Laboratório VISGRAF
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Discrete Exterior Calculus and Applications

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1 Introduction

The *Exterior Calculus of Differential Forms*, first introduced by E. J. Cartan [Cartan 1945], is a geometry-based calculus that has become the foundation of modern differential geometry. It is independent of a coordinate system and thus allows for expressing the geometric nature of many current physical theories. The *Discrete Exterior Calculus* (DEC) aims to offer a discrete counterpart of the exterior calculus on differentiable manifolds, now on simplicial manifolds (Definition 1.4.2) or, more exactly, on simplicial p-chains (Definition 1.4.3).

In this short report, we give a short introduction to the DEC, with the main goal to set the discipline into a broader perspective by drawing its connections to other areas of mathematics and different computational methods. We try to convince the reader about the importance of the theory and demonstrate its utility on the example of smoothing of curves and surfaces (Section 2).

1.1 The Objective of Discrete Exterior Calculus

As said above, exterior calculus explores geometric meaning of quantities in the continuous setting and uses geometric insight to model theories such as electromagnetism or fluid mechanics. DEC translates the tools of exterior calculus into the discrete world of simplicial or more general CW complexes[1], maintaining the consistency with the continuous setting. Preserving the essential structures at the discrete level leads to faster, simpler and more exact computations. To put it short, DEC is an extension of the exterior calculus to discrete spaces including graphs and simplicial complexes.

1.2 DEC vs. Other Computational Techniques

There have been many computational techniques offering discretization of differential equations, but they often fail to preserve the geometric structures they are simulating.

Finite difference or particle methods focus on accurate discretization of local laws, which often leads to loss of the global structures and invariants.

Finite element methods remedy this inadequacy to some extent by satisfying local conservation laws on average and preserve some important invariants. But there is some loss of fidelity following from a discretization process that does not preserve fundamental geometric and topological structures of the underlying continuous models, according to [Desbrun et al. 2008].

Discrete exterior calculus, unlike the other methods, maintains the separation of the topological (metric-independent) and geometric (metric-dependent) components of quantities at play. Moreover, it stores and manipulate quantities at their geometrically meaningful locations. Pointwise evaluations of an approximation are not appropriate discrete analogs of n-dimensional volume integrals. Instead, we consider values on vertices, edges, faces as proper discrete versions of pointwise functions, line and surface integrals, respectively.

[1] For definition of a CW complex, see §38 of [Munkres 1984].

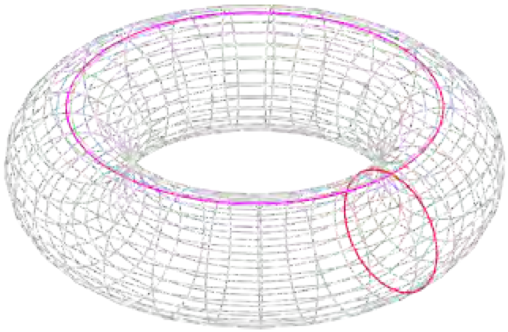


Figure 1: Simple torus. Image from https://categoricalounge.wordpress.com/tag/homology/

1.3 Related Mathematical Disciplines

Not surprisingly, DEC is closely related to other mathematical disciplines. *Differential geometry* studies problems in geometry using techniques of differential and integral calculus and algebra. *Exterior calculus* is a geometry based calculus that has become the modern language of differential geometry, where it is used to define differential forms. The exterior algebra of differential forms, together with the exterior derivative plays a vital role in the algebraic topology of differentiable manifolds.

Algebraic topology deals with topological spaces and aims to find algebraic invariants that classify topological spaces up to homeomorphism. Algebraic topology of simplicial and CW complexes (duals of simplicial complexes are not simplices in general, but the most reasonable duals of simplicial meshes are the so called CW complexes) then studies the topological invariants of the spaces of our interest.

Example 1.3.1. In algebraic topology, **Betti numbers** are used to classify topological spaces based on the connectivity of n-dimensional simplicial complexes. The first Betti number of an orientable closed finite surface fully characterizes its topology [Stillwell, 1993, p. 182-183].

