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RAY TRACING IN $\widetilde{SL_2(\mathbb{R})}$ GEOMETRY

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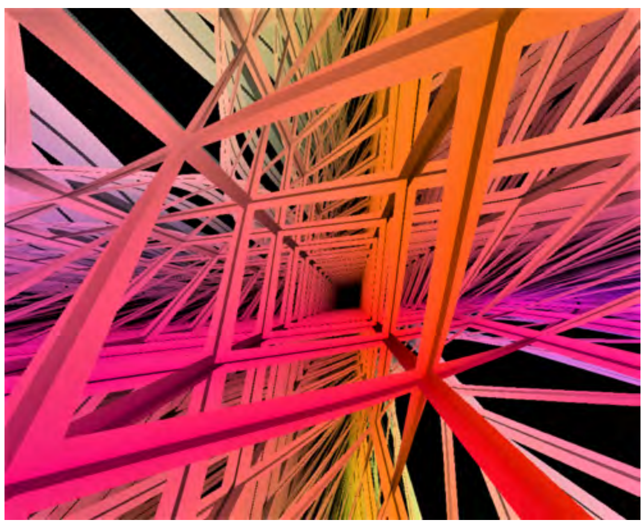
ABSTRACT. In this paper, we present an image-based "real-time" rendering of the space of all 2×2 matrices with unit determinant, the $SL_2(\mathbb{R})$. This manifold is locally modeled by $\widetilde{SL_2(\mathbb{R})}$, which is one of the eight Thurston's geometries.



1. RAY TRACING REQUIREMENTS

This paper deals with an immersive visualization of spaces modeled by the $\widetilde{SL_2(\mathbb{R})}$ *geometry* — one of the eight Thurston geometries [4] — using ray tracing. The space must satisfy at least three properties:

- Being locally similar to a Euclidean space — that is, a *manifold*. This allows us to model the viewer and scene inside the ambient;
- The *tangent space* endowed with an *inner product* at each point — that is, a *Riemannian metric*. Such a definition is used to simulate effects produced between the lights and the scene objects.
- The *ray* leaving a point in any direction — that is, the *geodesic*. Finally, the intersection between rays and the scene "objects" is required.

Geometric manifolds satisfies the above properties. Such objects are locally geometrical similar to special spaces called *model geometries*. In dimension two,

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for example, there are exactly three models: Euclidean, hyperbolic, and spherical spaces. In dimension three, there are five more model geometries, however, this work explores the $\widetilde{SL_2(\mathbb{R})}$ geometry. Great texts on this subject are Thurston [4] and Martelli [3]. In [Ray3D,Ray3D-Nil, Ray3D-Sol] the authors approached the *Euclidean*, *hyperbolic*, *spherical*, *Nil*, and *Sol* spaces.

2. $\widetilde{SL_2(\mathbb{R})}$ SPACE

A *Lie group* is a manifold G which is also a group and its operations are all smooth. The *special linear group* $SL_2(\mathbb{R})$ consisting of all 2×2 matrices with unit determinant is an example of a Lie group. Indeed, the product of two matrices with unit determinant has unit determinant, the same for the inverse matrix.

To understand the richness of $SL_2(\mathbb{R})$, we present two interpretations: one geometrical and another more topological. The familiarized reader can skip to the last paragraph of this section.

Geometrical interpretation: The elements of $SL_2(\mathbb{R})$ are matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad - bc = 1$. In other words $SL_2(\mathbb{R}) = \{(a, b, c, d) \in \mathbb{R}^4 |\ ad - bc = 1\}$, that is, a 3-manifold in $\mathbb{R}^4$. More precisely, rewriting the equation $ad - bc = 1$ we obtain:

$$\left(\frac{a+d}{2}\right)^2 - \left(\frac{a-d}{2}\right)^2 + \left(\frac{b-c}{2}\right)^2 - \left(\frac{b+c}{2}\right)^2 = 1,$$

which describes the equation of a 3-hyperbola in $\mathbb{R}^4$.

Topological interpretation: We also can identify (topologically) $SL_2(\mathbb{R})$ as the product of the *upper half plane* $\mathbb{H} = \{z \in \mathbb{C} |\ Im(z) > 0\}$ (hyperbolic model) and the circle $\mathbb{S}^1$. In other words, we identify, topologically, $SL_2(\mathbb{R})$ with a solid torus. Before, we need some additional definitions. Each element $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $SL_2(\mathbb{R})$ provides an *action* in $\mathbb{H}$:

$$gz = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az+b \\ cz+d \end{bmatrix} = \frac{az+b}{cz+d}. \tag{2.1}$$

We used homogeneous coordinates in Equation 2.1.

