Mathematical Optimization in Graphics and Vision

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Course Schedule

Part 1 - Optimization Problems in Graphics
Luiz Velho (45 minutes)

Why optimization is important for graphics?

Part 2 - Overview of Optimization Techniques
Paulo Cezar Pinto Carvalho (45 minutes)

How optimization can be used in graphics?

End - Questions and Answers (10 minutes)

Website

http://www.visgraf.impa.br/otim-02

- **Presentation Slides**
- **Additional Material**

Optimization Problems in Graphics

Luiz Velho

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Outline

- Concepts
  - Graphical Objects
  - Operators
- Graphics Problems
  - Direct / Inverse
  - Well-Posed / Ill-Posed
- Optimization
  - Representation Operator
  - Examples of Applications

Motivation

- Optimization is a Basic Tool for Graphics
  - Widely Used
  - Flexible
- What Makes Optimization Important ?
  - Need to Understand Graphics Problems
  - Conceptual View

Graphics and Vision

Data —Graphics→ Image
Data ←Vision— Image

- Relation with Physical Universe
  - *Photography*: (2D representation)
  - *Human Visual System*: (3D reconstruction)
- Concepts
  - What are the Models ?
  - What are the Problems ?

Graphical Object

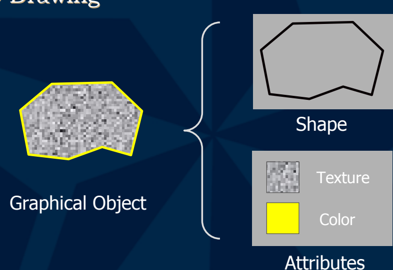
$o = (U, f), \quad f: U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$

- Geometric Support: U
  (shape)
- Attribute Function: f
  (properties)

* *Dimension*
  * Object: *dim(U)*
  * Space: n

Simple Example

- 2D Drawing



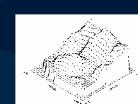
More Examples

- Images:
  - Grayscale, Color

  (simple shape, complex attributes)

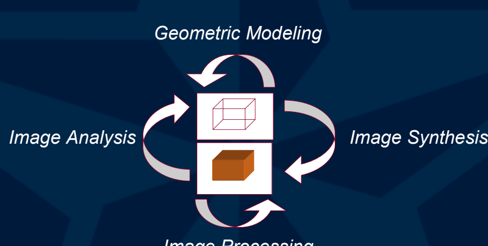
- Models (Data)
  - Surfaces, Solids

  (complex shape, simple attributes)

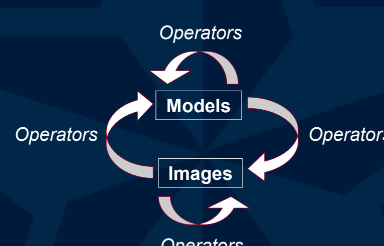


* *Graphical Object :- Comprehensive Concept*

Processing Graphical Objects



Operators on Graphical Objects



Graphical Problems

- Spaces of Graphical Objects
  (function spaces)

  $x \in O$

- Operators on Spaces of Graphical Objects

  $T: O_1 \rightarrow O_2$

  $T(x) = y$

Types of Problems

$$T(x) = y$$

- Direct Problems
  - Given T and x , find y
- Inverse Problems
  - Given T and y , find x such that $T(x) = y$
  - Given x and y , find T

Example: Visualization

- x is the scene
  - Geometry
  - Illumination
  - Camera
- T is the *rendering* operator

  $T x = y$

- y is the rendered image

* *Direct Problem (Vision - Inverse Problem)*

Problem Characterization

* Hadamard (1902)

- Well-Posed Problem
  - Existence of Solution
  - Uniqueness of Solution
  - Continuous Dependence on Initial Conditions
- Ill-Posed Problem

  (doesn't satisfy at least one of the above conditions)

Ill-Posed Problems

- Inherent in some Applications
  - Inverse Problems are usually Ill-Posed
- Sources of Ill-Posedness
  - Multiple Solutions
  - Numerical Errors

⇒Need to *get around* Ill-Posedness

- Optimization Methods
  - Best Solution (*unique*)

Example of Ill-Posed Problem

- Linear System: $T x = y$,

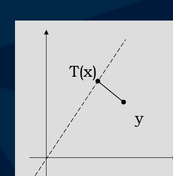
  $T = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$, $x = (u, v)$, $y = (a, 2a)$, and $a \in \mathbf{R}$

