Introduction to Computational Manifolds and Applications

Part 1 - Constructions

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Parametric Pseudo-Manifolds

Simplicial Surfaces

We will start investigating the construction of 2-dimensional PPM’s in $\mathbb{E}^3$.

In the previous lecture, we considered a polygon as a sketch of the shape of the curve we wanted to build. Now, we need another object to play the same role the polygon did.

We can think of a few choices, but the easiest one is arguably a polygonal mesh.

So, let us start with a triangle mesh, which is a formally known as a simplicial surface.
Definition 9.1. Given a finite family, \((a_i)_{i \in I}\), of points in \(\mathbb{E}^n\), we say that \((a_i)_{i \in I}\) is **affinely independent** if the family of vectors, \((a_ia_j)_{j \in (I-\{i\})}\), is linearly independent for some \(i \in I\).
Simplicial Surfaces

**Definition 9.2.** Let \( a_0, \ldots, a_d \) be any \( d + 1 \) affinely independent points in \( \mathbb{E}^n \), where \( d \) is a non-negative integer. The simplex \( \sigma \) spanned by the points \( a_0, \ldots, a_d \) is the convex hull of these points, and is denoted by \( [a_0, \ldots, a_d] \). The points \( a_0, \ldots, a_d \) are the vertices of \( \sigma \). The dimension, \( \text{dim}(\sigma) \), of the simplex \( \sigma \) is \( d \), and \( \sigma \) is also called a \( d \)-simplex.

In \( \mathbb{E}^n \), the largest number of affinely independent points is \( n + 1 \).

So, in \( \mathbb{E}^n \), we have simplices of dimension 0, 1, \ldots, \( n \). A 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. Furthermore, the convex hull of any nonempty subset of vertices of a simplex is a simplex.
Definition 9.3. Let $\sigma = [a_0, \ldots, a_d]$ be a $d$-simplex in $\mathbb{E}^n$. A face of $\sigma$ is a simplex spanned by a nonempty subset of $\{a_0, \ldots, a_d\}$; if this subset is proper then the face is called a proper face. A face of $\sigma$ whose dimension is $k$, i.e., a $k$-simplex, is called a $k$-face.
Definition 9.4. A simplicial complex $\mathcal{K}$ in $\mathbb{E}^n$ is a finite collection of simplices in $\mathbb{E}^n$ such that

(1) if a simplex is in $\mathcal{K}$, then all its faces are in $\mathcal{K}$;

(2) if $\sigma, \tau \in \mathcal{K}$ are simplices such that $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau$ is a face of both $\sigma$ and $\tau$.
Simplicial Surfaces

Definition 9.5. The dimension, $\dim(K)$, of a simplicial complex, $K$, is the largest dimension of a simplex in $K$, i.e., $\dim(K) = \max\{\dim(\sigma) \mid \sigma \in K\}$. We refer to a $d$-dimensional simplicial complex as simply a $d$-complex. The set consisting of the union of all points in the simplices of $K$ is called the underlying space of $K$, and it is denoted by $|K|$. The underlying space, $|K|$, of $K$ is also called the geometric realization of $K$. 

![a 2-complex](image-url)
A simplicial complex is a **combinatorial object** (i.e., a finite collection of simplices).

The underlying space of a simplicial complex is a **topological object**, a subset of some $\mathbb{E}^n$. 

A 2-complex
Simplicial Surfaces

**Definition 9.6.** Let $\mathcal{K}$ be a simplicial complex in $\mathbb{E}^n$. Then, for any simplex $\sigma$ in $\mathcal{K}$, we define two other complexes, the *star*, $\text{st}(\sigma, \mathcal{K})$, and the *link*, $\text{lk}(\sigma, \mathcal{K})$, of $\sigma$ in $\mathcal{K}$, as follows:

$$\text{st}(\sigma, \mathcal{K}) = \{ \tau \in \mathcal{K} \mid \exists \eta \text{ in } \mathcal{K} \text{ such that } \sigma \text{ is a face of } \eta \text{ and } \tau \text{ is a face of } \eta \}$$

and

$$\text{lk}(\sigma, \mathcal{K}) = \{ \tau \in \mathcal{K} \mid \tau \text{ is in } \text{st}(\sigma, \mathcal{K}) \text{ and } \tau \text{ and } \sigma \text{ have no face in common} \}.$$
Definition 9.7. A 2-complex $\mathcal{K}$ in $\mathbb{E}^n$ is called a simplicial surface without boundary if every 1-simplex of $\mathcal{K}$ is the face of precisely two simplices of $\mathcal{K}$, and the underlying space of the link of each 0-simplex of $\mathcal{K}$ is homeomorphic to the unit circle, $S^1 = \{x \in \mathbb{E}^2 \mid \|x\| = 1\}$.