The action in Equation 2.1 is well defined, that is $cz + d \neq 0$ and $Im(gz) > 0$. As $ad - bc = 1$, either c or d are non null, so is $cz + d$. To verify that the imaginary part of gz is greater than zero we multiply the numerator and denominator of gz by $c\bar{z} + d$, where $\bar{z}$ is the conjugate of z. After some calculations we get $Im(gz) = Im(z)/|cz + d|^2$, which is greater than zero.

We now provide a decomposition of g in two components AN and K, know as *Iwasawa decomsposition.* Where K will be responsible by rotations around the point $i \in \mathbb{H}$ and AN will be translations of i. This procedure provides the decomposition of $SL_2(\mathbb{R})$ into the product $\mathbb{S}^1 \times \mathbb{H}$. Specifically, define

$$AN = \begin{bmatrix} \frac{1}{\sqrt{c^2+d^2}} & \frac{ac+bd}{\sqrt{c^2+d^2}} \\ 0 & \sqrt{c^2+d^2} \end{bmatrix}, \text{and } K = \begin{bmatrix} d & -c \\ c & d \end{bmatrix} \cdot \frac{1}{\sqrt{c^2+d^2}}. \tag{2.2}$$

It is possible to verify that the product $AB \cdot K$ is g. Observe that K can be write as $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$, for some angle $0 \leq \phi \leq 2\pi$. Thus, the matrices K can be parameterized by $\mathbb{S}^1$. We now verify that AN parameterize $\mathbb{H}$.

Let $z = x + iy$ be a point in $\mathbb{H}$, then

$$z = \begin{bmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = x + iy. \tag{2.3}$$

That is, every element in $\mathbb{H}$ can be write as the action of some matrix in $SL_2(\mathbb{R})$ on i. Taking $y = c^2 + d^2$ and $x = ac + bd$ we have the desired correspondence between AN and $\mathbb{H}$.

The identification of $SL_2(\mathbb{R})$ with $\mathbb{H} \times \mathbb{S}^1$ provides a topological interpretation of $SL_2(\mathbb{R})$: its fundamental group is $\mathbb{S}^1$. That is, $SL_2(\mathbb{R})$ is not simple connected, which imply that it is not a model geometry. The *universal cover* $\widetilde{SL_2(\mathbb{R})}$ of $SL_2(\mathbb{R})$ is model geometry [4]. We focus on the visualization of $SL_2(\mathbb{R})$ since its geometry is locally modeled by $\widetilde{SL_2(\mathbb{R})}$.

We could use the above parameterization of $SL_2(\mathbb{R})$ by $\mathbb{H} \times \mathbb{S}^1$, however, we take an easier coordinate system. We parameterize a neighborhood of the identity of $SL_2(\mathbb{R})$ by a neighborhood of the origin of $\mathbb{R}^3$ using the map [2]:

$$X(x, y, z) = \begin{bmatrix} 1 + x & y \\ z & \dfrac{1 + yz}{1 + x} \end{bmatrix}. \tag{2.4}$$

Observe that $X(0, 0, 0)$ is the identity of $SL_2(\mathbb{R})$, and that in the plane $x = 1$ this map is not defined. We use the map X to push-back the metric of $SL_2(\mathbb{R})$ to $\mathbb{R}^3$. We describe this procedure with more details in the next section.

3. The geometry of $SL_2(\mathbb{R})$

The classical way to define a metric in a Lie group G is by fixing a scalar product $\langle \cdot, \cdot \rangle_e$ at the tangent space in the identity element $e \in G$. Then extend it by left multiplication. We construct a metric in the $SL_2(\mathbb{R})$ and then using the map defined in Equation 2.2 we push it to $\mathbb{R}^3$.

The element $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity of $SL_2(\mathbb{R})$. Let $T_e SL_2(\mathbb{R})$ be the tangent space at e endowed with the well known matrix inner product [2]

$$\langle u, v \rangle_e = Tr(u \cdot v),$$

between two 2×2 matrices at u and v in $T_e SL_2(\mathbb{R})$. Tr is the matrix trace. Let p be a point in $SL_2(\mathbb{R})$, we define a scalar product $\langle \cdot, \cdot \rangle_p$ in $T_p SL_2(\mathbb{R})$. To reach this, we define an isometry Φ, which will translate p to the origin e. Then we use the differential $d\Phi_p$ to transpose vectors from $T_p SL_2(\mathbb{R})$ to $T_e SL_2(\mathbb{R})$.