- Solution y must lie on

  $2u - v = 0$

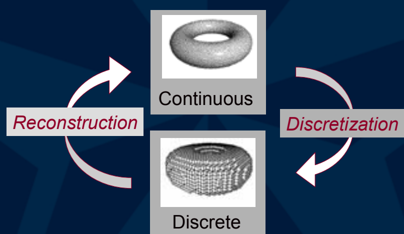
  - Ill-Conditioning / Perturbations
- Well-Posed Solution

  y = argmin $|y, Tx|$



Describing Graphical Objects

- Computer Representation from Mathematical Model



Ill-Posed Problems in Graphics

- Representation Operator

$$R : \mathbf{O} \rightarrow \mathbf{O}_d$$

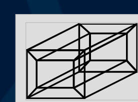
- Discretization of Graphical Objects
  - Direct Problem
- Reconstruction of Graphical Objetcs
  - Inverse Problem

Reconstruction Issues

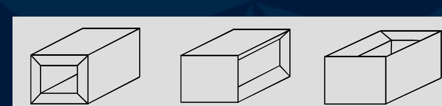
- Why do we need to reconstruct ?
  - Discretization gives incomplete information
  - Working in continuous domain avoid numerical errors
  - Semantics
- Reconstruction
  (Invertibility of the Representation Operator)
  - Exact
  - Non-Exact
- Ill-Posed Reconstruction
  - Ambiguous Representation

Ambiguity in Models

- *Wireframe* Model

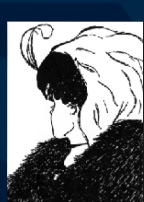


- Many Interpretations

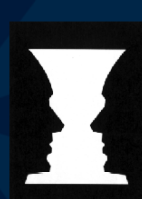


Ambiguity in Images

- Two possible interpretations



2D / 3D



Foreground / Background

Graphical Problems and Optimization

* *Typical problems from different areas*

- Selected Problems
  - Modeling
  - Visualization
  - Vision / Image Procesing
- Not Discussed
  - Animation

Variational Modeling

- Energy Minimization Model

$$\gamma(t) = \arg\min E_{total}(\gamma)$$

- Energy Functional
  - Shape Quality
  - Model Control

$$E_{total}(\gamma) = E_{internal}(\gamma) + E_{external}(\gamma)$$

Internal Energy

- *Fair* Shape (Physical Model)

$$E_{internal}(\gamma) = \lambda E_{bend} + (1-\lambda) E_{stretch}$$

- Thin Plate

$$E_{bend}(\gamma) = \int_{\gamma} k^2(t)\,dt$$

- Membrane

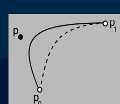
$$E_{stretch}(\gamma) = \int_{\gamma} \| \gamma'(t) \| \, dt$$

* *Spline Curves and Surfaces*

External Energy

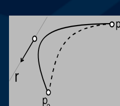
- Modeling Controls
  - Attraction / Repulsion Forces
- Punctual

$$E_{punctual}(\gamma) = \min_t \| \gamma(t) - p \|^2$$



- Directional

$$E_{directional}(\gamma) = \min_t \| \gamma'(t) \times p \|$$



Applications

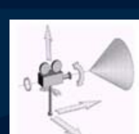
- Reconstruction from Points
- Interactive Modeling

Camera Calibration

- Projective Transformation

$$T(x) = p$$

- Inverse Problems
  - Compute Camera Transformation ***T***
  - Compute Scene Points ***x***



Pinhole Camera
(7 parameters)

* *Best Parameter Estimation*

Differential Camera Control

- Camera Motion with Constrains

minimize

$$E = \frac{1}{2} \left\| \frac{dc}{dt} - \frac{dc_0}{dt} \right\|^2$$

subject to

$$\frac{dp}{dt} - \frac{dp_0}{dt} = 0$$

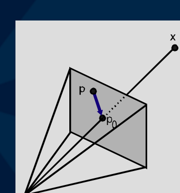
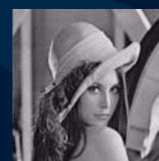


Image Analysis

- *Primal Sketch*
  - Edges
- David Marr's Conjecture
  - Image is Characterized by its Edges
- Boundary Operator
  (Computation of Edges)
  - Frequency Methods
  - Geometric Methods

Wavelets

- Frequency Methods for Computation of Edges
- Image Reconstruction and Compression
  (invertibility of boundary operator)




Snakes

- Geometric Computation of the Boundary
- Energy Minimization Approach


Conclusions

* *Optimization has many uses in Graphics*
- Representation
  - Solves Ambiguity
  - Incorporates User-defined Criteria
- Algorithms
  - Efficient
  - Robust
- User Interface
  - Intuitive Controls