The set consisting of the 0-, 1-, and 2-faces of a 3-simplex is a simplicial surface without boundary.
Simplicial Surfaces

The simplicial complex consisting of the proper faces of two 3-simplices (i.e., two tetrahedra) sharing a common vertex is not a simplicial surface without boundary as the link of the common vertex of the two 3-simplices is not homeomorphic to the unit circle, $S^1$. 
Parametric Pseudo-Manifolds

Simplicial Surfaces

From now on, we will refer to a simplicial surface without boundary as simply a simplicial surface. The underlying space of a simplicial surface is called its underlying surface.

The underlying surface of a simplicial surface is a topological 2-manifold in $\mathbb{E}^n$. 

\[ \mathcal{K} \quad |\mathcal{K}| \]
Simplicial Surfaces

**Definition 9.8.** Let $\mathcal{K}$ be a simplicial complex in $\mathbb{E}^n$. For each integer $i$, with $0 \leq i \leq \text{dim}(\mathcal{K})$, we define $\mathcal{K}^{(i)}$ as the simplicial complex consisting of all $j$-simplices of $\mathcal{K}$, for every $j$ such that $0 \leq j \leq i$. Moreover, if $\mathcal{L}$ is a simplicial complex in $\mathbb{E}^m$, then a map

$$f: \mathcal{K}^{(0)} \rightarrow \mathcal{L}^{(0)}$$

is called a *simplicial map* if whenever $[a_0, \ldots, a_d]$ is a simplex in $\mathcal{K}$, then $[f(a_0), \ldots, f(a_d)]$ is a simplex in $\mathcal{L}$. A simplicial map is a *simplicial isomorphism* if it is a bijective map, and if its inverse is also a simplicial map. Finally, if there exists a simplicial isomorphism from $\mathcal{K}$ to $\mathcal{L}$, then we say that $\mathcal{K}$ and $\mathcal{L}$ are *simplicially isomorphic*.
Parametric Pseudo-Manifolds

Simplicial Surfaces

$\mathbb{E}^3$ and $\mathbb{E}^2$ are simplicially isomorphic.

$\mathcal{K}$ and $\mathcal{L}$ are simplicially isomorphic.
Let $f : \mathcal{K}^{(0)} \to \mathcal{L}^{(0)}$ be given by $f(a_0) = b_5, \ f(a_1) = b_3, \ f(a_2) = b_2, \ f(a_3) = b_1, \ f(a_4) = b_0, \ f(a_5) = b_4$. 

Simplicial Surfaces
It is easily verified that \( f \) is a simplicial isomorphism.
Given a simplicial surface, $\mathcal{K}$, in $\mathbb{E}^3$, we are interested in building a parametric pseudo-surface, $\mathcal{M}$, in $\mathbb{E}^3$ such that the image, $\mathcal{M}$, of $\mathcal{M}$ is homeomorphic to the underlying surface, $|\mathcal{K}|$, of $\mathcal{K}$, and such that $\mathcal{M}$ also approximates the geometry of $|\mathcal{K}|$. 
Parametric Pseudo-Manifolds

Gluing Data

As we did before, let us first focus on the definition of a set of gluing data.

Unfortunately, this task is not as easy as it was in the one-dimensional case.

The key is to notice that the simplicial surface, $\mathcal{K}$, which is a combinatorial object, explicitly defines a topological structure on $|\mathcal{K}|$ (via the adjacency relations of all simplices).

So, we should define $p$-domains, gluing domains, and transition functions based on $\mathcal{K}$. 
Gluing Data

As we will see during the next lectures, there are many choices for $p$-domains. But, in general, $p$-domains are associated with simplices of $\mathcal{K}$. For instance, the vertices of $\mathcal{K}$.

