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Ray Tracing in NIL Geometry Spaces

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RAY TRACING IN NIL GEOMETRY SPACES

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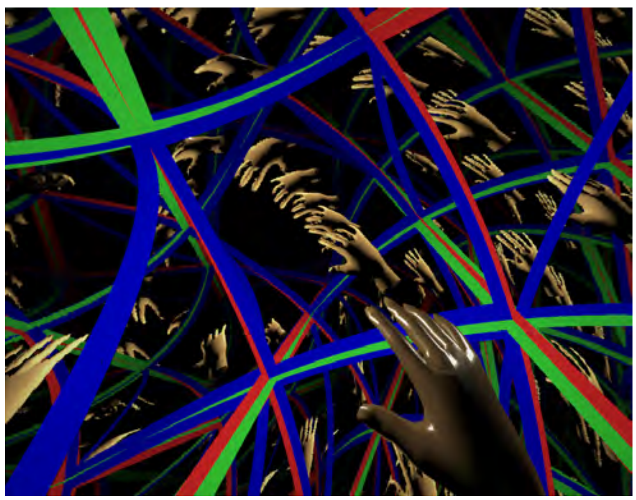
Abstract. In this paper we present an image–based real–time rendering of some manifolds locally modeled by the Nil geometry, which is one of the eight Thurston's geometries.



1. Ray tracing requirements

The paper deals with an immersive visualization of spaces locally modeled by *Nil geometry* — an example of a Non–Euclidean space — using ray tracing, thus we need at least three properties:

- Being locally similar to an Euclidean space — that is, a *manifold*. This allows us to model the viewer and scene inside the ambient;
- The *tangent space* endowed with an *inner product* at each point — that is, a *Riemannian metric*. Such definition is used to simulate effects produced between the lights and the scene objects.
- The *ray* leaving a point in any direction — that is, the *geodesic*. Finally, the intersection between rays and the scene "objects" are required.

Geometric manifolds satisfies the above properties. Such objects are locally geometrical similar to special spaces called *model geometries*. In dimension two, for example, there are exactly three models: Euclidean, hyperbolic, and spherical spaces. In dimension three, there are five more model geometries, however, in this

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work, we focus on Nil geometry. Great texts on this subject are Thurston [4] and Martelli [2]. In [Ray3D] the authors approached the three classical geometries: *Euclidean*, *hyperbolic*, and *spherical* spaces.

2. Nil space

A *Lie group* is a manifold G which is also a group and its operations are all smooth. *Nil space* is an example of a Lie group which consists of all matrices

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

with the multiplication operation. We denote such space by Nil. This space is diffeomorphic to $\mathbb{R}^3$ since we could parameterize it by

$$(x, y, z) \in \mathbb{R}^3 \to \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in Nil,$$

then we push-forward the differentiable structure of $\mathbb{R}^3$ to Nil. We identify $\mathbb{R}^3$ with Nil, which allows us to use $\mathbb{R}^3$ to set our scene for a ray tracing.

Let (x, y, z) and (x', y', z') be two elements in Nil, then their multiplication has the following form:

$$\begin{aligned} (x, y, z) \cdot (x', y', z') &= \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x + x' & z + z' + xy' \\ 0 & 1 & y + y' \\ 0 & 0 & 1 \end{bmatrix} \\ &= (x + x', y + y', z + z' + xy'). \end{aligned}$$

In other words, the multiplication of elements in Nil is the sum of the coordinates, with an additional term in the last one. This term makes all the difference, since in order to put a geometry in Nil we consider the *right and left multiplication*, $(x, y, z) \to (x, y, z) \cdot p$ and $(x, y, z) \to p \cdot (x, y, z)$, for all $p \in Nil$, being isometries. We describe this procedure with more details in the next section.

3. The geometry of Nil

The classical way to construct a metric in a Lie group G is by fixing a scalar product $\langle \cdot, \cdot \rangle_e$ at the tangent plane in some point $e \in G$ and then extend it by left multiplication. We construct a metric in the Nil space.