The k-th Betti number β_k of a 3D simplicial complex has the following intuitive interpretation: β_0 is the number of connected components, β_1 is the number of one-dimensional holes or tunnels (non-contractible circles), and β_2 is the number of two-dimensional voids or cavities.

A simple torus has one connected component, thus $\beta_0 = 1$, it has two one-dimensional holes (see the circles in Figure 1), hence $\beta_1 = 2$. And finally, it has one void (the connected empty space inside the torus), therefore $\beta_2 = 1$.

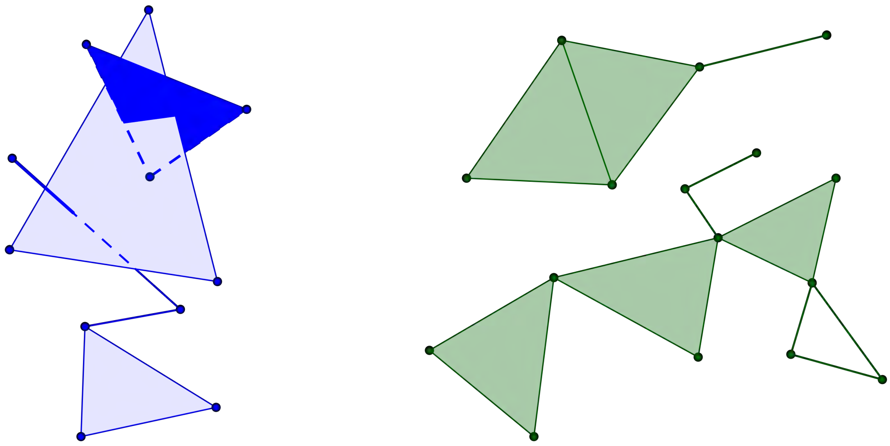
1.4 Discrete Differential Geometry

We search for discrete versions of forms and their domains that would be formally identical to its continuous counterparts. Forms are thus represented as cochains and domains as chains of simplicial or CW complexes.

Definition 1.4.1. Let σ be the ***n*-simplex** spanned by geometrically independent set of points $a_0, \ldots, a_n$ in $\mathbb{R}^d$. Any simplex spanned by a subset of $\{a_0, \ldots, a_n\}$ is called a **face** of σ.

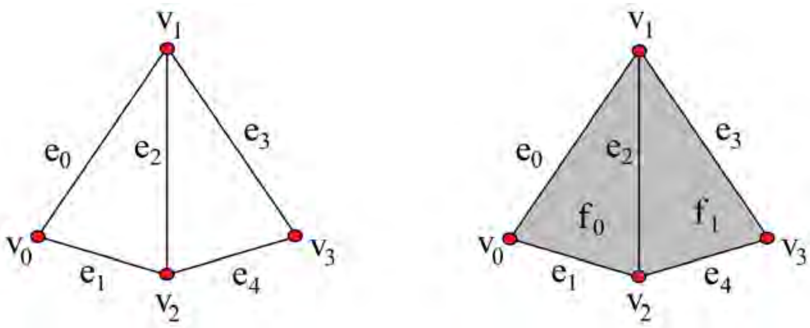
A **simplicial complex** K in $\mathbb{R}^d$ is a collection of simplices in $\mathbb{R}^d$ such that:

1. Every face of a simplex of K is in K.
2. The intersection of any two simplexes of K is either empty or it is a face of each of them.



The Definition 1.4.1 is illustrated in the figure above (created by author in GeoGebra). The set of simplices on the left is not a simplicial complex because the intersection of the upper triangles is not a face of each of them, among others. But the set on the right is a simplicial complex because the intersection of any two simplices is a face of both.

Definition 1.4.2. An n-dimensional **simplicial manifold** is an n-dimensional simplicial complex for which the geometric realization is homeomorphic to a topological manifold. That is, for each simplex, the union of all the incident n-simplices is homeomorphic to an n-dimensional ball, or half a ball if the simplex is on the boundary.



The simplicial complex on the left of the figure above, consisting of all the vertices $\{v_0, v_1, v_2, v_3\}$ and edges $\{e_0, e_1, e_2, e_3, e_4\}$ is not a simplicial manifold because the neighborhoods of the vertices v_1 and v_2 are not homeomorphic to 1D nor 2D balls. Adding the faces f_0, f_1, as shown on the right image, the simplicial complex becomes a 2-dimensional simplicial manifold with boundary. Figure was taken from [Desbrun et al. 2008].

