RAY TRACING IN CURVED SPACES

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Abstract. In this paper we present an image–based real–time rendering of some classical manifolds locally modeled by Euclidean, hyperbolic, and spherical geometries.

1. Ray tracing requirements

The paper deals with an immersive visualization of spaces locally modelled by Non–Euclidean geometries using ray tracing, thus we need at least three properties:

- Being locally similar to an Euclidean space — that is, a *manifold*. This allows us to put the viewer and the scene inside the ambient as in common approaches: some deformation may be allowed;
- For each point p we need vectors pointing in all directions: the *tangent vectors* in p. Moreover, the *inner product* between two tangent vector is required. These definitions are used to simulate effects produced between the lights and the scene objects.
- For a point p and a vector v tangent in p, we should be able to compute the *ray* leaving p in the direction of v. Finally, the intersection between rays and the scene "objects" are required.

Geometric manifolds satisfies the above properties. Such objects are locally geometrical similar to special spaces called *model geometries.* In dimension two, for example, there are exactly three models: Euclidean, hyperbolic, and spherical spaces. In dimension three, there are five more model geometries, however, in this work we focus in the first three spaces. We describe these topics in more details below. Great texts on this subject are Thurston [7] and Martelli [5].

2. Geometric Models

The spaces presented in this section will be very useful to model more complexes spaces which we should introduce later. The main ingredients for a ray tracing implementation are also present here.

Example 2.1. The *Euclidean space* $\mathbb{E}^3$ is the set $\mathbb{R}^3 = \{(x, y, z)|\ x, y, z \in \mathbb{R}\}$ endowed with the classical *inner product* $\langle u, v\rangle_{\mathbb{E}} = u_x \cdot v_x + u_y \cdot v_y + u_z \cdot v_z$ where $u = (u_x, u_y, u_z)$ and $v = (v_x, v_y, v_z)$ are vectors in $\mathbb{E}^3$. The *distance* between two points p and q is defined by $d_{\mathbb{E}}(p, q) = \sqrt{\langle p - q, p - q\rangle_{\mathbb{E}}}$. The curve $\gamma(t) = p + t \cdot v$ describes a *ray* leaving a point p in a direction v. Analogously, for any $n > 0$ the n-dimensional Euclidean space $\mathbb{E}^n$ is constructed.

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Example 2.2 (Hyperbolic space)**.** The *Lorentzian inner product* of the vectors v and u in $\mathbb{R}^4$ is defined as $\langle u, v\rangle_{\mathbb{H}} = u_x v_x + u_y v_y + u_z v_z - u_w v_w$. The vector space $\mathbb{R}^4$ endowed with the Lorentzian product is called the *Lorentzian space*. The *hyperbolic space* $\mathbb{H}^3$ is the hyperboloid $\{p \in \mathbb{R}^4 | \ \langle p, p\rangle_{\mathbb{H}} = -1\}$ endowed with a special metric $d_{\mathbb{H}}(p, q) = \cosh^{-1}(-\langle p, q\rangle_{\mathbb{H}})$, where p and q are two points in $\mathbb{H}^3$. Due to its remarkable similarity to the sphere definition (see next example), $\mathbb{H}^3$ is also known as *pseudo-sphere*.

A tangent vector v to a point p in $\mathbb{H}^3$ satisfies $\langle p, v\rangle_{\mathbb{H}} = 0$. Moreover, the *tangent space* $T_p\mathbb{H}^3$ coincides with the set $\{v \in \mathbb{R}^4 | \ \langle p, v\rangle_{\mathbb{H}} = 0\}$. The Lorentzian inner product is positive when restricted to the tangent space.

Rays in $\mathbb{H}^3$ are the intersections between $\mathbb{H}^3$ and the planes in $\mathbb{R}^4$ containing the origin. For instance, the ray leaving a point $p \in \mathbb{H}^3$ in a tangent direction v is the intersection between $\mathbb{H}^3$ and the plane spanned by the vectors v and p in $\mathbb{E}^4$. Such ray can be parameterized as $r(t) = \cosh(t)p + \sinh(t)v$.

The space $\mathbb{H}^3$ does not contain any straight line, thus its rays can not be straight. However, it is possible to model $\mathbb{H}^3$ in the unit open ball in $\mathbb{R}^3$ — known as *Klein model* $\mathbb{K}^3$— such that in this model the rays are straight lines. More precisely, each point $p \in \mathbb{H}^3$ is projected in the space $\{(x, y, z, w) \in \mathbb{R}^4 | \ w = 1\}$ by considering p/p_w, the space $\mathbb{K}^3$ is obtained by forgetting the coordinate w.

The hyperbolic space is a model of a *Non-Euclidean* geometry, since it does not satisfy only the Parallel Postulate: given a ray r and a point $p \notin r$, there is a unique ray parallel to r. For a ray r in the hyperbolic space and a point $p \notin r$ there are infinite rays going through p which do not intersect r.

Example 2.3. The 3-*sphere* $\mathbb{S}^3$ is the set $\{p \in \mathbb{E}^4 | \ \langle p, p\rangle_{\mathbb{E}} = 1\}$ endowed with the metric $d_{\mathbb{S}}(p, q) = \cos^{-1} \langle p, q\rangle_{\mathbb{E}}$, where p and q are in $\mathbb{S}^3$.