Let $e = (0, 0, 0)$ be the origin of Nil, and $T_e Nil = \mathbb{R}^3$ be the tangent space at e endowed with Euclidean scalar product. That is $\langle u, v \rangle_e = u_x v_x + u_y v_y + u_z v_z$, where u and v are vectors tangent at e.

Let p be a point in Nil, we define a scalar product $\langle \cdot, \cdot \rangle_p$ in $T_p Nil$. Defining an isometry Φ would allow us to translate p to the origin $e = (0, 0, 0)$. Then we use the differential $d\Phi_p$ to transpose vectors from $T_p Nil$ to $T_e Nil$. Let $\Phi : Nil \to Nil$ be an isometry such that $\Phi(p) = e$, that is, $\Phi(p) = q \cdot p = 0$, for some q in Nil. Solving this equation, we obtain $q = (-p_x, -p_y, -p_z + p_x p_y)$, and the desired isometry is $\Phi(x, y, z) = (x - p_x, y - p_y, z - p_z + p_x(p_y - y))$.

Let v be a tangent vector at p than $d\Phi_p(v)$ is a tangent vector at e. We compute $d\Phi_p(v)$. Note that $d\Phi_p(v) = v_x \cdot d\Phi_p(e_1) + v_y \cdot d\Phi_p(e_2) + v_z \cdot d\Phi_p(e_3)$, where $\{e_1, e_2, e_3\}$

is canonical base of $\mathbb{R}^3$. Computing, $d\Phi_p(e1) = (1,0,0)$, $d\Phi_p(e2) = (0,1,p_x)$, and $d\Phi_p(e3) = (0,0,1)$, we obtain $d\Phi_p(v) = v_x(1,0,0) + v_y(0,1,p_x) + v_z(0,0,1)$.

Then the scalar product between two tangent vectors u and v at p can now be defined as:

$$\langle u,v\rangle_p := \langle d\Phi_p(u), d\Phi_p(v)\rangle_e = u^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & p_x^2+1 & -p_x \\ 0 & -p_x & 1 \end{bmatrix} v.$$

The 3×3 matrix above defines a metric at p. Varying p we obtain a Riemannian metric $\langle\cdot,\cdot\rangle$, since each matrix entry is differentiable. The vectors $(1,0,0)$, $(0,1,x)$, and $(0,0,1)$ form a orthogonal basis at (x,y,z). Also, the volume form of Nil coincides with the standard one from $\mathbb{R}^3$, since $\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & p_x^2+1 & -p_x \\ 0 & -p_x & 1 \end{bmatrix} = 1$.

As Nil has a Riemannian metric we can use the *Levi–Civita theorem* from Riemannian geometry [1] to compute the *Christoffel symbols* Γ_{ij}^k of Nil. Such symbols are important for geodesic computations. In fact, a *geodesic* (a *ray*) is a curve $(x_1(t), x_2(t), x_3(t))$ satisfying:

$$\frac{d^2x_k}{dt^2}(t) + \sum_{i,j=1}^{3} \Gamma_{ij}^k(t)\frac{dx_i}{dt}(t)\frac{dx_j}{dt}(t) = 0, \quad k = 1,2,3.$$

Here, for convenience, we use the notation of (x_1,x_2,x_3) instead of (x,y,z).

The Christoffel symbols at point (x,y,z) are all zero except for

$$\Gamma_{22}^1 = -x,\ \Gamma_{23}^1 = \Gamma_{32}^1 = \frac{1}{2},$$

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{x}{2},\ \Gamma_{13}^2 = \Gamma_{31}^1 = -\frac{1}{2},$$

$$\Gamma_{12}^3 = \Gamma_{21}^3 = \frac{(x^2-1)}{2},\ \Gamma_{13}^3 = \Gamma_{31}^3 = -\frac{x}{2}.$$

The calculation can be done using the software *Maple*, for example [3]. We prefer the term ray instead of geodesic since we are dealing with ray tracing.

