Ray Tracing in NIL Geometry Spaces

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Technical Report TR-19-06 Relatório Técnico

October - 2019 - Outubro

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RAY TRACING IN NIL GEOMETRY SPACES

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ABSTRACT. In this paper we present an image–based real–time rendering of some manifolds locally modeled by the Nil geometry, which is one of the eight Thurston’s geometries.

1. Ray tracing requirements

The paper deals with an immersive visualization of spaces locally modeled by Nil geometry — an example of a Non–Euclidean space — using ray tracing, thus we need at least three properties:

- Being locally similar to an Euclidean space — that is, a manifold. This allows us to model the viewer and scene inside the ambient;
- The tangent space endowed with an inner product at each point — that is, a Riemannian metric. Such definition is used to simulate effects produced between the lights and the scene objects.
- The ray leaving a point in any direction — that is, the geodesic. Finally, the intersection between rays and the scene “objects” are required.

Geometric manifolds satisfies the above properties. Such objects are locally geometrical similar to special spaces called model geometries. In dimension two, for example, there are exactly three models: Euclidean, hyperbolic, and spherical spaces. In dimension three, there are five more model geometries, however, in this

Date: June 24, 2019.
Key words and phrases. Nil geometry.
work, we focus on Nil geometry. Great texts on this subject are Thurston [4] and Martelli [2]. In [Ray3D] the authors approached the three classical geometries: Euclidean, hyperbolic, and spherical spaces.

2. Nil space

A Lie group is a manifold $G$ which is also a group and its operations are all smooth. Nil space is an example of a Lie group which consists of all matrices

$$
\begin{bmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1 
\end{bmatrix}
$$

with the multiplication operation. We denote such space by $Nil$. This space is diffeomorphic to $\mathbb{R}^3$ since we could parameterize it by

$$(x, y, z) \in \mathbb{R}^3 \mapsto \begin{bmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1 
\end{bmatrix} \in Nil,$$

then we push-forward the differentiable structure of $\mathbb{R}^3$ to $Nil$. We identify $\mathbb{R}^3$ with $Nil$, which allows us to use $\mathbb{R}^3$ to set our scene for a ray tracing.

Let $(x, y, z)$ and $(x', y', z')$ be two elements in $Nil$, then their multiplication has the following form:

$$(x, y, z) \cdot (x', y', z') = \begin{bmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1 
\end{bmatrix} \begin{bmatrix}
1 & x' & z' \\
0 & 1 & y' \\
0 & 0 & 1 
\end{bmatrix} = \begin{bmatrix}
1 & x + x' & z + z' + xy' \\
0 & 1 & y + y' \\
0 & 0 & 1 
\end{bmatrix}$$

$$(x + x', y + y', z + z' + xy').$$

In other words, the multiplication of elements in $Nil$ is the sum of the coordinates, with an additional term in the last one. This term makes all the difference, since in order to put a geometry in $Nil$ we consider the right and left multiplication, $(x, y, z) \mapsto (x, y, z) \cdot p$ and $(x, y, z) \mapsto p \cdot (x, y, z)$, for all $p \in Nil$, being isometries. We describe this procedure with more details in the next section.

3. The geometry of Nil

The classical way to construct a metric in a Lie group $G$ is by fixing a scalar product $\langle \cdot, \cdot \rangle_e$ at the tangent plane in some point $e \in G$ and then extend it by left multiplication. We construct a metric in the Nil space.

Let $e = (0, 0, 0)$ be the origin of $Nil$, and $T_e Nil = \mathbb{R}^3$ be the tangent space at $e$ endowed with Euclidean scalar product. That is $\langle u, v \rangle_e = u_x v_x + u_y v_y + u_z v_z$, where $u$ and $v$ are vectors tangent at $e$.

