral meshes for instance, these meshes are also from the class of cell complexes. In general, all polytopal meshes are cell complexes.

2.1.1 Simplicial Complexes

Definition 2.1.1. Let σ be the ***n*-simplex** spanned by geometrically independent set of points $a_0, \dots, a_n$ in $\mathbb{R}^d$. An **oriented *n*-simplex** is an ordered set of points $[a_0, \dots, a_n]$. Any simplex spanned by a (proper) subset of $\{a_0, \dots, a_n\}$ is called a (proper) **face** of σ. The union of proper faces of σ is called the **boundary** of σ and denoted Bd σ.

Now we can define a simplicial complex:

Definition 2.1.2. A **simplicial complex** K in $\mathbb{R}^d$ is a collection of simplices in $\mathbb{R}^d$ such that:

1. Every face of a simplex of K is in K.
2. The intersection of any two simplexes of K is either empty or it is a face of each of them.

The definition of a simplicial complex is illustrated in the Figure 2.1 (created by author in GeoGebra). The set of simplices on the left is not a simplicial complex because the intersection of the upper triangles is not a face of each of them, among others. But the set on the right is a simplicial complex because the intersection of any two simplices is a face of both.

Definition. The collection of all simplices of K of dimension at most p is called the ***p*-skeleton** of K and is denoted $K^{(p)}$. The points of the collection $K^{(0)}$ are called the **vertices** of K.

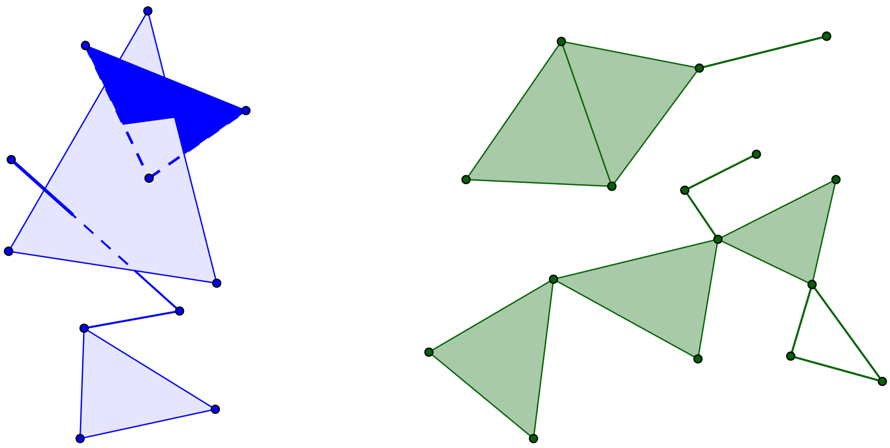

Figure 2.1: A set of simplices (left) and a simplicial complex (right).

2.1.2 Finite Regular CW complexes

Now we will go through some notions from algebraic topology which will peak in the definition of a *CW pseudomanifold* as stated in [Massey 1991]. Some important properties of CW pseudomanifolds can be found also in [Geoghegan 2008, Section 12.3].

Definition. A space is called a ***k*-cell** if it is homeomorphic with the unit k-ball B^k. It is called an **open cell** of dimension k if it is homeomorphic with Int B^k. The set $\dot{e}_i^k := \bar{e}_i^k - e_i^k$ for $k > 0$ is called the **boundary** of the k-cell e_i^k.

We remark that an isolated point is an open and closed 0-cell since it is the whole space of dimension 0. Obviously, every k-simplex is a k-cell, but not vice versa.

Next we give the definition of a special class of cell complexes, called finite CW complexes. The letter C stands for closure-finite and W stands for weak topology. As we mentioned earlier, common primal and dual meshes are from this category of cell complexes. For more details about CW complexes, see for example [Munkres 1984, §38].

Definition 2.1.3. A **finite CW complex** is a space X and a finite collection of disjoint open cells e_i whose union is X such that:

1. X is Hausdorff.

2. For each open m-cell e_i of the collection, there exist a continuous map $f_i : B^m \to X$ that maps Int B^m homeomorphically onto e_i and carries Bd B^m into a union of open cells, each of dimension less than m.

If the maps f_i can be taken to be homeomorphisms, and each set $\dot{e}_i := \bar{e}_i - e_i$ equals the union of some open cells of X, then X is called a **finite regular CW complex (cell complex)**.

Example 2.1.1. The Figure 2.2 shows an example of a regular and non-regular CW complex. The complex on the right consisting of a point on a sphere is not regular because both the endpoints of the 1-cell (circle) get mapped to the single 0-cell (the point), therefore the map mapping boundary of the 1-cell is not a homeomorphism. But the complex on the left consisting of two points and two semicircles is regular.

Definition. We say that e^m is a **face** of e^n if $e^m \subset \bar{e}^n$, and denote as $e^m \leq e^n$. If $e^m \neq e^n$, then e^m is a **proper face** of e^n ($e^m < e^n$).

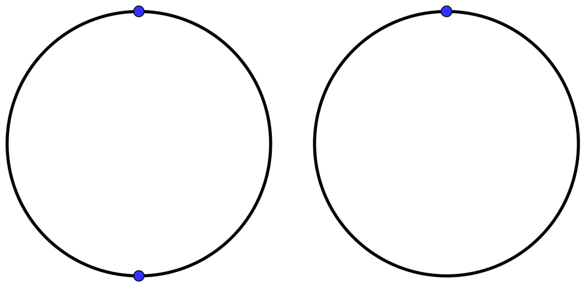

Figure 2.2: A regular (left) and a non-regular CW complex (right).

Next we state some properties of cell complexes without proving them, the interested reader is referred to [Massey 1991, Chapter IX]:

1. If $m < n$ and e^m, e^n are cells such that $e^m \cap \dot{e}^n \neq \emptyset$, then $e^m \subset \dot{e}^n$.
2. For any n-cell $e^n, n \geq 0$, $\dot{e}^n$ is the union of closures of $(n-1)$-cells.
3. Let e^n and e^{n+2} be cells of a regular cell complex such that e^n is a face of e^{n+2}. Then there are exactly two $(n+1)$-cells e^{n+1} such that $e^n < e^{n+1} < e^{n+2}$.

The subspace X^p of X that is the union of the open cells of X of dimension at most p is called the ***p*-skeleton** of X and it is a CW complex in its own right.

Definition 2.1.4. A **CW *n*-pseudomanifold** is an n-dimensional finite regular CW complex which satisfies the following three conditions:

1. Every cell is a face of some n-cell.
2. Every $(n-1)$-dimensional cell is a face of exactly two n-cells.
3. Given any two n-cells, e^n_a and e^n_b, there exist a sequence of n-cells

$$e^n_0, e^n_1, \ldots, e^n_k$$

such that $e^n_a = e^n_0$, $e^n_b = e^n_k$, and e^n_{i-1} and e^n_i have a common $(n-1)$-dimensional face $(i = 1, \ldots, k)$.

As mentioned in [Massey 1991, Chapter IX], it can be shown that any regular CW complex on a compact connected n-manifold without boundary is an n-dimensional CW pseudomanifold. It is known that every compact n-manifold admits a subdivision so as to define a regular CW complex structure on it if $n \leq 3$. However, there exist compact 4-manifolds which do not admit such a subdivision. But then again, in computer graphics we are mainly interested in 2- and 3-dimensional manifolds.

2.2 Orientation and the Boundary Homomorphism

The next definitions are actually a result of a set of theorems in [Massey 1991, Chapter IX], but for our purposes their implications are sufficient.