Definition 1.4.3. A ***p*-chain** on a simplicial complex K is a function c from the set of oriented p-simplices of K to the integers, such that:

1. $c(\sigma) = -c(\bar{\sigma})$ if σ and $\bar{\sigma}$ are opposite orientations of the same simplex.

2. $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

We add p-chains by adding their values, the resulting group is denoted $C_p(K)$.

Definition 1.4.4. Let K be a simplicial complex and G an abelian group G, e.g. real numbers under addition. The p-dimensional **cochain** ω is the dual of a p-chain c_p in the sense that ω is a linear mapping that takes p-chains to G:

$$\omega: \; C_p(K) \to G, \;\; c_p \to \omega(c_p).$$

The group of p-dimensional cochains of K, with coefficients in G is denoted $C_p(K, G)$.

As said before, we formally identify differential p-forms with p-cochains. Notice that in the definition of a p-cochain above we could have substituted the abstract abelian group G by the pair $(\mathbb{R}, +)$. But we did not do so for a purpose, because a p-form, and thus also a p-cochain, can be vector-valued or it can have complex coefficients.

The general setting of differential forms is on a differentiable manifold, which allows one to use the exterior calculus. Similarly, cochains are defined on chains and we use discrete exterior calculus to manipulate given quantities.

For a more profound treatment of the groups of chains and cochains, but also for the definition of the related notions of boundary and coboundary operators, which are of a big importance in DEC, see for instance [Munkres 1984] or [Desbrun et al. 2008].

2 Application in Smoothing of Curves and Surfaces

The general approach in fairing is to define an energy E that measures the smoothness of the given manifold. The energy E is either

- a function of the immersion (vertex positions) f of the plane curve/surface in $\mathbb{R}^3$, or
- a function of curvature.

Subsequently, we reduce E by gradient descent. If E is a function of vertex positions, we have to solve partial differential equations. On the other hand, E as a function of curvature leads to ordinary differential equations.

In this section we provide a brief overview of four algorithms. Classical curvature flow on positions of a curve (Subsection 2.1) and isometric curvature flow on curves in curvature space (Subsection 2.2), the latter proposed by K. Crane in [Crane et al. 2013a].

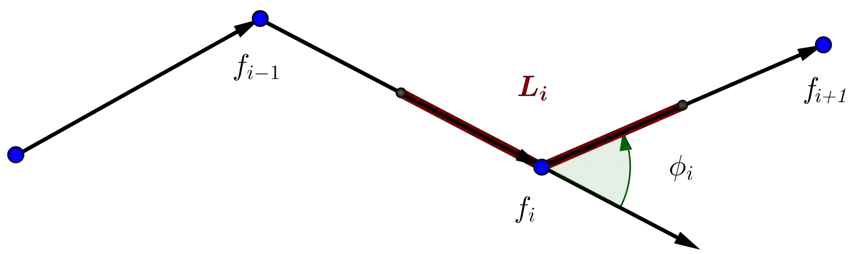
Next, we outline implicit mean curvature flow on surfaces (Subsection 2.3), where E is a function of vertex positions, a method that was suggested by M. Desbrun in [Desbrun et al. 1999]. The last technique for surface fairing, advertised in Section 2.4, is conformal curvature flow in curvature space first presented also in [Crane et al. 2013a].

2.1 Curvature Flow on Positions of a Curve

A discrete curve f is given by an ordered set of vertices $f_i \in \mathbb{R}^2$, i.e., $f = (f_0, \ldots, f_n)$. We define the pointwise curvature κ at a vertex i as

$$\kappa_i = \frac{\phi_i}{L_i}, \tag{1}$$

where $L_i = \frac{1}{2}(|f_{i+1} - f_i| + |f_{i-1} - f_i|)$ is the length of the dual edge of the vertex i, and ϕ_i is the exterior angle at the corresponding vertex (see the image bellow).