The isometry $\Phi : SL_2(\mathbb{R}) \to SL_2(\mathbb{R})$, satisfying $\Phi(p) = e$, is the left multiplication by the inverse element of p, which is

$$p^{-1} = \begin{bmatrix} \dfrac{1+p_y p_z}{1+p_x} & -p_y \\ \\ -p_z & 1+p_x \end{bmatrix}.$$

Note that we are making an abuse of notation by considering $\Phi(p)$ equal to p. Finally, we define $\Phi(x,y,z) = p^{-1} \cdot X(x,y,z)$.

We now compute the differential of Φ at p. Let v be a tangent at p, then $d\Phi_p(v)$ is a tangent vector at e. Note that $d\Phi_p(v) = v_x \cdot d\Phi_p(e_1) + v_y \cdot d\Phi_p(e_2) + v_z \cdot d\Phi_p(e_3)$, where $\{e_1, e_2, e_3\}$ is canonical base of $\mathbb{R}^3$. Computing,

$$\frac{\partial}{\partial x} := d\Phi_p(e1) = \begin{bmatrix} \dfrac{1+p_y p_z}{1+p_x} & \dfrac{p_y + p_y^2 p_z}{(1+p_x)^2} \\ \\ -p_z & -\dfrac{1+p_y p_z}{1+p_x} \end{bmatrix},$$

$$\frac{\partial}{\partial y} := d\Phi_p(e2) = \begin{bmatrix} 0 & \dfrac{1}{1+p_x} \\ \\ 0 & 0 \end{bmatrix}, \text{ and } \frac{\partial}{\partial z} := d\Phi_p(e3) = \begin{bmatrix} -p_y & -\dfrac{p_y^2}{1+p_x} \\ \\ -1+p_x & p_y \end{bmatrix}.$$

After some computations using the inner product at $T_e SL_2(\mathbb{R})$ we obtain a metric tensor at $\Phi(p) \in SL_2(\mathbb{R})$.

$$\begin{bmatrix} 2\dfrac{(1+p_y p_z)}{(1+p_x)^2} & -\dfrac{p_z}{1+p_x} & -\dfrac{p_y}{1+p_x} \\ \\ -\dfrac{p_z}{1+p_x} & 0 & 1 \\ \\ -\dfrac{p_y}{1+p_x} & 1 & 0 \end{bmatrix}. \tag{3.1}$$

4. Rays in $SL_2(\mathbb{R})$ space

In this section, we investigate the geodesics (rays) in $SL_2(\mathbb{R})$. In Euclidean space, these curves are those that have null acceleration (second derivative). More precisely, a curve $\gamma(t)$ in Euclidean space is a geodesic if $\gamma''(t) = 0$, in other words, a straight line. The analogous concept of "acceleration" in the $SL_2(\mathbb{R})$ geometry is defined using the concept of *covariant derivative*.

For convenience, we use the notation of (x_1, x_2, x_3) instead of (x, y, z). Let $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ be a curve in $SL_2(\mathbb{R})$, the *covariant derivative* of $\gamma'(t)$ is defined as [1]:

$$\frac{D}{dt}\left(\gamma'(t)\right) = \sum_{k=1}^{3} \left(x_k''(t) + \sum_{i,j=1}^{3} \Gamma_{ij}^k x_i' x_j' \right) \frac{\partial}{\partial x^k}. \tag{4.1}$$

Γ_{ij}^k are the *Christoffel symbols* of the $\widetilde{SL_2(\mathbb{R})}$, which are obtained using *Levi–Civita theorem* from Riemannian geometry [1]. Finally, $\gamma(t)$ is a *geodesic* (*ray*) if $\frac{D}{dt}(\gamma'(t)) = 0$, in other words

$$x_k'' + \sum_{i,j=1}^{3} \Gamma_{ij}^k x_i' x_j' = 0, \;\; k = 1, 2, 3. \tag{4.2}$$

Observe that the unique difference between this definition and the Euclidean is the addition of the term $\sum_{i,j=1}^{3} \Gamma_{ij}^k x_i' x_j'$.

