Voronoi based clustering
Douglas Cedrim
Summary

- Motivation
- Voronoi clustering
- Topological clustering
- Results
- Directions
Motivation

Given a mesh $M$, we want to obtain another mesh that approximates $M$ with a lower number of elements (vertices / faces)
Motivation

Manifold base mesh construction

Given a mesh $M$, we want to obtain another mesh that approximates $M$ with a lower number of elements (vertices / faces)
Motivation

• Clustering

- Packing
- Selecting
- Meshing
Summary

- Motivation
- Voronoi clustering
  - Overview
  - Lloyd's relaxation
  - Some extensions for unstructured meshes
- Topological clustering
- Results
- Directions
Voronoi Diagram

• Continuous setting:

Given an open set $\Omega$ of $\mathbb{R}^d$, and $(x_i)_{i=1}^n$ points (seeds), a Voronoi tessellation is defined by

$$C_i = \{ w \in \Omega : \| w - x_i \| \leq \| w - x_j \|, j = 1, 2, \ldots, n, j \neq i \}$$
Voronoi Diagram

• Continuous setting:

Given an open set $\Omega$ of $\mathbb{R}^d$, and $(x_i)_{i=1}^n$ points (seeds), a Voronoi tessellation is defined by

• 2D case: Dual of a triangulation

$$C_i = \{ w \in \Omega : \| w - x_i \| \leq \| w - x_j \|, j = 1, 2, \ldots, n, j \neq i \}$$
Centroidal Voronoi Diagram (CVD)

- How to avoid skinny triangles?

An intuition...
Centroidal Voronoi Diagram (CVD)

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An intuition...
Centroidal Voronoi Diagram (CVD)

• More formally...

  Given a region $C \subset \mathbb{R}^n$. The mass centroid $z^*$ minimizes the functional

  $\int_C \| z^* - y \| \, dy$

  and is given by

  $z^* = \frac{\int_C y \, dy}{\int_C dy}$
Centroidal Voronoi Diagram (CVD)

• More generally...

Given a density function $\rho$, the mass centroid is given by

$$z^* = \frac{\int_C y\rho(y)dy}{\int_C \rho(y)dy}$$

and minimizes

$$\int_C \rho(y) \| z^* - y \| dy$$
Centroidal Voronoi Diagram (CVD)

- Definition: A CVD of $\Omega$ tesselates it in disjoint regions

$$C_i = \{ w \in \Omega : \|w - z_i\| \leq \|w - z_j\|, j = 1, 2, \ldots, n, j \neq i \}$$

with

$$z_i = \frac{\int_{C_i} y \rho(y) dy}{\int_{C_i} \rho(y) dy}$$

minimizing

$$E = \sum_{i}^{n} \int_{C_i} \rho(y) \|z_i - y\| dy$$
Centroidal Voronoi Diagram (CVD)

\[ \rho \equiv 1 \]
Centroidal Voronoi Diagram (CVD)

\[ \rho \equiv 1 \]
Centroidal Voronoi Diagram (CVD)

\[
\rho(x, y) = \exp(-10(x^2 + y^2))
\]
Centroidal Voronoi Diagram (CVD)

- Lloyd's algorithm:
  1. Select an initial set of $k$ points (seeds);
  2. Construct a Voronoi tesselation associated with these seeds;
  3. Compute the mass centroids of each region;
  4. If this new set of points meets some convergence criterion, terminate; otherwise, return to step 1.
Centroidal Voronoi Diagram (CVD)

- Wide range of applications
  - Clustering
  - Optimal quadrature rules
  - Image processing
  (...)

Centroidal Voronoi Diagram (CVD)

• And for meshes?
  • Constrained Centroidal Voronoi Diagram (CCVD) [Du 2003]
    - Centroids problem
  • Geodesic metric on the mesh [Cedrim 2011]
    - High cost for a relaxation process
Centroidal Voronoi Diagram (CVD)

• And for meshes?
  • Valette [2008]
    - Initial clustering by average densities over $n$ clusters
    - Cleaning process for disconnected cells
    - Adjusts cells according to tests on its boundaries
    - Centroids defined in order to minimize QEM
    - Polynomial fitting on local neighbourhood for densities
Summary

- Motivation
- Voronoi clustering
- Topological clustering
  - Cell definition
  - Centroids
  - Relaxation
  - Quality control
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Topological clustering

- Our approach aims to
  - Generate a base (coarse) mesh
  - Maintain properties of previously processed meshes

![Diagram showing the relationship between Uniform mesh, General mesh, Adaptive mesh, and Topological clustering, leading to a good quality coarser mesh.](image-url)
Topological clustering

• Main idea
  1. Define a cell decomposition
  2. Calculate each cell centroid
  3. Nested convergence
     1. Relaxation process
     2. Quality control
Topological clustering

• Cell definition
Topological clustering

• Cell definition
Topological clustering

• Cell definition
Topological clustering

- Cell definition
Topological clustering

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Topological clustering

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Topological clustering

- Cell definition

![Diagram of topological clustering with red and blue cells and white points]
Topological clustering

• Cell definition
Topological clustering

- Cell definition
Topological clustering

- Cell definition
Topological clustering

• Centroids
Topological clustering

• Centroids
Topological clustering

- Centroids
**Topological clustering**

- Centroids
Topological clustering

• Centroids
Topological clustering

- Centroids
Topological clustering

- Centroids
  - Remain as elements of the mesh
  - Connected cells
Topological clustering

• Relaxation process

“(...)meets some convergence criterion(...)”
Topological clustering

- Relaxation process
  “(...)meets some convergence criterion(...)”
- Cell converges locally if centroids and seeds became neighbours since $k$ previous iterations
Topological clustering

- Relaxation process
  "(...)meets some convergence criterion(...)"
- Cell converges locally if centroids and seeds became neighbours since $k$ previous iterations
Topological clustering

- Quality control
  - Average area deviation
Topological clustering

• Quality control
  • Average area deviation
  • Elements per cluster
  • Number of clusters
Results

- Uniform input mesh (1.9x average area)

file:genus3_1.9_24c
Results

- Uniform input mesh (1.5x average area)
Results

- Uniform input mesh (500 tri per cluster)
Results

- Adaptive input mesh (100 tri per cluster)
Results

- Adaptive input mesh (3.5x average area)
Results

- Seeds size
  Variable X Fixed
  - 0.5% from original mesh
- Isotropy and anisotropy

100 clusters

~ 10min

< 1min
Results

- Uniform input mesh (500 clusters – 0.33% of original)

149K

339s
Results

20K

2%
Directions

• New ways to control quality
• Speedup process
• Topological guarantees
Main references


- **SACHT, L. K. & PEREIRA, T. S.** - *Centroidal Voronoi Tessellation on Meshes*
Thank's for your attention!

More on:
www.lcad.icmc.usp.br/~cedrim/courses/cma_impa