Let $p$ be a point in $Nil$, we define a scalar product $\langle \cdot, \cdot \rangle_p$ in $T_p Nil$. Defining an isometry $\Phi$ would allow us to translate $p$ to the origin $e = (0, 0, 0)$. Then we use the differential $d\Phi_p$ to transpose vectors from $T_p Nil$ to $T_e Nil$. Let $\Phi : Nil \rightarrow Nil$ be an isometry such that $\Phi(p) = e$, that is, $\Phi(p) = q \cdot p = 0$, for some $q$ in $Nil$. Solving this equation, we obtain $q = (-p_x, -p_y, -p_z + p_x p_y)$, and the desired isometry is $\Phi(x, y, z) = (x - p_x y - p_y z - p_x p_y, y, z - p_z + p_x p_y)$.

Let $v$ be a tangent vector at $p$ than $d\Phi_p(v)$ is a tangent vector at $e$. We compute $d\Phi_p(v)$. Note that $d\Phi_p(v) = v_x \cdot d\Phi_p(e_1) + v_y \cdot d\Phi_p(e_2) + v_z \cdot d\Phi_p(e_3)$, where $\{e_1, e_2, e_3\}$
The calculation can be done using the software Maple.

\[ \Phi d \] in the direction \( p \) is canonical base of \( x, y, z \) using the isometry \( \Phi(0) = 0 \) and \( \Phi(p) = p \) with \( p \in \mathbb{R}^3 \), we obtain \( \Phi_\ast(v) = v \cdot (0, 1) + v_3(0, 0, 1) \).

Then the scalar product between two tangent vectors \( u \) and \( v \) at \( p \) can now be defined as:

\[ \langle u, v \rangle_p := \langle d\Phi_p(u), d\Phi_p(v) \rangle_e = u^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & p_x^2 + 1 & -p_x \\ 0 & -p_x & 1 \end{bmatrix} v. \]

The \( 3 \times 3 \) matrix above defines a metric at \( p \). Varying \( p \) we obtain a Riemannian metric \( \langle \cdot, \cdot \rangle \), since each matrix entry is differentiable. The vectors \( (1, 0, 0) \), \( (0, 1, 0) \), and \( (0, 0, 1) \) form an orthogonal basis at \( (x, y, z) \). Also, the volume form of \( \mathbb{R}^3 \) coincides with the standard one from \( \mathbb{R}^3 \), since \( \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & p_x^2 + 1 & -p_x \\ 0 & -p_x & 1 \end{bmatrix} = 1. \)

As \( \mathbb{R}^3 \) has a Riemannian metric we can use the Levi–Civita theorem from Riemannian geometry \cite{1} to compute the Christoffel symbols \( \Gamma^k_{ij} \) of \( \mathbb{R}^3 \). Such symbols are important for geodesic computations. In fact, a geodesic \( \gamma \) is a curve \( (x_1(t), x_2(t), x_3(t)) \) satisfying:

\[ \frac{d^2 x_k}{dt^2} + \sum_{i,j=1}^3 \Gamma^k_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt} = 0, \quad k = 1, 2, 3. \]

Here, for convenience, we use the notation of \( (x_1, x_2, x_3) \) instead of \( (x, y, z) \).

The Christoffel symbols at point \( (x, y, z) \) are all zero except for

\[ \Gamma^1_{22} = -x, \quad \Gamma^1_{23} = \Gamma^1_{32} = \frac{1}{2}, \]

\[ \Gamma^2_{12} = \Gamma^2_{31} = \frac{x}{2}, \quad \Gamma^2_{13} = \Gamma^2_{31} = -\frac{1}{2}, \]

\[ \Gamma^3_{12} = \Gamma^3_{21} = \frac{(x^2 - 1)}{2}, \quad \Gamma^3_{13} = \Gamma^3_{31} = -\frac{x}{2}. \]

The calculation can be done using the software Maple, for example \cite{3}. We prefer the term ray instead of geodesic since we are dealing with ray tracing.

