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Ray Tracing in Sol Geometry Spaces

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Technical Report TR-19-08 Relatório Técnico

October - 2019 - Outubro

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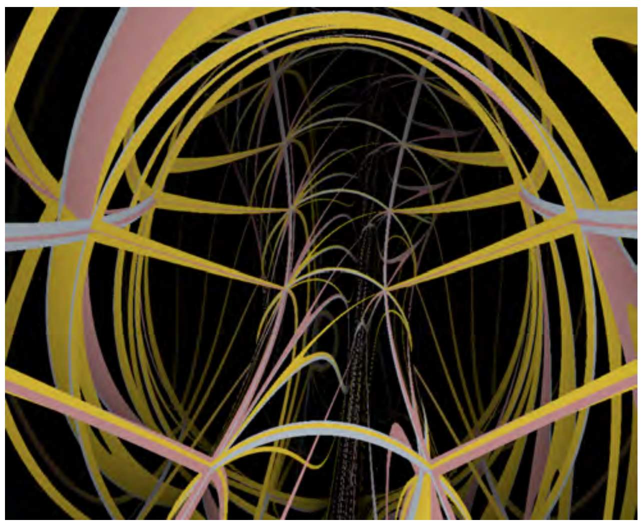
Abstract. In this paper we present an image–based "real–time" rendering of some manifolds locally modeled by the Sol geometry, which is one of the eight Thurston's geometries.



1. Ray tracing requirements

The paper deals with an immersive visualization of spaces locally modeled by the *Sol geometry* — one of the eight Thurston geometries [4] — using ray tracing. Then the space must satisfy at least three properties:

- Being locally similar to a Euclidean space — that is, a *manifold*. This allows us to model the viewer and scene inside the ambient;
- The *tangent space* endowed with an *inner product* at each point — that is, a *Riemannian metric*. Such a definition is used to simulate effects produced between the lights and the scene objects.
- The *ray* leaving a point in any direction — that is, the *geodesic*. Finally, the intersection between rays and the scene "objects" is required.

Geometric manifolds satisfies the above properties. Such objects are locally geometrical similar to special spaces called *model geometries*. In dimension two, for example, there are exactly three models: Euclidean, hyperbolic, and spherical spaces. In

Date: June 24, 2019.
Key words and phrases. Sol geometry.

dimension three, there are five more model geometries, however, this work explores the Sol geometry. Great texts on this subject are Thurston [4] and Martelli [2]. In [Ray3D,Ray3D-Nil] the authors approached the *Euclidean*, *hyperbolic*, *spherical*, and *Nil* spaces.

2. Sol space

A *Lie group* is a manifold G which is also a group and its operations are all smooth. The *Sol space* is an example of a Lie group which consists of all matrices

$$\begin{bmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{bmatrix}$$

with the multiplication operation. We denote such space by Sol. This space is diffeomorphic to $\mathbb{R}^3$ since we could parameterize it by

$$(x,y,z) \in \mathbb{R}^3 \to \begin{bmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{bmatrix} \in Sol,$$

then we push-forward the differentiable structure of $\mathbb{R}^3$ to Sol. We identify $\mathbb{R}^3$ with Sol, which allows us to use $\mathbb{R}^3$ to set our scene for a ray tracing.

Let (x,y,z) and (x',y',z') be two elements in Sol, then their multiplication has the following form:

$$\begin{aligned} (x,y,z)\cdot(x',y',z') &= \begin{bmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{z'} & 0 & x' \\ 0 & e^{-z'} & y' \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{z+z'} & 0 & x'e^z + x \\ 0 & e^{-z-z'} & y'e^{-z} + y \\ 0 & 0 & 1 \end{bmatrix} \\ &= (x'e^z + x, y'e^{-z} + y, z + z'). \end{aligned}$$

That is, the multiplication two elements in Sol is the sum of their coordinates controlled by an additional exponential term in the first two coordinates. In order to put a geometry in Sol we consider the *left multiplication* $(x,y,z) \to p\cdot(x,y,z)$, for all $p \in Sol$, being isometries. We describe this procedure with more details in the next section.