Definition 2.2.1. Let $K = \{K^n\}$ be a cell complex on the topological space X. And let $[e^n_\lambda : e^{n-1}_\mu]$ be the **incidence number** of the cells e^n_λ and e^{n-1}_μ, for $n > 0$, such that:

1. If e^{n-1}_μ is not a face of e^n_λ, then $[e^n_\lambda : e^{n-1}_\mu] = 0$.

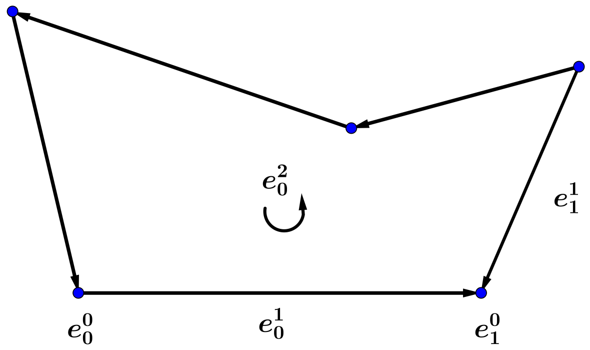

Figure 2.3: Specifying the orientation of a 2-cell e_0^2.

2. If e_μ^{n-1} is not a face of e_λ^n, then $[e_\lambda^n : e_\mu^{n-1}] = \pm 1$.

3. If e_α^0 and e_β^0 are the two vertices of the 1-cell e_λ^1, then $[e_\lambda^1 : e_\alpha^0] + [e_\lambda^1 : e_\beta^0] = 0$.

4. Let e_λ^n and e_ρ^{n-2} be cells such that $e_\rho^{n-2} < e_\lambda^{n-1}$; let e_α^n and e_β^{n-1} denote the unique $(n-1)$-cells e^{n-1} such that $e_\rho^{n-2} < e^{n-1} < e_\lambda^{n-1}$. Then

$$[e_\lambda^n : e_\alpha^{n-1}][e_\alpha^{n-1} : e_\rho^{n-2}] + [e_\lambda^n : e_\beta^{n-1}][e_\beta^{n-1} : e_\rho^{n-2}] = 0.$$

With these conditions it is possible to choose an **orientation** for each cell e_λ^n in one and only one way.

Thus we can specify orientations for the cells of a cell complex by specifying a set of incidence numbers for the complex. Even though the definition may look slightly cumbersome, it actually gives the intuitive way to specify the orientation of cells, as we demonstrate in the Example 2.2.1.

Example 2.2.1. When using the definition, we assume we have given a list of cells of K together with the information as to whether $e_i^{n-1} < e_j^n$ for any two cells e_i^{n-1} and e_j^n.

1. For each 1-cell e^1, choose incidence numbers between it and its two vertices such that conditions (2) and (3) hold, all other incidence numbers will be zero.

2. Now assume, inductively, that incidence numbers have been chosen between all cells of dimension $< n$. Let e^n be an n-cell. Choose a face e_0^{n-1} of e^n, and choose $[e^n : e_0^{n-1}]$ to be +1 or -1. Using condition (4), determine $[e^n : e_i^{n-1}]$ for all $(n-1)$-cells $e_i^{n-1} < e^n$ which have an $(n-2)$-face in common with e_0^{n-1}. Spread over the boundary e^n by repeating this process. All other incidence numbers between e^n and $(n-1)$-cells will be zero by condition (1). And repeat this process for each n-cell of K.

For example in Figure 2.3, in the first step the incidence numbers between the 1-cell e_0^1 and its vertices were chosen to be $[e_0^1 : e_0^0] = -1$ and $[e_0^1 : e_1^0] = 1$. Next, we have chosen $[e_0^2 : e_0^1]$ to be $+1$, therefore the orientation of e_0^2 will be the same as that of e_0^1. But $[e_0^2 : e_1^1] = -1$.

Definition 2.2.2. Let K be an n-dimensional cell complex, and let e_1^n, e_2^n be n-cells with a common $(n-1)$-cell e^{n-1}. We define orientations for e_1^n and e_2^n to be **coherent** (with respect to e^{n-1}) if:

$$[e_1^n : e^{n-1}] + [e_2^n : e^{n-1}] = 0.$$

A set of orientations for all the n-cells of K is said to be **coherent** if it is coherent in the above sense for any pair of n-cells with a common $(n-1)$-face.

K is said to be **orientable** if all its n-cells can be oriented such that any pair of n-cells sharing an $(n-1)$-dimensional face are oriented coherently. Otherwise it is called **nonorientable**.

The orientability or nonorientability of an n-dimensional cell complex K only depends on the underlying topological space involved, and not on the choice of the regular cell complex K.

Before we define the boundary operator, we need one more definition:

Definition 2.2.3. Let K be a cell complex. A ***p*-chain** on K is a function c from the set of oriented p-cells of K to the integers, such that:

1. $c(e) = -c(e')$ if e and e' are opposite orientations of the same cell.
2. $c(e) = 0$ for all but finitely many oriented p-cells e.

We add p-chains by adding their values, the resulting group is denoted $C_p(K)$ and is called the **chain group** of K. If $p < 0$ or $p >$ *dim* K, we let $C_p(K)$ denote the trivial group.

The incidence numbers, or equivalently, the set of orientations of p-cells of K, give us such a p-chain.

Definition 2.2.4. Let $C_n(K), n \geq 0$, be the chain groups of K. The **boundary homomorphism**

$$\partial_n : C_n(K) \to C_{n-1}(K),\ n > 0,$$

is defined to be

$$\partial_n(e^n) = \sum_{\lambda} [e^n : e_{\lambda}^{n-1}] e_{\lambda}^{n-1}, \tag{2.1}$$

where e^n is an *oriented* n-cell of K with $n > 0$.

From the definition of incidence numbers it follows that ∂_n is well-defined and that $\partial_n(-e^n) = -\partial_n(e^n)$.

Next we give an example of computation of the boundary homomorphism in a polygon:

Example 2.2.2. Let e_0^2 and e_1^2 be two 2-cells with opposite orientation as in Figure 2.4. We will check that $\partial_2 e_0^2 = -\partial_2 e_1^2$ and $\partial_2 \partial_1 = 0$. We have:

$$\begin{aligned} \partial_2 e_0^2 &= +e_0^1 - e_1^1 - e_2^1 + e_3^1, \\ \partial_2 e_1^2 &= -e_0^1 + e_1^1 + e_2^1 - e_3^1, \end{aligned}$$

thus $\partial_2 e_0^2 = -\partial_2 e_1^2$. And for $\partial_1 \partial_2 e_0^2$, omitting the indexes of ∂, we can write

$$\begin{aligned} \partial \partial e_0^2 &= \partial(+e_0^1 - e_1^1 - e_2^1 + e_3^1) = +\partial e_0^1 - \partial e_1^1 - \partial e_2^1 + \partial e_3^1 \\ &= e_1^0 - e_0^0 - (e_1^0 - e_2^0) - (e_2^0 - e_3^0) + e_0^0 - e_3^0 = 0. \end{aligned}$$

Remark (Boundary operator for simplicial complexes)**.** Let K be a simplicial complex and $C_n(K)$ the group of oriented n-chains of K for $n > 0$. Then if $\sigma = [v_0, \dots, v_n]$ is an oriented simplex of K, we can define

$$\partial_n \sigma = \partial_n [v_0, \dots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \bar{v}_i, \dots, v_n],$$

where the symbol $\bar{v}_i$ means that the vertex v_i was omitted from the array.