The curvature energy is given by

$$E(f) = \sum_i \kappa_i^2 L_i = \sum_i \frac{\phi_i^2}{L_i}.$$

And the curvature flow becomes

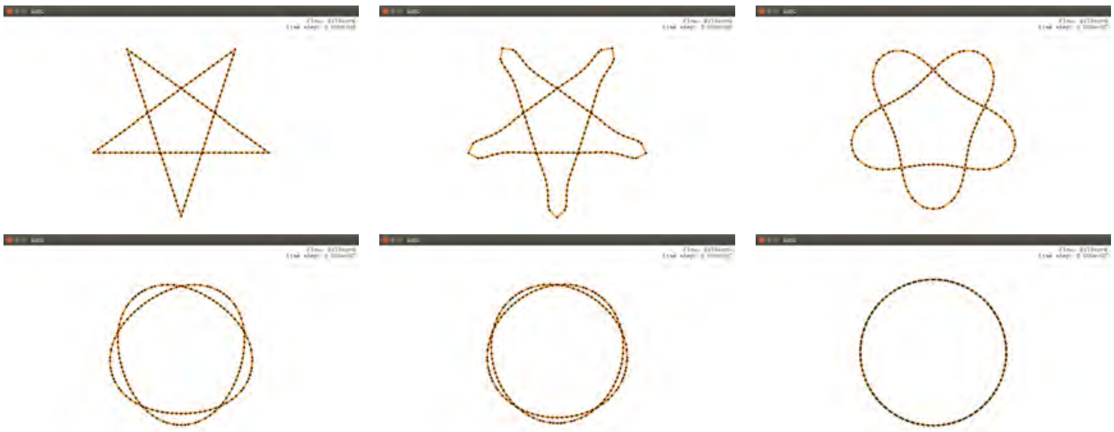
$$\dot{f} = -\nabla E(f).$$

We integrate the flow using the forward Euler scheme, i.e.,

$$f^t = f^0 + t \cdot \dot{f},$$

where f^0 are the initial vertex positions and f^t are the new vertex positions after applying the forward Euler step.

The images bellow show the curvature flow on a star-shaped curve.



You can notice that the flow is not isometric, the distance between two consecutive points is changing during the flow. This can be a problem for some applications. In the next subsection, we remedy this drawback.

The images were generated by a program implemented by the author, its skeleton code can be found in the course notes of [Crane and Schroder 2012], Homework 4. In the same source you can find details about the method just presented.

2.2 Isometric Curvature Flow in Curvature Space

We start again with a discrete curve $f = (f_0, \ldots, f_n), f_i \in \mathbb{R}^2$. But we define the curvature energy as a function of the curvature κ:

$$E(\kappa) = \kappa^2 = \sum_i \kappa_i^2,$$

where κ_i is given by equation (1). The curvature flow is now a simple function of the curvature:

$$\dot{\kappa} = -\nabla E(\kappa) = -2\kappa.$$

We integrate the flow using the forward Euler scheme again and obtain new vertex curvatures κ^t :

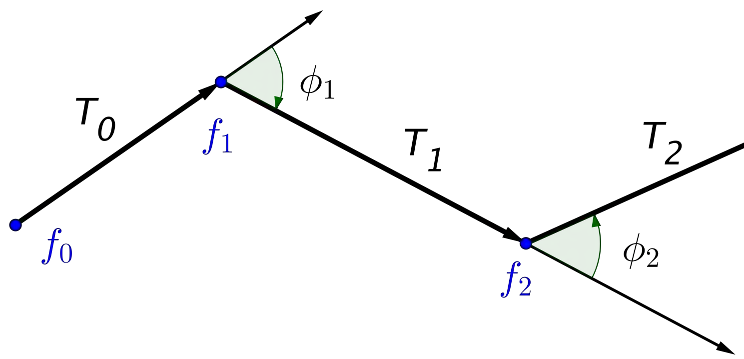
$$\kappa^t = \kappa^0 + t\,\dot{\kappa}\,. \tag{2}$$

To recover the curve, we first integrate curvatures to get tangents:

$$T_i = L_i(\cos\theta_i, \sin\theta_i), \text{ where } \theta_i = \sum_{k=0}^{i} \phi_k.$$

And then we integrate tangents to get the new vertex positions:

$$f_i = \sum_{k=0}^{i} T_k.$$



Notice that the length of each edge is preserved by construction, i.e., the curvature flow is isometric.