Therefore, to compute a ray in $SL_2(\mathbb{R})$ we need the Christoffel symbols at each point $\Phi(p)$, which are all zero except for

$$\Gamma_{11}^1 = -\frac{1+p_y p_z}{1+p_x}, \; \Gamma_{12}^1 = \Gamma_{21}^1 = \frac{p_z}{2}, \; \Gamma_{13}^1 = \Gamma_{31}^1 = \frac{p_y}{2}, \; \Gamma_{23}^1 = \Gamma_{32}^1 = -\frac{1+p_x}{2}$$

$$\Gamma_{11}^2 = -\frac{p_y + p_y^2 p_z}{(1+p_x)^2}, \; \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{p_y p_z}{2+2p_x}, \; \Gamma_{13}^2 = \Gamma_{31}^2 = \frac{p_y^2}{2+2p_x}, \; \Gamma_{23}^2 = \Gamma_{32}^2 = -\frac{p_y}{2}$$

$$\Gamma_{11}^3 = -\frac{p_z + p_z^2 p_y}{(1+p_x)^2}, \; \Gamma_{12}^3 = \Gamma_{21}^3 = \frac{p_z^2}{2+2p_x}, \; \Gamma_{13}^3 = \Gamma_{31}^3 = \frac{p_y p_z}{2+2p_x}, \; \Gamma_{23}^3 = \Gamma_{32}^3 = -\frac{p_z}{2}$$

Equation 4.2 has order two, to solve it is common to add a new variable been its first derivative. Specifically, to compute a ray $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ we consider the *tangent bundle* $TSL_2(\mathbb{R})$ of $SL_2(\mathbb{R})$. This space consists of all pairs of points and tangent direction of $SL_2(\mathbb{R})$, that is $\{(p, v) | p \in SL_2(\mathbb{R}), \; v \in T_p SL_2(\mathbb{R})\}$. In this space the ray $\gamma(t)$ can be expressed by

$$t \to (\gamma(t), \gamma'(t)) = (x_1(t), x_2(t), x_3(t), y_1(t), y_2(t), y_3(t)).$$

The numbers y_k are the coordinates of the tangent vectors. Rewriting Equation 4.2:

$$\begin{cases} x_k' &= y_k \\ y_k' &= -\sum_{i,j=1}^{3} \Gamma_{ij}^k x_i' x_j', \;\; k = 1, 2, 3. \end{cases} \tag{4.3}$$

Equation 4.3 define a vector field on $TSL_2(\mathbb{R})$ which is called *geodesic flow* of $SL_2(\mathbb{R})$. Replacing the Christoffel symbols of $SL_2(\mathbb{R})$ in this equation we obtain.

$$\begin{cases} x_k' &= y_k, \;\; k = 1, 2, 3. \\ y_1' &= \frac{(1+p_y p_z) y_1^2}{1+p_x} - p_z y_1 y_2 - p_y y_1 y_3 + (1+p_x) y_2 y_3 \\ y_2' &= \frac{(1+p_y p_z) p_y y_1^2}{(1+p_x)^2} - \frac{p_z p_y}{1+p_x} y_1 y_2 - \frac{p_y^2}{1+p_x} y_1 y_3 + p_y y_2 y_3 \\ y_3' &= \frac{(1+p_y p_z) p_z y_1^2}{(1+p_x)^2} - \frac{p_z^2}{1+p_x} y_1 y_2 - \frac{p_y p_z}{1+p_x} y_1 y_3 + p_z y_2 y_3 \end{cases} \tag{4.4}$$

To integrate Equation 4.4 we use Euler's numerical method.

A ray γ starting at p in the direction v (that is $\gamma(0) = p$ and $\gamma'(0) = v$) can be approximated by Euler's method. The result is a polygonal curve $\{p_i\}$, defined by:

$$\begin{cases} p_{i+1} = p_i + h \cdot \widetilde{\gamma}'(0) \\ \\ v_{i+1} = v_i + h \cdot \widetilde{\gamma}''(0) \end{cases} \tag{4.5}$$

where h is the integration step and $\widetilde{\gamma}(t)$ is the ray leaving p_i in the direction v_i, that is $\widetilde{\gamma}(0) = p_i$ and $\widetilde{\gamma}'(0) = v_i$. We use Equation 4.4 to compute $\widetilde{\gamma}''(0)$.

We use the polygonal curve $\{p_i\}$ to ray trace a scene in the $SL_2(\mathbb{R})$ geometry. When $h \to 0$ the scene is rendered with more accuracy.

To compute intersection between ray a the scene objects we test the intersection between each segment given by the polygonal approximation given by the above computation.

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