Definition 2.2.5. A **chain complex** $K = \{K_n, \partial_n\}$ is a sequence of abelian groups $K_n, n \in \mathbb{Z}$, and a sequence of homomorphisms $\partial_n : K_n \to K_{n-1}$ which are required to satisfy the condition

$$\partial_{n-1} \partial_n = 0 \ \ \forall n.$$

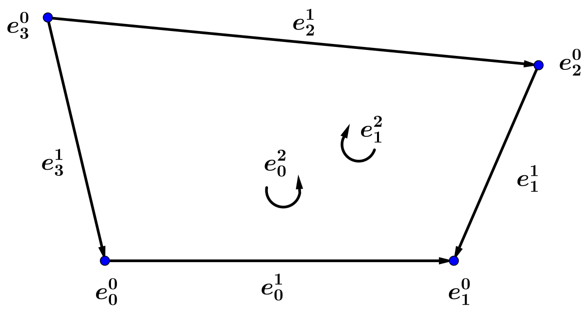

Figure 2.4: Boundary homomorphism on 2-cells e_0^2 and e_1^2.

For any such chain complex $K = \{K_n, \partial_n\}$ we have that $\operatorname{im} \partial_{n+1} \subset \ker \partial_n \subset K_n$ and we can define

$$H_n(K) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}},$$

called the nth **homology group of** K.

Let K be an n-dimensional cell complex, then obviously $K^i = \emptyset$ for $i < 0$ and $i > n$, thus $C_{n+1}(K) = 0$ and we get the following chain complex:

$$0 \longrightarrow C_n(K) \longrightarrow^{\partial_n} \cdots \longrightarrow^{\partial_{k+1}} C_k(K) \longrightarrow^{\partial_k} \cdots \longrightarrow^{\partial_1} C_0(K) \longrightarrow 0.$$

2.3 Differential Forms and the Exterior Derivative

In this section we go through some more notions from algebraic topology, specifically that of cochains and coboundary operator, which will eventually lead to definition of discrete differential forms and the exterior derivative on them.

We can think of a cochain group $C^n(K)$ as dual of a chain group $C_n(K)$ with coefficients group G, i.e., as a group $C^n(K) = \text{ Hom } (C_n(K), G)$,

Definition. Let K be a cell complex and $C_n(K)$ the group of oriented n-chains of K. Let G be an Abelian group. The group of ***n*-dimensional cochains** of K, with coefficients in G, is the group

$$C^n(K) = \text{ Hom } (C_n(K), G).$$

The **coboundary operator** δ is defined to be the dual of the boundary operator $\partial_n : C_{n+1}(K) \to C_n(K)$, i.e., it is the homomorphism

$$\delta : C^n(K) \to C^{n+1}(K),$$

such that $\delta\delta = 0$.

The most common choices of the coefficient group G are $(\mathbb{R}, +)$, $(\mathbb{C}, +)$, or a vector space V together with addition.

If $K = \{K^n\}$ is a cell complex and $u \in C_n(K, G)$ is a cochain on K, then u has a unique expression of the form

$$u = \sum_i g_i e_i^n, \tag{2.2}$$

where $g_i \in G$ and e_i^n are oriented n-cells of K. Thus we sometimes use the notation $\langle c^p, c_p \rangle$ to denote the value of a p-dimensional cochain c^p on the chain c_p. In [Desbrun et al. 2005a] this bilinear pairing of a p-cochain c^p and a simplicial p-chain c_p is called the **natural pairing**.

With this new notation, the definition of the coboundary operator becomes

$$\langle \delta c^p, c_{p+1} \rangle = \langle c^p, \partial c_{p+1} \rangle. \tag{2.3}$$

As suggested in [Geoghegan 2008, Section 12.1], we can see the coboundary operator as a homomorphism that algebraically sums up the cells of which a given cell is a face, which becomes obvious in the following definition. Moreover, equation (2.4) gives us a useful formula which shows the geometrical meaning of the coboundary operator.

Definition 2.3.1. Let X be an oriented cell complex. Then the **coboundary homomorphism** $\delta : C_n(X;R) \to C_{n+1}(X;R)$ is defined by

$$\delta\Big(\sum_i m_i e_i^n\Big) = \sum_j \Big(\sum_i m_i [e_j^{n+1} : e_i^n]\Big) e_j^{n+1}. \tag{2.4}$$

Similarly to the definition of chain complexes (Definition 2.2.5), we have the following:

Definition 2.3.2. A **cellular cochain complex** $(C^*(K,R), \delta)$ consists of a sequence of the Abelian groups of n-dimensional cochains $C^n(K)$ together with the coboundary operators $\delta_n : C^n(K) \to C^{n+1}(K)$, i.e.,

$$0 \longleftarrow C^n(K) \xleftarrow{\delta_{k-1}} \cdots \xleftarrow{\delta_k} C^k(K) \xleftarrow{\delta_{k-1}} \cdots \xleftarrow{\delta_0} C^0(K) \longleftarrow 0.$$

We further define $Z^q(C^*) = \ker \delta^q$, the set of ***q*-dimensional cocycles**, $B^q(C^*) = \operatorname{im} \delta^{q-1}$, the set of ***q*-dimensional coboundaries**, and

$$H^q(C^*) = \frac{Z^q(C^*)}{B^q(C^*)},$$

the ***q*-dimensional cohomology group.**

We now give the formal definition of discrete forms and the discrete exterior derivative.

Definition 2.3.3. A **discrete *n*-form** ω on a cell complex K is an element of $C^n(K)$, the group of n-dimensional cochains of K. We will denote the group of n-forms on K as $\Omega^n(K)$, that is

$$\Omega^n(K) = C^n(K) = \text{ Hom } (C_n(K), G).$$

The **discrete exterior derivative** denoted by $d : \Omega^n(K) \to \Omega^{n+1}(K)$ is defined to be the coboundary operator δ^n.

Next we state an important theorem in differential geometry, called Stokes' theorem, whose equivalent is valid also in the discrete setting.

Theorem 2.3.1 (Stokes)**.** *If M^n is an oriented n-manifold with boundary ∂M^n and ω is an $(n-1)$-form on M^n with compact support, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

The Stokes' theorem tells us that the value of a n-form $d\omega$ over an oriented n-manifold is equal to the value of ω over its whole boundary. Comparing this result to the definition of the coboundary operator (and thus the discrete exterior derivative) in equation (2.3), we can see that the Stokes' theorem on discrete forms on an n–dimensional cell complex is true by definition.

Having defined the derivative d, we can introduce the following standard terminology:

Definition. A k-form α is said to be a **closed form** if $d\alpha = 0$, and it is called an **exact form** if there exist a $(k-1)$-form β such that $d\beta = \alpha$. Two closed k-forms are **cohomologous** if they differ by an exact k-form.
Let M be a manifold, the complex $\Omega^*(M)$ together with the differential operator d is called the **de Rham cohomology complex** on M. The quotient of the real vector space of closed k-forms by the subspace of exact k-forms on M is called the ***k*-th de Rham cohomology group** of K and is denoted $H_{dR}(M)$.

Note that each exact k-form $\alpha = d\beta$ is closed, since $d\alpha = dd\beta = 0$. Furthermore, looking back at the Definition 2.3.2, we conclude that a cocycle is a closed form and a coboundary is an exact form.

2.4 The Cup and the Wedge Product

In this section we first define the wedge product in the continuous world. Next we study cup products of cochains because there is a clear analogy between these two products as we will see later. And we review some possible definitions of a discrete wedge product.

The Wedge Product

There are some slightly different definitions of the wedge product of exterior forms, but we will stick to [Abraham et al. 1988] in the following. A differential k-form α on a smooth manifold is thus defined as tensor field of type $(0, k)$ that is completely antisymmetric, i.e., $\alpha \in T_k^0$.