But there is a shortcoming. Changing the curvature arbitrarily can lead to some undesirable artifacts, e.g., closed curve do not close back up any more. The recovered tangents do not "integrate up" to form a closed loop.

Fortunately, we can easily force the curve to form a closed loop again by applying the following **integrability constraints**:

1. A closed loop f must satisfy
$$\sum_i \kappa_i L_i = 2\pi k,$$
for some turning number $k \in \mathbb{Z}$. Which is equivalent to
$$T_1 = T_n \iff \sum_i \dot{\kappa}_i = 0.$$

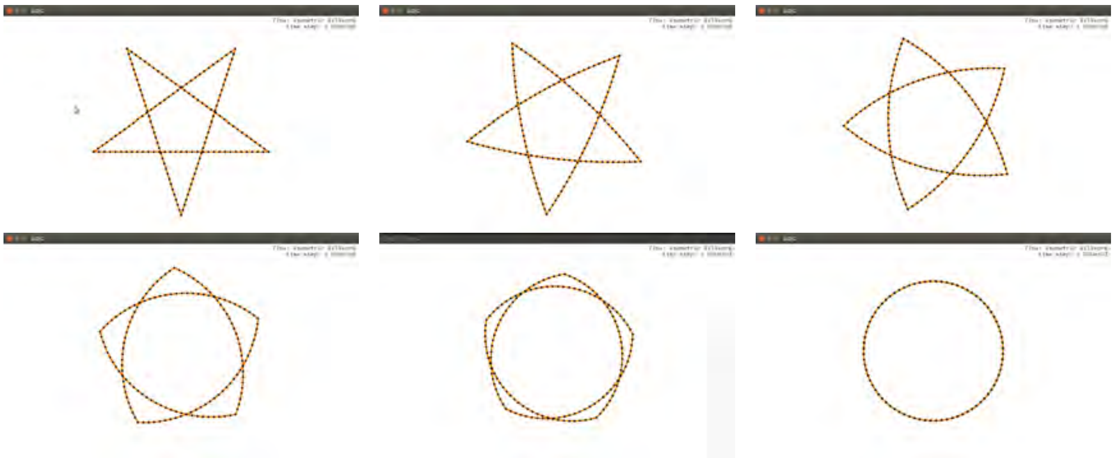
2. The endpoints must meet up, i.e., $f_0 = f_n$, which leads to:
$$\sum_i \dot{\kappa}_i f_i = 0.$$

Overall, then, the change in curvature must avoid a three-dimensional subspace of directions:

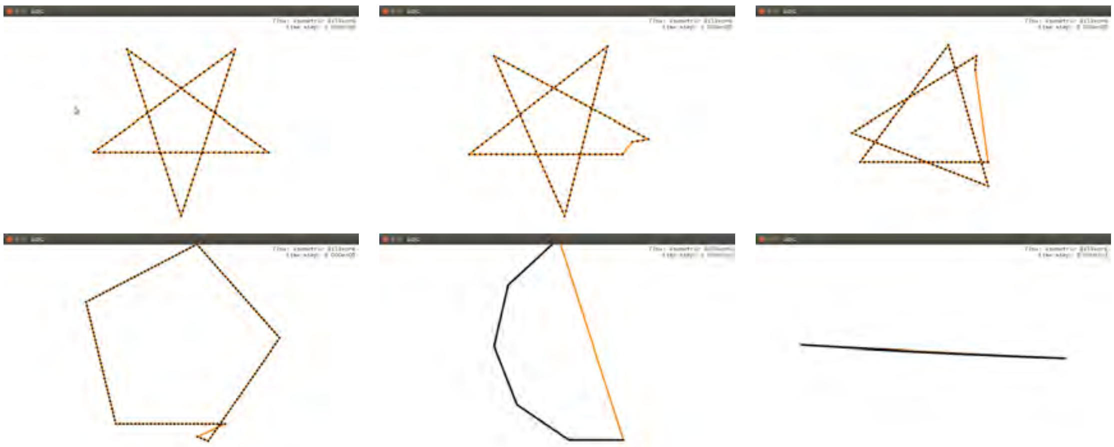
$$\langle \dot{\kappa}, 1 \rangle = \langle \dot{\kappa}, f_x \rangle = \langle \dot{\kappa}, f_y \rangle = 0,$$

where f_x and f_y are x and y coordinates of the vertices, respectively.