3. The geometry of Sol

The classical way to define a metric in a Lie group G is by fixing a scalar product $\langle \cdot,\cdot \rangle_e$ at the tangent space in the identity element $e \in G$. Then extend it by left multiplication. We construct a metric in the Sol space.

The element $e = (0,0,0)$ is the identity of Sol. Let $T_eSol = \mathbb{R}^3$ be the tangent space at e endowed with the Euclidean scalar product. That is $\langle u,v \rangle_e = u_xv_x + u_yv_y + u_zv_z$, where $u,v \in T_eSol$. Let p be a point in Sol, we define a scalar product $\langle \cdot,\cdot \rangle_p$ in T_pSol. To reach this, we define an isometry Φ, which will translate p to the origin $e = (0,0,0)$. Then we use the differential $d\Phi_p$ to transpose vectors from T_pSol to T_eSol.

The isometry $\Phi : Sol \to Sol$, satisfying $\Phi(p) = e$, is the left multiplication by the inverse element of p, which is $p^{-1} = (-p_x e^{p_z}, -p_y e^{-p_z}, -p_z)$. Then we define $\Phi(x, y, z) = p^{-1} \cdot (x, y, z) = (xe^{-p_z} - p_x e^{p_z}, ye^{p_z} - p_y e^{-p_z}, z - p_z)$.

We now compute the differential of Φ. Let v be a tangent vector at p, then $d\Phi_p(v)$ is a tangent vector at e. Note that $d\Phi_p(v) = v_x \cdot d\Phi_p(e_1) + v_y \cdot d\Phi_p(e_2) + v_z \cdot d\Phi_p(e_3)$, where $\{e_1, e_2, e_3\}$ is canonical base of $\mathbb{R}^3$. Computing, $d\Phi_p(e1) = (e^{-p_z}, 0, 0)$, $d\Phi_p(e2) = (0, e^{p_z}, 0)$, and $d\Phi_p(e3) = (0, 0, 1)$. Therefore, $d\Phi_p(v) = (v_x e^{-p_z}, v_y e^{p_z}, v_z)$.

Then the scalar product between two tangent vectors u and v at p can now be defined as

$$\langle u, v\rangle_p := \langle d\Phi_p(u), d\Phi_p(v)\rangle_e = u^T \begin{bmatrix} e^{2p_z} & 0 & 0 \\ 0 & e^{-2p_z} & 0 \\ 0 & 0 & 1 \end{bmatrix} v.$$

The 3×3 matrix above defines a metric at p. Varying p we obtain a Riemannian metric $\langle \cdot, \cdot \rangle$, since each matrix entry is differentiable. The volume form of Sol coincides with the standard one from $\mathbb{R}^3$, since $\det \begin{bmatrix} e^{2p_z} & 0 & 0 \\ 0 & e^{-2p_z} & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$.

4. Rays in Sol space

In this section, we investigate the geodesics (rays) in Sol. In Euclidean space, these curves are those that have null acceleration (second derivative). More precisely, a curve $\gamma(t)$ in Euclidean space is a geodesic if $\gamma''(t) = 0$, in other words, a straight line. The analogous concept of "acceleration" in the Sol geometry is defined using the concept of *covariant derivative*.

For convenience, we use the notation of (x_1, x_2, x_3) instead of (x, y, z). Let $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ be a curve in the Sol, the *covariant derivative* of $\gamma'(t)$ is

$$\frac{D}{dt}\left(\gamma'(t)\right) = \sum_{k=1}^{3} \left(x_k''(t) + \sum_{i,j=1}^{3} \Gamma_{ij}^k x_i' x_j' \right) \frac{\partial}{\partial x^k}. \tag{4.1}$$

$\partial/\partial x^k$ is the tangent vector of the k-coordinate curve, which in Sol coincides with the axis k. Γ_{ij}^k are the *Christoffel symbols* of the Sol, which are obtained using *Levi–Civita theorem* from Riemannian geometry [1]. Finally, $\gamma(t)$ is a *geodesic* (*ray*) if $\frac{D}{dt}\left(\gamma'(t)\right) = 0$, in other words

$$x_k'' + \sum_{i,j=1}^{3} \Gamma_{ij}^k x_i' x_j' = 0, \;\; k = 1, 2, 3. \tag{4.2}$$

Observe that the unique difference between this definition and the Euclidean is the addition of the term $\displaystyle\sum_{i,j=1}^{3} \Gamma_{ij}^k x_i' x_j'$.