The **exterior *k*-forms** are elements of $\bigwedge^k(\mathbf{E})$, the vector space of skew symmetric $\mathbb{R}$-valued multilinear continuous alternating maps on a Banach space $\mathbf{E}$.

The **alternation mapping** $\mathbf{A} : T_k^0(\mathbf{E}) \to T_k^0(\mathbf{E})$ is defined by

$$\mathbf{A}\alpha(e_1, \ldots, e_k) = \frac{1}{k!} \sum_{\tau \in S_k} (\operatorname{sign} \tau)\alpha(e_{\tau(1)}, \ldots, e_{\tau(k)}),$$

where the sum is over all $k!$ elements of the **permutation group** S_k, i.e., the set of bijections $\tau : \{1, \ldots, k\} \to \{1, \ldots, k\}$. And for the homomorphism sign : $S_k \to \{-1, 1\}$ we have that $\operatorname{sign} \tau = +1$ if the permutation τ is even and $\operatorname{sign} \tau = -1$ if it is odd.

Definition 2.4.1. Let $\alpha \in T_k^0(\mathbf{E})$ and $\beta \in T_l^0(\mathbf{E})$, we define their **wedge product** $\alpha \wedge \beta \in \bigwedge^{k+l}(\mathbf{E})$ by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!}\mathbf{A}(\alpha \otimes \beta)(e_1, \ldots, e_k, e_{k+1}, \ldots, e_{k+l}),$$

where $\alpha \otimes \beta$ is the **tensor product**, i.e.,

$$\alpha \otimes \beta(e_1, \ldots, e_{k+l}) = \alpha(e_1, \ldots, e_k)\beta(e_{k+1}, \ldots, e_{k+l}).$$

For exterior forms it holds

$$(\alpha \wedge \beta) = \sum_{\bar{\tau}} \operatorname{sign} \bar{\tau} \alpha(e_{\tau(1)}, \ldots, e_{\tau(k)})\beta(e_{\tau(k+1)}, \ldots, e_{\tau(k+l)}), \tag{2.5}$$

where the sum is over all **shuffles**, that is, permutations $\bar{\tau}$ of $\{1, \ldots, k+l\}$ such that $\tau(1) < \cdots < \tau(k)$ and $\tau(k+1) < \cdots < \tau(k+l)$.

Examples of computing the wedge products can be found in [Abraham et al. 1988, Section 7.1]. We now list several properties of the wedge product of differential forms, which we desire to maintain in the discrete setting as well.

Proposition 2.4.1. *For $\alpha \in T^0_k(\mathbf{E})$, $\beta \in T^0_l(\mathbf{E})$, and $\gamma \in T^0_m(\mathbf{E})$, we have*

1. *$\wedge$ is bilinear: $\alpha \wedge (c_1\beta + c_2\gamma) = c_1(\alpha \wedge \beta) + c_2(\alpha \wedge \gamma)$ and $(c_1\alpha + c_2\beta) \wedge \gamma = c_1(\alpha \wedge \gamma) + c_2(\beta \wedge \gamma)$ for some constants c_1, c_2,*
2. *$\wedge$ is skew commutative: $\alpha \wedge \beta = (-1)^{kl}\beta \wedge \alpha$,*
3. *$\wedge$ is associative: $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$,*
4. *Leibniz rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$.*

For the proof see [Abraham et al. 1988, Proposition 7.1.5 and Theorem 7.4.1].

Example 2.4.1. In linear algebra, the determinant of an $n \times n$ matrix is a skew-symmetric function of its rows or columns. For columns $x_1, \ldots, x_n \in \mathbb{R}^n$ we define ω by

$$\omega(x_1, \ldots, x_n) = \det[x_1, \ldots, x_n],$$

then $\omega \in \bigwedge^n(\mathbb{R}^n)$. And ω gives us the oriented volume of the parallelepiped spanned by the vectors $x_1, \ldots, x_n$.

The Cup Product

The wedge product allows for building higher degree forms from lower degree ones, similarly a cup product is a product of cochains of arbitrary degree p and q that returns a cochain of degree $p + q$.

Further, the cup product makes cohomology into a graded ring, thus cohomology equipped with cup product provides a more powerful tool than homology as it encodes more information about the topology of given objects.

Whitney in [Whitney 1957] gives the following abstract definition of the cup product, which also appears in [Arnold 2012].

Definition 2.4.2. Let X be a cell complex, the **cup product** of two cochains c^p and c^q is a bilinear operation $\smile : C^p(X) \times C^q(X) \to C^{p+q}(X)$ that satisfies the following three properties:

1. Let $\sigma_p \in C_p(X)$ and $\sigma_q \in C_q(X)$. Then $\sigma^p \smile \sigma^q$ is a $(p+q)$-cochain in $St(\sigma_p) \cdot St(\sigma_q)$, where $St(\sigma_i)$ is the union of all cells in which c_i is a face, and $A \cdot B$ denotes the union of all cells in A and B.
2. $d(c^p \smile c^q) = dc^p \smile c^q + (-1)^p c^p \smile dc^q$ (Leibniz rule).
3. If X is connected, then there exist a real number $\gamma_\smile$ such that $I^0 \smile c^p = c^p \smile I^0 = \gamma_\smile c^p$, where I^0 is the constant 0-cochain that takes value 1 on the 0-cells of X.

Whitney also asserts the following properties of the cup product (for details, see [Massey 1991, Chapter XIII]):

Proposition 2.4.2. *For $\alpha \in H^k(X)$, $\beta \in H^l(X)$, and $\gamma \in H^m(X)$), we have*

1. *$\smile$ is skew commutative: $\alpha \smile \beta = (-1)^{kl}\beta \smile \alpha$,*
2. *$\smile$ is associative: $\alpha \smile (\beta \smile \gamma) = (\alpha \smile \beta) \smile \gamma$.*

Comparing Proposition 2.4.1 with Definition 2.4.2 and Proposition 2.4.2, we affirm that both the wedge and the cup product, respectively, are bilinear operations that take as input two forms, resp. cochains, of arbitrary degree p and q and return a form, resp. cochain, of degree $p + q$. Moreover, they both satisfy the Leibniz rule. Also, the cup products is associative and skew-commutative on cohomology.

The Cup Product on Cell Complexes

If X is a cell complex, then also the product $X \times X$ has a natural cell structure, and we can describe the cellular (co)chain complex of $X \times X$ in terms of the one for X. To compute the cup product on X, one first has to find a cellular map f homotopic to the diagonal map $X \to X \times X$. Rognes in [Rognes 2011, Section 4.1] affirms that there always exists such a cellular map (by *Cellular Approximation Theorem*) $D : X \to X \times X$ that is homotopic to a diagonal approximation. Unfortunately, the choice of diagonal approximation D is not explicit.

Second, we use the cross product isomorphisms $C^p(X) \otimes C^q(X) \approx C^n(X \times X)$ for $n = p + q$ and we can form the composite homomorphism

$$C^n \to^{D_n} C^n(X \times X) \approx [C^*(X) \otimes C^*(X)]^n \to^{\alpha \cdot \beta} R$$

to get a cochain $\alpha \smile \beta$ in $C^n(X; R)$ with values in a ring R. Thus we get the desired pairing

$$\smile : C^p(X) \otimes C^q(X) \to C^{p+q}(X). \tag{2.6}$$

Even though the explicit formulas for computing cup product on a general cell complex are unknown, there are well known explicit formulas for cup products on simplicial and cubical complexes [Arnold 2012].