Having removed the forbidden directions from the flow $\dot{\kappa}$, we get the constrained curvature flow $\dot{\kappa_c}$ which we employ in equation (2) and proceed in the same manner as for the unconstrained version.



The six pictures above show the constrained curvature flow, the set of pictures bellow shows the unconstrained one. Both versions are isometric, but in the latter the curve is opening and finally converges to a perfectly straight line.



The images were generated by a program implemented by the author, its skeleton code can be found in the course notes of [Crane and Schroder 2012], Homework 4. In the same source or in [Crane et al. 2013a] can be found details about the method.

2.3 Implicit Mean Curvature Flow

On a discrete surface $f = (f_0, \ldots, f_n)$, $f_i \in \mathbb{R}^3$, we consider the flow

$$\dot{f} = 2HN = \triangle f,$$

that is, we move the points in the direction of vertex normals with magnitude proportional to the mean curvature. The Laplace operator $\triangle f$ reads:

$$(\triangle f)_i = \frac{1}{2} \sum_j (\cot \alpha_j + \cot \beta_j)(f_j - f_i). \tag{3}$$

We now use the backward Euler scheme to integrate the flow:

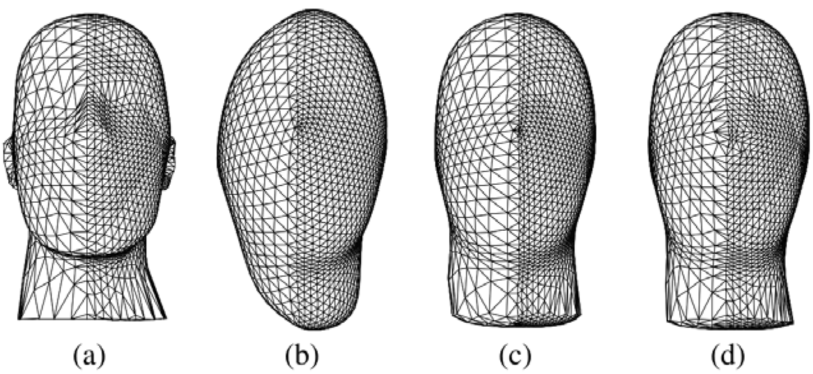
$$(I - t\triangle) f^t = f^0.$$

The matrix $A = (I - t\triangle)$ is highly sparse, therefore it is not too expensive to solve the given linear system.

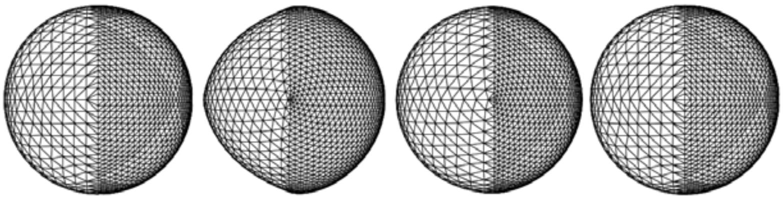
The quality of the resulting process highly depends on the approximation of the Laplace operator:

- linear approximation, the so called **umbrella operator** expects the edges to be of equal length, which leads to distortion of the shape,
- **scale-dependent umbrella operator** almost keeps the original distribution of triangle sizes,
- **cotangent discretization of the Laplace operator** (equation (3)) achieves the best smoothing with respect to the shape, no drifting occurs.

For a more thorough review of these three methods, see [Desbrun et al. 1999]. The following two set of images were taken from the same source.



Smoothing of a mesh (a) with different sampling rates, (b) the umbrella operator creates a significant distortion of the shape, (c) the scale-dependent umbrella operator does not create distortion but it introduces some sliding of the mesh, whereas in (d) the curvature flow smooths the small features but they stay in place, the smoothing looks more natural. The same holds for the mesh of the sphere bellow.