4. Rays in Nil space

A ray $\gamma(t) = (x(t), y(t), z(t))$ starting at $(0,0,0)$ in the unit tangent direction $v = (c\cos(\alpha), c\sin(\alpha), w)$ has the following form [3]

$$\begin{aligned} x(t) &= \frac{c}{w}(\sin(wt+\alpha) - \sin(\alpha)) \\ y(t) &= -\frac{c}{w}(\cos(wt+\alpha) - \cos(\alpha)) \\ z(t) &= t(w + \frac{c^2}{2w}) \quad -\frac{c^2}{4w^2}(\sin(2wt+2\alpha) - \sin(2\alpha)) \\ &\qquad + \frac{c^2}{2w^2}(\sin(wt+2\alpha) - \sin(2\alpha) - \sin(tw)). \end{aligned}$$

We now can use the isometries of Nil to compute a geodesic $\beta(t)$ starting at p in the direction v. That is, $\beta(0) = p$ and $\beta'(0) = v$. Translate β to the origin using the isometry $\Phi(x,y,z) = (-p_x, -p_y, -p_z + p_x p_y)\cdot(x,y,z)$. Observe that, $\Phi(0) = e$ and $d\Phi_p(v) = v - (0,0,p_x v_y)$. Therefore, $\Phi\circ\beta(t) = (x(t), y(t), z(t))$ is a ray starting at e in the direction $v - (0,0,p_x v_y)$, and it can be computed using the above closed formula. To compute $\beta(t)$ we simply apply Φ^{-1} to $\Phi \circ \beta(t)$. As $\Phi^{-1}(x, y, z) = p \cdot (x, y, z)$, we obtain the desired formula

$$\beta(t) = p \cdot (x(t), y(t), z(t)) = (p_x + x(t), p_y + y(t), p_z + z(t) + p_x y(t)).$$

5. Example of compact manifold

We present a compact manifold $M = Nil/\Gamma$ with the geometry modeled by Nil space. The construction of this manifolds will be "analogous" of the torus.

Let Γ be the discrete group generated by $\Phi_1(p) = e_1 \cdot p = (x + 1, y, y + z)$, $\Phi_2(p) = e_2 \cdot p = (x, y + 1, z)$, and $\Phi_3(p) = e_3 \cdot p = (x, y, z + 1)$. That is, the "translations" in the direction of axis x, y, and z. The manifold $M = Nil/\Gamma$ inherits the geometry of Nil space and for each fixed x it provides a two dimension torus, thus M admits a foliation by torus. The unit cube it the fundamental domain.

We set the scene inside the unit cube. For ray trace the scene, we use the ray definition given in Section 4. Each time a ray intersects (next section) a cube face we update the ray using the discrete group Γ.

6. Intersection between rays and planes

Let $\beta(t)$ be a ray, such that $\beta(0) = p \in Nil$ and $\beta'(0) = v \in T_pNil$. In section 4 we learned that $\beta(t) = (p_x+x(t), p_y+y(t), p_z+z(t)+p_x y(t))$, where $(x(t), y(t), z(t))$ is a ray leaving the origin.

We compute the intersection between $\beta(t)$ and the planes $x = c$ and $y = c$, where c is a constant. We start with the planes $x = c$. We have to solve the equation $x(t) + p_x = c$, which is equivalent to $\frac{c}{w}(\sin(wt + \alpha) - \sin(\alpha)) + p_x = c$. After some calculation, we obtain

$$t = \frac{\arcsin((c - p_x)\frac{w}{c} + \sin(\alpha)) - \alpha}{w}.$$

In an analogous way we compute the intersection between $\beta(t)$ and the plane $y = c$. The parameter is

$$t = \frac{\arccos((c - p_y)\frac{-w}{c} + \cos(\alpha)) - \alpha}{w}.$$

To compute the intersection of $\beta(t)$ and the plane $z = c$, we have to solve the equation $p_z + z(t) + p_x y(t) = c$. This question still open. For now we are avoiding this problem by considering a ray pair-wise linear (in Euclidean metric).

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