We can define a one-to-one correspondence between $p$-domains and vertices of $\mathcal{K}$. 

\[ \Omega_v \] 
\[ \Omega_w \] 
\[ \Omega_u \] 

\[ \mathbb{E}^3 \] 
\[ \mathbb{E}^2 \]
Gluing Data

The previous correspondence implies that the number of $p$-domains is equal to the number of vertices of $\mathcal{K}$. A distinct choice of correspondence may yield a different number.

The choice of a geometry for the $p$-domains is a key decision too.
Gluing Data

Intuitively, each $p$-domain is an open "disk" that is consistently glued to other $p$-domains in order to define the topology of the image, $M$, of the parametric pseudo-surface.

Since a vertex $u$ of $\mathcal{K}$ is connected only to the vertices of $\mathcal{K}$ that belong to the link, $\text{lk}(u, \mathcal{K})$, of $u$ in $\mathcal{K}$, it is natural to think of the $p$-domain, $\Omega_u$, which is associated with vertex $u$, as the interior of a polygon in $\mathbb{E}^2$ with the same number of vertices as $\text{lk}(u, \mathcal{K})$. 
Gluing Data

To simplify calculations, we can assume that $\Omega_u$ is a regular polygon inscribed in a unit circle centered at the origin of a local coordinate system of $\mathbb{E}^2$. We can also assume that one vertex of $\Omega_u$ is located at the point $(0, 1)$. Now, $\Omega_u$ is uniquely defined.
Gluing Data

Formally, let $I = \{ u \mid u \text{ is a vertex in } K \}$, $n_u$ be the number of vertices of the link, $\text{lk}(u, K)$, of $u$ in $K$, and $P_u$ be the regular, $n_u$-polygon whose vertices are located at the points

$\left( \cos \left( i \cdot \frac{2\pi}{n_u} \right), \sin \left( i \cdot \frac{2\pi}{n_u} \right) \right)$,

for all $i = 0, 1, \ldots, n_u - 1$. Then, we can define $\Omega_u = \hat{P}_u$, where $\hat{P}_u$ is the interior of $P_u$. 
Parametric Pseudo-Manifolds

Gluing Data

Checking...

(1) For every $i \in I$, the set $\Omega_i$ is a nonempty open subset of $\mathbb{E}^n$ called parametrization domain, for short, $p$-domain, and any two distinct $p$-domains are pairwise disjoint, i.e.,

$$\Omega_i \cap \Omega_j = \emptyset,$$

for all $i \neq j$.

Our $p$-domains are (connected) open subsets of $\mathbb{E}^2$. If we assume that they live in distinct copies of $\mathbb{E}^2$, then they will not overlap, and hence condition (1) of Definition 7.1 holds.
Gluing Data

What about gluing domains? The following picture should help us find a good choice:
Gluing Data

As we can see, the intersection of the stars, \(\text{st}(u, \mathcal{K})\) and \(\text{st}(w, \mathcal{K})\), of \(u\) and \(w\) consists of exactly two triangles. These triangles share an edge in both \(\text{st}(u, \mathcal{K})\) and \(\text{st}(w, \mathcal{K})\). So, we can think of defining the gluing domains as \textit{diamond-shaped}, open subsets of the \(p\)-domains.
Gluing Data

To precisely define gluing domains, we associate a 2-dimensional simplicial complex, $K_u$, with each $p$-domain $\Omega_u$. The complex $K_u$ satisfies the following two conditions: (1) $|K_u|$, is the closure, $\overline{\Omega_u}$, of $\Omega_u$ and (2) $K_u$ is isomorphic to the star, $\text{st}(u, K)$, of $u$ in $K$.

An obvious choice for $K_u$ is the canonical triangulation of $\overline{\Omega_u}$:
Gluing Data

Fix any counterclockwise enumeration, $u_0, u_1, \ldots, u_m$, of the vertices in $\text{lk}(u, \mathcal{K})$. 
Gluing Data

Let $u'_0$ be the vertex of $\mathcal{K}_u$ located at the point $(0, 1)$.