### 4. Rays in \( \mathbb{R}^3 \) space

A ray \( \gamma(t) = (x(t), y(t), z(t)) \) starting at \( (0, 0, 0) \) in the unit tangent direction \( v = (c \cos(\alpha), c \sin(\alpha), w) \) has the following form \cite{3}

\[ x(t) = \frac{c}{w} (\sin(wt + \alpha) - \sin(\alpha)) \]

\[ y(t) = -\frac{c}{w} (\cos(wt + \alpha) - \cos(\alpha)) \]

\[ z(t) = t(w + \frac{c^2}{2w}) - \frac{c^2}{4w^2} (\sin(2wt + 2\alpha) - \sin(2\alpha)) + \frac{c^2}{2w^2} (\sin(wt + \alpha) - \sin(\alpha) - \sin(tw)). \]

We now can use the isometries of \( \mathbb{R}^3 \) to compute a geodesic \( \beta(t) \) starting at \( p \) in the direction \( v \). That is, \( \beta(0) = p \) and \( \beta'(0) = v \). Translate \( \beta \) to the origin using the isometry \( \Phi(x, y, z) = (-p_x, -p_y, -p_z + pxpy) \cdot (x, y, z) \). Observe that, \( \Phi(0) = e \) and \( d\Phi_p(v) = v - (0, 0, pxvy) \). Therefore, \( \Phi \circ \beta(t) = (x(t), y(t), z(t)) \) is a ray starting at \( e \) in the direction \( v - (0, 0, pxvy) \), and it can be computed using
the above closed formula. To compute $\beta(t)$ we simply apply $\Phi^{-1}$ to $\Phi \circ \beta(t)$. As $\Phi^{-1}(x,y,z) = p \cdot (x,y,z)$, we obtain the desired formula

$$\beta(t) = p \cdot (x(t), y(t), z(t)) = (p_x + x(t), p_y + y(t), p_z + z(t) + p_x y(t)).$$

5. Example of compact manifold

We present a compact manifold $M = \text{Nil}/\Gamma$ with the geometry modeled by Nil space. The construction of this manifold will be "analogous" of the torus.

Let $\Gamma$ be the discrete group generated by $\Phi_1(p) = e_1 \cdot p = (x + 1, y, y + z)$, $\Phi_2(p) = e_2 \cdot p = (x, y + 1, z)$, and $\Phi_3(p) = e_3 \cdot p = (x, y, z + 1)$. That is, the "translations" in the direction of axis $x$, $y$, and $z$. The manifold $M = \text{Nil}/\Gamma$ inherits the geometry of Nil space and for each fixed $x$ it provides a two dimension torus, thus $M$ admits a foliation by torus. The unit cube is the fundamental domain.

We set the scene inside the unit cube. For ray trace the scene, we use the ray definition given in Section 4. Each time a ray intersects (next section) a cube face we update the ray using the discrete group $\Gamma$.

6. Intersection between rays and planes

Let $\beta(t)$ be a ray, such that $\beta(0) = p \in \text{Nil}$ and $\beta'(0) = v \in T_p\text{Nil}$. In section 4 we learned that $\beta(t) = (p_x + x(t), p_y + y(t), p_z + z(t) + p_x y(t))$, where $(x(t), y(t), z(t))$ is a ray leaving the origin.

We compute the intersection between $\beta(t)$ and the planes $x = c$ and $y = c$, where $c$ is a constant. We start with the planes $x = c$. We have to solve the equation $x(t) + p_x = c$, which is equivalent to $\frac{c}{w}(\sin(wt + \alpha) - \sin(\alpha)) + p_x = c$. After some calculation, we obtain

$$t = \frac{\arcsin((c - p_x) \frac{w}{c} + \sin(\alpha)) - \alpha}{w}.$$

In an analogous way we compute the intersection between $\beta(t)$ and the plane $y = c$. The parameter is

$$t = \frac{\arccos((c - p_y) \frac{w}{c} + \cos(\alpha)) - \alpha}{w}.$$

To compute the intersection of $\beta(t)$ and the plane $z = c$, we have to solve the equation $p_z + z(t) + p_x y(t) = c$. This question still open. For now we are avoiding this problem by considering a ray pair-wise linear (in Euclidean metric).

References

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