Also, [Gonzalez–Diaz et al. 2011] gives a formula for computing cup products of 1–cocycles on polygonal chains, and [Kravatz 2008] introduces cup products on 0– and 1–cochains on a polygon. They give different explicit formulas for cellular maps homotopic to a diagonal approximation. Moreover, Kravatz provides a geometric representation of his diagonal approximation as a tessellation of a polygon P. See Subsection 2.4.3.

2.4.1 The Cup and the Wedge Product on Simplicial Complexes

The cup product on a simplicial complex can be computed by an explicit formula stated in the following definition.

Definition 2.4.3. Let K be a simplicial complex with partial ordering of the vertices that linearly orders the vertices of each simplex of K. Let $C^*(K; R)$ be cohomology groups with coefficients in a ring R. The **simplicial cup product**

$$\smile : C^p(K; R) \times C^q(K; R) \to C^{p+q}(K; R)$$

is defined by the formula

$$(c^p \smile c^q)([v_0, \ldots, v_{p+q}]) = c^p([v_0, \ldots, v_p])c^q([v_p, \ldots, v_{p+q}]), \tag{2.7}$$

i.e., it is the value of a cochain c^{p+q} on an oriented simplex $\sigma^{p+q} = [v_0, \ldots, v_{p+q}]$.

By definition, the simplicial cup product is bilinear and it satisfies the Leibniz rule. Moreover, it shares other properties with the wedge product of differential forms (Proposition 2.4.1) as shown in the following proposition, which is a result of Theorem 49.1. and Corollary 61.4. of [Munkres 1984].

Proposition 2.4.3. *Let $C^k(K; R), k \geq 0$ be oriented k-cochains of a simplicial complex K and let R be a commutative ring with unity element 1. Then the corresponding simplicial cup product satisfies the following:*

1. *If $c^p \in H^p(K; R)$ and $c^q \in H^q(K; R)$, then $c^p \smile c^q \in H^{p+q}(K; R)$ and the cup product is anti-commutative: $c^p \smile c^q = (-1)^{pq} c^q \smile c^p$.*

2. $\smile$ *is associative.*

3. The cochain c^0 *whose value is 1 on each vertex of* K *acts as a unity element* I^0*.*

Thus it seams reasonable to define the wedge product on simplicial chains as a cup product.

Definition 2.4.4. Let K be an oriented simplicial complex with partial ordering of the vertices that linearly orders the vertices of each simplex of K. Let $\alpha \in \Omega^p(K)$ and $\beta \in \Omega^q(K)$, we define the **simplicial wedge product** by the formula (2.7), that is,

$$\alpha \wedge \beta = \alpha([v_0, \dots, v_p])\beta([v_p, \dots, v_{p+q}]). \tag{2.8}$$

Thus $\alpha \wedge \beta$ is a simplicial differential $(p+q)$-form on a $(p+q)$-simplex.

Naturally all the statements of Proposition 2.4.3 hold for our newly defined wedge product. But the anti–commutativity of the cup product holds only "up to cohomology".

There is another interesting result due to Kock [Kock 2009, Proposition 3.5.3]:

Proposition 2.4.4. *Let* α *and* β *be simplicial differential forms of degree* k *and* l*, respectively, with values in a commutative algebra R, whose underlying vector space is KL. Then*

$$\alpha \cup \beta = (-1)^{pq} \alpha \cup \beta.$$

The wedge product of simplicial differential forms (2.8) differs from the wedge product of the corresponding "classical" differential forms by a factor $(p+q)!/p!q!$, as stated in Theorem 4.7.3 of [Kock 2009], which reads:

Theorem 2.4.1. *Let* $\bar{\alpha}$ *and* $\bar{\beta}$ *be classical* p*- and* q*-forms, respectively, on a manifold* M*, and let* α *and* β *be the corresponding simplicial forms. Then the classical* $(p+q)$*-form* $\bar{\alpha} \wedge \bar{\beta}$ *given by equation (2.5) corresponds to the simplicial form* $(p+q)!/p!q! \cdot (\alpha \cup \beta)$ *of equation (2.7).*

We are aware of two other definitions of a simplicial wedge product. One of them uses metric and the other does not, both of them can be found in [Hirani 2003, Section 7.2]. The first one, involving metric, is *anti-commutative*, satisfies the *Leibniz rule* and is *associative for closed forms.* We state here just the latter (metric-independent) one, which is due to Castrillon Lopez.

Definition 2.4.5 (Castrillon Lopez, 2003)**.** Let K be a simplicial complex of dimension $\geq k+l$ and let $\alpha \in \Omega^k(K), \beta \in \Omega^l(K)$ be discrete forms on K. We define the **discrete wedge product** $\wedge : \Omega^k(K) \times \Omega^l(K) \to \Omega^{k+l}(K)$ on a $(k+l)-$simplex σ^{k+l} as

$$\langle \alpha \wedge \beta, \sigma^{k+l} \rangle = \frac{1}{(k+l+1)!} \sum_{\tau \in S_{k+l+1}} \operatorname{sign}(\tau)(\alpha \smile \beta)(\tau(\sigma^{k+l})), \tag{2.9}$$

where

$$(\alpha \smile \beta)(\tau(\sigma^{k+l})) = \langle \alpha, [v_{\tau(0)}, \dots, v_{\tau(k)}] \rangle \langle \beta, [v_{\tau(k)}, \dots, v_{\tau(k+l)}] \rangle.$$

In [Hirani 2003, Section 7.2] can be found also some suggestions for the definition of the "discrete dual-dual wedge product" and "discrete primal-dual wedge product", where the primal complex is a simplicial complex and the dual complex is the circumcentric dual of the given simplicial complex. For details see the discussion therein.

2.4.2 Cubical Complexes

We first define the cubical complex as it is a new notion in this paper. See also [Arnold 2012, Section 2.1].

Definition 2.4.6. The standard ***p*-cube** e_p is the subset of $\mathbb{R}^n$ consisting of points $(x_1, \ldots, x_n)$ such that $x_i = 0$ for $i > p$, and $0 \leq x \leq 1$ for $1 \leq i \leq p$. We will denote the p-cube by its nonzero variables, i.e., $e_p = (x_1, \ldots, x_p)$.

Definition. A **cubical complex** X in $\mathbb{R}^n$ is a regular cell complex whose cells are cubes such that

1. Every face of a cube of X is in X.
2. The intersection of any two cubes of X is a face of each of them.

Definition. Let $\tau = (x_1, \ldots, x_p)$ be a p-cube for some $p > 0$. We regard the direction of increase of each variable as fixed in the direction from 0 to 1. Two orderings of the variables of τ are equivalent if they differ by an even permutation. For all $p \geq 2$, the orderings split into two equivalence classes. Each of these classes is called an **orientation** on τ.
The **standard orientation of a *p*-cube** is the one that agrees with the ordering of its variables $x_1, \ldots, x_p$ for all $p \geq 1$.

Definition. The **cubical boundary map** $\partial : C_p(X) \to C_{p-1}(X)$ for $p \geq 1$ is the homomorphism

$$\partial_p \tau = \sum_{i=1}^{p} (-1)^{i+1} (\tau|_{x_i=1} - \tau_{x_i=0}), \tag{2.10}$$

where $\tau|_{x_i=1} = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_p)$ is called the **front *i*-face** and $\tau|_{x_i=0} = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_p)$ is the **back *i*-face** of the p-cube τ.

The definitions of cubical cup products require the introduction of some notation, which we give bellow and which can be found also in [Arnold 2012, Section 2.3], the definitions themselves are Definitions 3.2.6 and 3.2.1 therein.