Let

$$u'_0, u'_1, \ldots, u'_m$$

be the counterclockwise enumeration of the vertices of $\text{lk}(u', \mathcal{K}_u)$ starting with $u'_0$. 
Gluing Data

Let  

\[ f_u : \text{st}(u, \mathcal{K})^{(0)} \rightarrow \mathcal{K}_u^{(0)} \]

be the simplicial map given by  

\[ f_u(u) = u' \]  

and  

\[ f_u(u_i) = u'_i, \]  

for  \( i = 0, \ldots, m \).

It is easily verified that  \( f_u \) is a simplicial isomorphism.
Gluing Data

Let $u$ and $w$ be any two vertices of $\mathcal{K}$ such that $[u, w]$ is an edge in $\mathcal{K}$.

Let $x$ and $y$ be the other two vertices of $\mathcal{K}$ that also belong to both $\text{st}(u, \mathcal{K})$ and $\text{st}(w, \mathcal{K})$.

Assume that $x$ precedes $w$ in a counterclockwise traversal of the vertices of $\text{lk}(u, \mathcal{K})$ starting at $y$. 
Gluing Data

We can now define the gluing domains, $\Omega_{uw}$ and $\Omega_{wu}$, as $\Omega_{uw} = \overset{\circ}{Q}_{uw}$ and $\Omega_{wu} = \overset{\circ}{Q}_{wu}$, where

$$Q_{uw} = [f_u(u), f_u(x), f_u(w), f_u(y)] \quad \text{and} \quad Q_{wu} = [f_w(w), f_w(y), f_w(u), f_w(x)]$$

are the quadrilaterals given by the vertices $f_u(u), f_u(x), f_u(w), f_u(y)$ of $\mathcal{K}_u$ and the vertices $f_w(w), f_w(y), f_w(u), f_w(x)$ of $\mathcal{K}_w$, and $\overset{\circ}{Q}_{uw}$ and $\overset{\circ}{Q}_{wu}$ are the interiors of $Q_{uw}$ and $Q_{wu}$. 

Formally, for every \((u, w) \in I \times I\), we let

\[
\Omega_{uw} = \begin{cases} 
\Omega_u & \text{if } u = w, \\
\emptyset & \text{if } u \neq w \text{ and } [u, w] \text{ is not an edge of } K, \\
\circ & \text{if } u \neq w \text{ and } [u, w] \text{ is an edge of } K.
\end{cases}
\]
Gluing Data

(2) For every pair \((i, j) \in I \times I\), the set \(\Omega_{ij}\) is an open subset of \(\Omega_i\). Furthermore, \(\Omega_{ii} = \Omega_i\) and \(\Omega_{ji} \neq \emptyset\) if and only if \(\Omega_{ij} \neq \emptyset\). Each nonempty subset \(\Omega_{ij}\) (with \(i \neq j\)) is called a gluing domain.

By definition, the sets \(\Omega_u, \emptyset\), and \(\check{Q}_{wu}\) are open in \(E^2\). Furthermore, the sets \(\check{Q}_{uw}\) and \(\check{Q}_{wu}\) are well-defined and nonempty, for every \(u, w \in I\) such that \([u, w]\) is an edge of \(K\).

So, for every \(u, w \in I\), we have that \(\Omega_{uw} \neq \emptyset\) iff \([u, w]\) is an edge of \(K\). Thus, for every \(u, w \in I\), \(\Omega_{uw} \neq \emptyset\) iff \(\Omega_{wu} \neq \emptyset\), and hence condition (2) of Definition 7.1 also holds.
Gluing Data

Our definitions of $p$-domain and gluing domain naturally lead us to a gluing process induced by the gluing of the stars of the vertices of $\mathcal{K}$ along their common edges and triangles.

The gluing strategy we adopted here does not depend on the geometry of the $p$-domains and gluing domains, but on the adjacency relations of vertices and edges of $\mathcal{K}$.

However, the geometry of the $p$-domains and gluing domains have a strong influence in the level of difficulty of the transition maps and parametrizations we choose to use.

Despite of our commitment to a particular geometry, we will present next an axiomatic way of defining the transition maps. Our axiomatic definition should be as much independent of the geometry of the $p$-domains and gluing domains as possible.