Therefore, to compute a ray in Sol geometries we need the Christoffel symbols at each point (x, y, z), which are (see [2]) all zero except for

$$\Gamma_{13}^1 = \Gamma_{31}^1 = 1, \;\Gamma_{23}^2 = \Gamma_{32}^2 = -1,$$

$$\Gamma^3_{11} = -e^{2z},\ \Gamma^3_{22} = \Gamma^3_{31} = e^{-2z}.$$

Equation 4.2 has order two, to solve it is common to add a new variable been its first derivative. Specifically, to compute a ray $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ we consider the *tangent bundle* $TSol$ of Sol. This space consists of all pairs of points and tangent direction of Sol, that is $\{(p, v)|p \in Sol,\ v \in T_pSol\}$. In this space the ray $\gamma(t)$ can be expressed by $t \to (\gamma(t), \gamma'(t)) = (x_1(t), x_2(t), x_3(t), y_1(t), y_2(t), y_3(t))$. The numbers y_k are the coordinates of the tangent vectors. We rewrite Equation 4.2:

$$\begin{cases} x'_k &= y_k \\ \\ y'_k &= -\displaystyle\sum_{i,j=1}^{3} \Gamma^k_{ij} x'_i x'_j \end{cases}, \quad k = 1, 2, 3. \tag{4.3}$$

Equation 4.3 define a vector field on $TSol$ which is called *geodesic flow* of Sol. Replacing the Christoffel symbols of Sol in this equation we obtain.

$$\begin{cases} x'_k &= y_k,\ k = 1, 2, 3. \\ \\ y'_1 &= -2y_1y_3 \\ \\ y'_2 &= 2y_2y_3 \\ \\ y'_3 &= e^{2p_z}y_1^2 - e^{-2p_z}y_2^2 \end{cases} \tag{4.4}$$

Let $p \in Sol$ and $v \in T_pSol$ be an initial condition for Equation 4.4, there is no solution for this problem in terms of elementary functions [3]. To overcome such difficult we use Euler's numerical method to integration.

A ray γ starting at p in the direction v (that is $\gamma(0) = p$ and $\gamma'(0) = v$) can be approximated by Euler's method. The result is a polygonal curve $\{p_i\}$, defined by:

$$\begin{cases} p_{i+1} = p_i + h \cdot \widetilde{\gamma}'(0) \\ \\ v_{i+1} = v_i + h \cdot \widetilde{\gamma}''(0) \end{cases} \tag{4.5}$$

where h is the integration step and $\widetilde{\gamma}(t)$ is the ray leaving p_i in the direction v_i, that is $\widetilde{\gamma}(0) = p_i$ and $\widetilde{\gamma}'(0) = v_i$. We use Equation 4.4 to compute $\widetilde{\gamma}''(0)$.

We use the polygonal curve $\{p_i\}$ to ray trace a scene in the Sol geometry. When $h \to 0$ the scene is rendered with more accuracy.

To compute intersection between ray a the scene objects we test the intersection between each segment given by the polygonal approximation given by the above computation.

5. Example of compact manifold

We present a compact manifold $M = Sol/\Gamma$ with the geometry modeled by Sol space. The construction of this manifolds will be close of the torus.

Let Γ be the discrete group generated by spanning the isometries $\Phi_1(p) = e_1 \cdot p = (x + 1, y, z)$, $\Phi_2(p) = e_2 \cdot p = (x, y + 1, z)$, and $\Phi_3(p) = (2x + y, x + y, z + 1)$. That is the "translations" in the direction of axis x, y, and z. The manifold $M = Sol/\Gamma$

inherits the geometry of Sol space and for each fixed z it provides a two dimensional torus, thus M admits a foliation by torus. The unit cube it the fundamental domain.

We set the scene inside the unit cube. For ray trace the scene, we use the ray definition given in Section 4. Each time a ray intersects (next section) a cube face we update the ray using the discrete group Γ.

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