Let e_n be the standard n-cube and H an ordered subset $h_1, \ldots, h_p$ of the integers $1, \ldots, n$. Define $\lambda_H^\epsilon : e_p \to e_n, \epsilon \in \{0, 1\}$ by

$$\lambda_H^\epsilon(u_1, \ldots, u_p) = (v_1, \ldots, v_n),$$

where $v_i = \epsilon$ if $i \notin H$ and $v_i = u_r$ if $i = h_r$ for some $r \in \{1, \ldots, p\}$. Thus, λ_H^0 is an isometry of e_p onto the p-face in e_n which contains the origin and lies in the subspace spanned by $u_{h_1}, \ldots, u_{h_p}$. And λ_H^1 is an isometry of e_p onto the p-face in e_n which contains the point $(1, \ldots, 1)$ and is parallel to the subspace spanned by $u_{h_1}, \ldots, u_{h_p}$.

Definition 2.4.7. For cellular cochains $\alpha \in C^p(X)$ and $\beta \in C^q(X)$, the **standard cubical cup product**

$$\smile : C^p(X) \times C^q(X) \to C^{p+q}(X)$$

is defined by

$$(\alpha \smile \beta)(x_1, \ldots, x_{p+q}) = \sum_H \rho_{HK} \alpha(\lambda_H^0(x_1, \ldots, x_{h_p})) \cdot \beta(\lambda_K^1(x_{k_1}, \ldots, x_{k_q})), \tag{2.11}$$

where $\rho_{HK} = \mathrm{sign}(h_1, \ldots, h_p, k_1, \ldots, k_q)$.

In [Arnold 2012, Section 3.2], the author also introduces a new cubical cup product $\smile_c$ (Definition 3.2.1 therein) that she later uses to define a cubical discrete ***Hodge star operator*** over $\mathbb{R}$, see also the Subsection 2.5.2 here. The new cup product is degenerate on the cochain level, but is nondegenerate on cohomology. More interestingly, the cup product of two cochains agrees with the wedge product of their **cubical Whitney forms** (see again [Arnold 2012, Section 3.2]). Although the new cup product $\smile_c$ does not agree with the standard cup product on cochains, they agree on cohomology.

The following introduction of $\smile_c$ is taken from [Arnold 2012, Subsection 3.2.1]. We present it here because it appears in the definition of the Hodge star defined without the notion of the dual complex (Definition 2.5.1).

Let K be a cubical structure on a closed, oriented n-dimensional manifold. Suppose $\sigma \in C_{p+q}(K)$ has standard orientation following the order of the variables $\{x_1, \ldots, x_{p+q}\}$. Let $F_p = \{x_{i_1}, \ldots, x_{i_p}\}$ denote an arbitrary collection of p free basis variables from $\{x_1, \ldots, x_{p+q}\}$ ordered by ascending index values. Let $\{F_p\}$ denote the collection of all possible collections of p free basis variables. Note: $F_p^c = \{x_{i_1}, \ldots, x_{i_{p+q}}\}$ with variables in ascending index value order. Let $V = \{v = (x_1, \ldots, x_{p+q}) : x_i \in \{0,1\}\ \forall i\}$.

Let $v \in V$. Suppose $F_p = \{x_{i_1}, \ldots, x_{i_p}\}$ is given. Let $y_p(v)$ denote the p-face with free variables $x_{i_1}, \ldots, x_{i_p}$ and with remaining variables in F_p^c held constant according to their values at vertex v. Note: all variables and constants are assigned in their standard positions. Let $y_p^c(v)$ denote the q-face with free variables $x_{i_{p+1}}, \ldots, x_{i_{p+q}}$ and with remaining variables in F_p held constant with their values at vertex v. Again, all variables and constants are assigned in their standard positions.

Definition 2.4.8. Let $\alpha \in C^p(X)$, $\beta \in C^q(X)$, and $\sigma \in C_{p+q}(K)$ with standard orientation $(x_1, \ldots, x_{p+q})$. Then the **cubical cup product**

$$\smile_c : C^p(X) \times C^q(X) \to C^{p+q}(X)$$

is defined by

$$(\alpha \smile_c \beta)(\sigma) = \frac{1}{2^{p+q}} \sum_{\{x_{i_1}, \ldots, x_{i_{p+q}}\} \in \{F_p\}} \sum_{v \in V} \operatorname{sign}(x_{i_1}, \ldots, x_{i_{p+q}}) \alpha(y_p(v)) \beta(y_p^c(v)). \tag{2.12}$$

Arnold also shows that the new cup product $\smile_c$ of two cochains agrees with the wedge product of their cubical Whitney forms. Thus the Whitney map provides a connection of cubical cohomology with de Rham cohomology.

2.4.3 Polygonal Complexes

Both [Gonzalez–Diaz et al. 2011] and [Kravatz 2008] define a cup product on a polygonal mesh of a compact orientable surface, but they differ by the diagonal approximation used.

The cup product of [Kravatz 2008] has the advantage to be defined on general 0-,1-,2- cochains, whereas the one of [Gonzalez–Diaz et al. 2011] is defined on 1-cocycles only. On the other hand, [Kravatz 2008] assumes a particular ordering of the vertices and the result depends on the choice of the minimal and maximal vertex. [Gonzalez–Diaz et al. 2011] introduces a formula for computing cup products which is independent of the ordering of vertices.

With some simplification of their approaches we state the next definition.

Definition 2.4.9. Let X be a polygonal cell complex with homology groups $H_*(X)$. Let $\alpha \in C_n(X)$ and F be a ring, consider the **elementary n-cochain** $\bar{\alpha} \in C^n(X)$

$$\bar{\alpha} : C_n(X) \to F \text{ such that for } \mu \in C_n, \bar{\alpha} := \begin{cases} 1 & \text{if } \mu = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Then for $\alpha, \beta, \gamma \in H_*(X)$, the **polygonal cup product** $\bar{\alpha} \smile \bar{\beta}$ is given by

$$(\bar{\alpha} \smile \bar{\beta})(\gamma) = m(\bar{\alpha} \otimes \bar{\beta})\Delta(\gamma), \tag{2.13}$$

where m denotes multiplication in F and Δ is the **diagonal approximation** on X.

Gonzalez–Diaz shows that his diagonal approximation induces a cup product that is bilinear, commutative, associative, and independent of the ordering of the vertices. Kravatz proves that his cup product on closed orientable surfaces forms a *graded commutative ring with identity*, thus his cup product is also bilinear, commutative, and associative.

2.5 The Hodge Star and the Codifferential

Suppose that M is an n-dimensional oriented Riemannian manifold and k is an integer such that $0 \leq k \leq n$, then the Hodge star operator establishes a one–to–one mapping between k-forms and their dual $(n-k)$-forms. In the exterior calculus on smooth manifolds the Hodge star operator can be defined in the following manner:

Definition. Let M be a smooth n-manifold endowed with a pseudo-Riemannian metric μ, and let $vol_\mu \in \Omega^n(M)$ denote the volume form induced by μ. The metric μ naturally induces a non-degenerate symmetric bilinear form $(\ ,\ ) : \Omega^p(M) \otimes \Omega^p(M) \to \Omega^0(M)$, called inner product.

The **Hodge star operator** $* : \Omega^p(M) \to \Omega^{n-p}(M)$, which maps any p-form α into its dual $(n-p)$-form $*\alpha$ on M, is the unique isomorphism satisfying

$$\alpha \wedge *\beta = (\alpha, \beta) vol_\mu \quad \forall\, \alpha, \beta \in \Omega^p(M). \tag{2.14}$$

Next we state some properties of the Hodge star, proof of the following preposition can be found in [Abraham et al. 1988, Proposition 7.2.13].