(a) (b) (c) (d)

On the following pictures we can see *implicit mean curvature flow with cotangent discretization of the Laplace operator* on a mesh of an armadillo. They were generated by a program implemented by the author, its skeleton code can be found in the course notes of [Crane and Schroder 2012], Homework 3.

Neither this method is without a drawback, the implicit mean curvature flow produces sharp geometric discontinuities (the sharp cusps on the ears or fingers of the armadillo). This can be a problem, which will be remedied by the next method - conformal curvature flow.

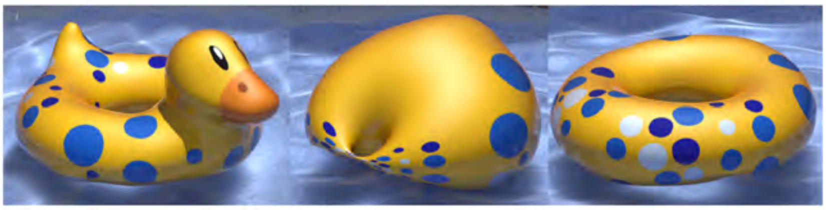


Figure 2: Duck with nontrivial topology (left). Center: unconstrained flow yields distortion of both geometry and texture. Right: an exactness constraint prevents distortion.

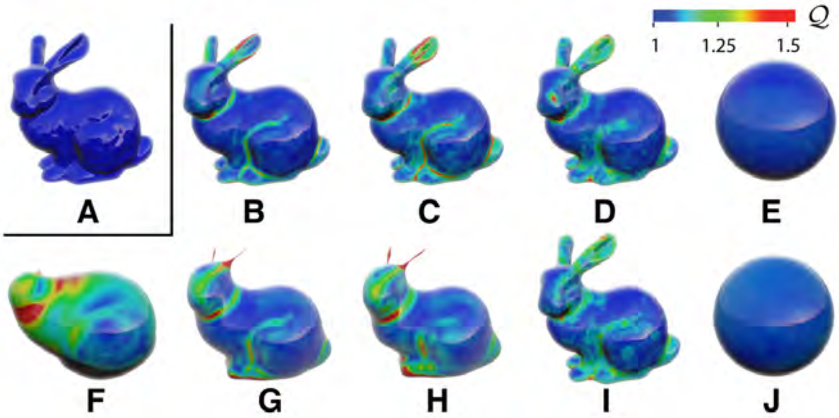


Figure 3: Various flows applied to (A) for 10s of CPU time at the maximum stable time step. (G) implicit mean curvature flow [Desbrun et al. 1999]. (J) conformal Willmore flow presented in this section.

2.4 Conformal Curvature Flow

Keenan Crane in [Crane et al. 2013a] suggests a curvature flow in curvature space that yields conformal smoothing of surfaces. Instead of using the potential energy $E(f)$ as a function of vertex positions, he uses Willmore energy $E_W(\mu)$ as a function of mean curvature half density:

$$E_W(\mu) = ||\mu||^2.$$

Gradient flow with respect to μ then reads $\dot{\mu} = -2\mu = -H$. Applying forward Euler scheme yields:

$$\mu^t = \mu^0 - 2tH,$$

where H is the pointwise mean curvature of the current mesh computed via the cotangent Laplace operator (3), that is, $\triangle f = 2HN$.

In order to obtain conformal curvature flow and avoid distortion or cracks, the flow must satisfy several linear constraints that are explained in [Crane et al. 2013a, Sec. 6.2]. We limit ourselves to showing some advantages of this new method in Figures 2 and 3, that were taken from the same article.

3 Summary and Literature

The goal of this brief introductory survey was to provide a general image about what a Discrete Exterior Calculus is and convince the reader that discrete differential forms can be extremely useful in computational science.

For theoretical treatment of DEC we indicate papers, other than the papers already cited, [Hirani 2003], [Desbrun et al. 2005], and [Desbrun et al. 2008]. Duals of simplicial meshes are studied in [De Goes 2014], [Mullen et al. 2011], and [De Goes et al. 2014b]. Geometry processing uses abundantly discrete representations of tangent vector fields [De Goes et al. 2014a] and connections on discrete surfaces [Crane et al. 2010].

There is a vast literature dealing with theoretical aspects of DEC and its applications in Computer Graphics, the reader is referred to the cited literature and references given therein.

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