Proposition 2.5.1. *Let M be a smooth n-manifold, the Hodge star operator satisfies the following properties for $\alpha, \beta \in \Omega^p(M)$:*

$$\begin{aligned} \alpha \wedge *\beta &= \beta \wedge *\alpha = (\alpha, \beta) vol_\mu, \\ *1 &= vol_\mu, \quad *vol_\mu = 1, \\ **\alpha &= (-1)^{p(n-p)}\alpha, \\ (\alpha, \beta) &= (*\alpha, *\beta). \end{aligned}$$

2.5.1 Discrete Hodge Star Operators

Having a primal complex K (often a simplicial complex) and its dual $\star K$ (a cellular complex), where $\dim K = \dim \star K = n$, we define the discrete Hodge Star as an operator which takes primal discrete p-forms to its dual discrete $(n-p)$-forms, taking in account the ratios between the volumes of dual and primal elements (cells).

Several authors have defined the discrete Hodge star in different manners. For example, in [Desbrun et al. 2005a] we find the following definition:

Definition. The **discrete Hodge star** is a map $* : \Omega^k(K) \to \Omega^{n-k}(\star K)$, defined by its action on simplices. For a k-simplex σ^k, and a discrete k-form α,

$$\langle *\alpha, \star\sigma^k \rangle = \frac{|\star \sigma^k|}{|\sigma^k|} \langle \alpha, \sigma^k \rangle, \tag{2.15}$$

where $\star\sigma^k \in \star K$ is the circumcentric dual of σ^k.

Discussion about this choice can be found therein. The Hodge star defined in this way gives us one of many possible diagonal approximations of the operator, called diagonal because in practice such an operator is represented by a diagonal matrix. More specifically, the discrete k-th Hodge star is encoded as a diagonal matrix $*^k$ with

$$\forall i,\ \ (*^k)_{ii} := \frac{|\star \sigma_i^k|}{|\sigma_i^k|}. \tag{2.16}$$

While computationally convenient, *diagonal Hodge stars* are not very accurate and they are generally only exact for constant forms, the authors of [Mullen et al. 2011] suggest to quantify the induced inaccuracy as the average difference between the discrete approximation and the exact Hodge star value. This approach leads them to designing meshes which minimize formal error bounds of diagonal Hodge stars.

As mentioned in [De Goes et al. 2014b], higher order accuracy can be achieved also by the use of higher-order approximations of the Hodge star, e.g., higher-order Galerkin Hodge stars. In the same paper, the authors proposed weighted versions of the Hodge star, defined on weighted triangulations. They also show that the space of *weighted Hodge stars* for one-forms is strictly larger than the space of unweighted Hodge stars.

2.5.2 Discrete Hodge Stars Defined without the Notion of the Dual Cell Complex

Scott Wilson in [Wilson 2007] defines a discrete Hodge star over $\mathbb{R}$ in a simplicial setting without reference to a dual cell complex. Arnold in [Arnold 2012] defines an analogous cubical discrete Hodge star over $\mathbb{R}$ (Section 3.3.1 therein) and over $\mathbb{Z}$ (Section 3.3.4 therein), using the new cup product $\smile_c$ and the standard cubical cup product $\smile$, respectively. She then uses the discrete Hodge stars to prove the Poincaré Duality over $\mathbb{R}$ and $\mathbb{Z}$, resp.

Definition 2.5.1. Let M be a closed oriented n-dimensional manifold and K a cubical structure on M ([Arnold 2012, Definition 3.1.1]). Denote $[M]$ the **fundamental class** of M, i.e., the sum of the n-dimensional cubes in K. Then the **discrete cubical Hodge star** $* : C^p(K) \to C^{n-p}(K)$ over $\mathbb{R}$, resp. $\mathbb{Z}$, reads

$$\langle *\alpha, \beta \rangle = (\alpha \smile_c \beta)[M], \text{ resp. } \langle \alpha, \beta \rangle = (\alpha \smile \beta)[M], \tag{2.17}$$

where

$$\langle \alpha, \beta \rangle = \sum_{p\text{-faces } c \in K} \alpha(c) \cdot \beta(c)$$

is the standard inner product on cochains.

In [Arnold 2012, Section 4.3] we can also find the analogous *simplicial discrete Hodge star operators* over $\mathbb{R}$ and $\mathbb{Z}$ defined without the notion of the dual cell complex.

2.5.3 The Codifferential

The exterior derivative and the Hodge star operator enable us to introduce a linear operator δ called the codifferential (do not confuse with the symbol δ for the coboundary operator). Given the (discrete) Hodge star, the definition of codifferential is formally the same for the discrete and continuous manifolds.

Definition 2.5.2. Given a (pseudo–)manifold K, the **(discrete) codifferential operator** δ : $\Omega^{k+1}(K) \to \Omega^k(K)$ is defined by $\delta(\Omega^0(K)) = 0$ and on $(k+1)$-forms β by

$$\delta\beta = (-1)^{n(k-1)+1} * d * \beta. \tag{2.18}$$

The codifferential is the adjoint operator of the exterior derivative (in the continuous world) with respect to the inner product of forms, that is,

$$(d\alpha, \beta) = (\alpha, \delta\beta) \quad \forall \alpha \in \Omega^{k-1}(M), \beta \in \Omega^k(M), \tag{2.19}$$

for M compact manifold without boundary. This identity follows from Stokes' theorem for smooth forms. Equation (2.19) holds also for discrete forms on a simplicial complex and its circumcentric dual, where the Hodge star is defined by equation (2.15), as proved in [Desbrun et al. 2008, Subsection 5.5].

Since we have a formula for discrete exterior derivative d for both the simplicial and cellular complexes, so if we are able to approximate the discrete Hodge star $*$ then the codifferential is fully determined by equation (2.18).

2.6 The Laplace Operator

The Laplace operator is a differential operator that occurs in differential equations that describe many physical phenomena but it is also used in geometry processing. In computer graphics, the Laplacian has been used for various tasks such as shape analysis, edge detection, or smoothing of curves and surfaces.

The Laplace operator on differential forms is usually called the *Laplace–de Rham operator* or the *Hodge Laplacian.* Since it involves the codifferential and thus the Hodge star operator, it is metric dependent and is defined on any manifolds equipped with a metric.

Definition 2.6.1. Given a (pseudo-)manifold K, the **(discrete) Laplace–de Rham operator** $\Delta : \Omega^k(K) \to \Omega^k(K)$ is defined by

$$\Delta = d\delta + \delta d. \tag{2.20}$$

A form for which $\Delta\alpha = 0$ is called **harmonic**.

For a scalar function f, i.e., a 0-form, the term $d\delta f = 0$, thus $\Delta f = \delta d f = -\nabla^2 f$, where ∇^2 is the **Laplace–Beltrami operator**. Using the notation of [Reuter et al. 2009], the common discrete approximations of the Laplace–Beltrami operator may be represented as

$$\Delta f(v_i) = \frac{1}{m_i} \sum_{j \in N(i)} w_{ij} [f(v_i) - f(v_j)], \tag{2.21}$$

where $f(v_i)$ is the function value at vertex v_i, $N(i)$ denotes the index set of the 1-ring of the vertex v_i, i.e., the indices of all neighbors connected with v_i by an edge. The masses m_i are associated to a vertex v_i and the w_{ij} are the symmetric edge weights.

For $w_{ij} = 1$ and $m_i = |N(i)|$, i.e., the number of neighbors of v_i, the approximation of Δ is called the **umbrella operator** in [Desbrun et al. 1999].

Pinkall and Polthier in [Pinkall and Polthier 1993] discretize the Laplace–Beltrami operator using constant masses (i.e., $m_i = 1$) and weights

$$w_{ij} = \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2},$$

where α_{ij} and β_{ij} denote the two angles opposite to the edge (i, j), which leads to the famous **cotangent formula** for the Laplace–Beltrami operator.

2.7 Vector Fields

A Riemannian manifold is a real manifold M equipped with an inner product g called a Riemannian metric. One elementary property of Riemannian metrics is that they allow us to convert vectors to covectors (1-forms) and vice versa. In [Lee 1997, Chapter 3] can be found a nice introduction to this topic.

Definition. Given a metric g on M, define a map called **flat operator** $\flat : TM \to T^*M$ by sending a vector X to the 1-form $X^\flat$ defined by

$$X^\flat(Y) := g(X, Y).$$

In coordinates, $X^\flat(Y) := g(X^i \partial_i, \bullet) = g_{ij} X^i dx^j$, where $dx^1, \ldots, dx^n$ represents an orthonormal 1-form basis in $\mathbb{R}^n$ and g_{ij} are the components of the covariant metric tensor g.
So the 1-form $X^\flat$ has components

$$X_j := g_{ij} X^i.$$

We say that $X^\flat$ is obtained from X by **lowering an index**.

Since the matrix of g is invertible, so is the flat operator. Its inverse is called the **sharp operator** $\sharp : T^*M \to T^{**}M = TM$, which takes a 1-form ω into a vector $\omega^\sharp$. In coordinates, $\omega^\sharp$ has components

$$\omega^i := g^{ij} \omega_j,$$

where g^{ij} are the components of the inverse matrix $(g_{ij})^{-1}$, i.e., of the contravariant metric tensor. We say that $\omega^\sharp$ is obtained by **raising an index**.

The flat and sharp operators can be applied to tensors of any rank. Let $T^a_b(M)$ denote the tensor field of type (a, b) on a manifold M, then

$$\begin{aligned} \flat :\ T^a_b(M) &\to T^{a-1}_{b+1}(M), \\ \sharp :\ T^a_b(M) &\to T^{a+1}_{b-1}(M). \end{aligned} \tag{2.22}$$

In the equation (2.22) we see that $\flat$ lowers an index, this is why the operator is designated by the musical notation $\flat$ = "flat". And on the contrary, $\sharp$ raises an index, that is why the musical notation $\sharp$ = "sharp". Therefore the flat and sharp operator are called the **musical isomorphism**.

An exhaustive treatment of discrete flat and sharp operators on primal simplicial meshes and its circumcentric dual cellular meshes can be found in [Hirani 2003, Chapter 5].

With these operators in hand we can extend the classical gradient, divergence, and curl operators to Riemannian manifolds. If f is a smooth, real–valued function and X is a vector field on a Riemannian manifold (M, g), then

$$\begin{aligned} \operatorname{grad} f &:= (df)^\sharp, \\ \operatorname{div} X &:= -\delta X^\flat = *d(*X^\flat), \\ \operatorname{curl} X &:= [*(dX^\flat)]^\sharp. \end{aligned} \tag{2.23}$$

For the proof of equation (2.23) in the smooth case see [Abraham et al. 1988]. For the proof in the discrete version see [Hirani 2003, Chapter 6].

Chapter 3

Some Applications and Open Problems

We have been mainly preoccupied with the theory of DEC, not with applications so far, our main interest was to define the main operators in harmony with the continuous theory, with the help of algebraic topology. We tried to go as far as possible sticking to the notions from algebraic topology before settling down with the common approximations used in computer graphics and other areas. This motivation leads us to defining problems of the wedge product (discussed in Section 3.2) and the Hodge star (Section 3.3).

We also wish to extend this theory to more general meshes than the simplicial ones (Section 3.1).

3.1 Extension of DEC to Polytopal Meshes

When we started to write this report, the idea was to explore the possibility of DEC in setting as much general as possible – on cell complexes. But cell complexes are unnecessarily ample and for the use in computer graphics we would be satisfied with polytopal meshes.

We have seen that for simplicial and cubical complexes some of the most important operators are already known in explicit form, such as the boundary, the wedge product, and the Hodge star. For simplicial meshes, some approximations of the Hodge star or the Laplacian have been widely studied. We wonder if it is possible to find good *approximations of the Hodge star and the Laplacian in the cubical setting* and whether these can be useful in computer graphics or other areas. Obviously, later one we may ask the same questions for *polytopal meshes* too.

Fernando de Goes also thinks that the extension of DEC for polygonal and polyhedral meshes in 2D/3D is an interesting problem. However I wish to extend DEC to more general, i.e., polytopal, complexes.

3.2 The Wedge Product

In Section 2.4 we introduced different cup products on simplicial and cubical complexes of any dimension, and also on polygonal complexes. Yet we believe there is more to be investigated in the case of polygonal complexes, i.e., we would like to *study further the possible definitions of a wedge product on polygonal meshes.*

Also, we would like to have a definition of *cup product on general polytopal complexes of any dimension.* To best of our knowledge, such a definition is unknown.

We have noticed the following. Given a simplex $\sigma_n, 0 \leq n \leq 3$, its barycentric dual cell $c_k, 0 \leq k \leq n$ restricted to the simplex area forms a **kite**[1]. If all circumcenters lay inside their associated (sub)simplices, then the same holds also for the circumcentric dual. Kites are diffeomorphic to cubes and thus the formula for their cup products is already known.

Another natural direction is to think about *possible applications* of these cup (wedge) products.

Furthermore, Fernando de Goes raised the question of the *location of the result of the wedge product* of two forms.

3.3 The Hodge Star Operator and the Laplacian

The Laplace operator, eq. (2.20), is a common operator in geometry processing and thus to find a good approximation is an important task, see discussion in Section 2.6. In the continuous setting, it involves the differential and the Hodge star operator. The differential in the discrete setting is given exactly, thus the only loss of accuracy arises from the Hodge star used.

The quest for good discretization of Hodge stars comes also from other applications. For example, authors of [Mullen et al. 2011] aim to design meshes to minimize formal error bounds of **diagonal Hodge stars** (equation (2.16)). Hodge star serves also for Hodge decomposition of discrete vector fields [Desbrun et al. 2008] and tensor fields [De Goes et al. 2014a], the authors of the latter asserts that the choice of diagonal Hodge star is computationally more attractive, even though the most natural discrete Hodge star is arguably the **Galerkin Hodge star**.

The so called **Delaunay Hodge star** was introduced in [Hirani et al. 2013] and it stays for the diagonal Hodge star on a class of meshes for which the signed dual volumes and hence Hodge star entries are positive. For a primal complex that is completely well centered, every dual element has a positive signed volume. Authors define a new sign convention that leads to signed dual of a Delaunay triangulation and thus allows for positive volumes of a broader class of meshes.

In Subsection 2.5.2 we presented a Hodge star on a cubical or a simplicial complex defined without a reference to its dual complex. The definition was precise but it does not gave us an explicit form of the operator, thus interesting direction of next research might be to *find good approximations of the Hodge star on cubical and also polytopal complexes.*

3.4 Other Problems

We find the above problems the most interesting ones. Yet, for completeness, we state three problems suggested by Fernando de Goes:

Forms on primal/dual meshes, point-collocated, FEM – localization of forms.

Hodge decomposition of tensors – studied in SIGGRAPH Course 2013 [de Goes et al. 2013b], [De Goes et al. 2014a], and [Desbrun et al. 2008].

Metric on discrete surfaces – studied already in [De Goes et al. 2014b], [Solomon et al. 2015], and [De Goes 2014].

[1] Kite complexes are studied in [Arnold 2012].

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