As in the hyperbolic case, a tangent vector v to a point in $\mathbb{S}^3$ satisfies $\langle p, v\rangle_{\mathbb{E}} = 0$. The *tangent space* $T_p\mathbb{S}^3$ corresponds to the set $\{v \in \mathbb{S}^3 | \langle p, v\rangle_{\mathbb{E}} = 0\}$. The space $T_p\mathbb{S}^3$ inherits the Euclidean inner product of $\mathbb{E}^4$.

A *ray* in $\mathbb{S}^3$ passing through a point p in a tangent direction v is the arc produced by the intersection between $\mathbb{S}^3$ and the plane spanned by v, p, and the origin of $\mathbb{E}^4$. Such ray can be parameterized as $r(t) = cos(t)p + sin(t)v$.

Again, the 3-sphere $\mathbb{S}^3$ is an example of a Non-Euclidean geometry, since its fails the Parallel Postulate: given a ray r and a point $p \notin r$, there is a unique ray parallel to r. As the rays in $\mathbb{S}^3$ are the big circles, thus choosing two of then in $\mathbb{S}^2 \subset \mathbb{S}^3$, they always intersect in exactly two points.

3. Manifolds

We now present the concept of a manifold, which generalizes the Euclidean, hyperbolic, and spherical spaces. In this work we are, particularly, interested in manifolds which can be modelled by those three geometric model, since they provide the required ingredient for a ray tracing algorithm.

A *n-manifold* M is a topological space which is locally identical (topologically speaking) to the Euclidean space $\mathbb{E}^n$; n is the dimension of M. More precisely, there is a neighborhood of every point in M mapped homeomorphically to the open ball of $\mathbb{E}^n$. These maps are called *charts* of M. The change of charts between two neighborhoods in M must be continuous. Thus, informally, the manifold definition generalizes the concept of Euclidean spaces. This work focus on manifolds of dimension 3. Examples of 3-manifolds include the Euclidean, hyperbolic, and spherical spaces.

Straight lines are fundamental objects when working with ray tracing algorithms, since light travels along them. A manifold M admits a generalization of such notion, the *geodesics*. To define them we need two additional tools. The first is the calculus framework, which is done by requesting changes of charts in M to be diffeomorphisms — M is called *differentiable*. This allows us to define for each point a *tangent space* and work with calculus on it. The second tool is the attribution of an appropriate metric on each tangent space — M is called *Riemannian*. Then we can compute angles between *vectors* in tangent spaces (crucial in ray tracing), and distances between two points in M. Finally, a *geodesic* in M is a curve such that locally it is the shortest path. We keep using the term *ray* instead of geodesics since the paper deals with ray tracing.

Hopefully, there are many combinatorial and algebraic constructions of manifolds which allow us to represent such exotic structures in computers. We focus on two notions: identifying faces of a polyhedra and quotient by discrete groups. Both constructions depart from a purely topological point of view, so we add a geometry, which we consider to be Euclidean, hyperbolic, and spherical (the other five geometries are left for future works).

We now remind a combinatorial way to build topological and geometrical surfaces. We start with a topological construction, then consider the geometric case.

4. Geometric surfaces

A *surface* is a 2-dimensional manifold. Such objects can be topologically constructed by gluing edges of polygons. Specifically, let there be given a finite collection of convex polygons with their edges divided into disjoint pairs. Identifying each couple of edges through a homeomorphism gives rise to space K. The well-known classification theorem of compact 2-manifolds states that K is a manifold. The *torus* is obtained by gluing, in the same orientation, opposite edges of a square, we obtain the *projective plane* if we reverse all the gluing orientations.

4.1. **Topological construction of surfaces.** The classical way to state the classification theorem of compact surfaces is by the concept of *connect sum* of surfaces. Let S_1 and S_2 be compact connected surfaces, we define the connected sum $S_1 \# S_2$ as follows. Remove a disk D_1 and D_2 from S_1 and S_2, then $S_1 \# S_2$ is the surface given by the identification of the boundaries ∂D_1 and ∂D_2 through a homeomorphism. Finally, the theorem says that any compact surface is homeomorphic to a sphere, a connected sum of tori, or a connected sum of projective planes.

The prove of the classification theorem give to us a combinatorial way of representing any compact surface in a computer. It states that any topological surface is produced by an appropriate pair-wise identification of the edges of an oriented polygon with an even number of edges. To present the fundamental ideas, we need some additional results and definitions.

Let S be a compact surface.

- It is well-known that every compact surface can be triangulated. Let T be a triangulation of S;
- Cutting along each edge in T we obtain a collection of triangles which can be all placed in the plane without intersection; the edge identification must

be remembered. The result is a collection of triangles in plane with their edges divided into disjoint pairs;
- We label each triangle edge with a letter according to its gluing orientation;
- Gluing the triangle collection through its pair-wise edge identification without leaving the plane produces a polygon P. The boundary ∂P is an oriented sequence of letters, a *word*. Each letter has exactly one couple;
- Let a and b be a couple of edges in ∂P. If the identification of a and b reverses the orientation of ∂P we denote b by a^{-1}, and simply a otherwise;
- There is a technical result which states that by cutting and gluing P leads us to an equivalent polygon Q (that is both produces the same surface) such that its boundary has one of the following configurations:
  - aa^{-1}, which is a sphere;
  - $\sum aba^{-1}b^{-1}$, a connected sum of tori $aba^{-1}b^{-1}$;
  - $\sum aa$, a connected sum of projective planes aa.

Therefore, each surface can be topologically represented as an even polygon with a special identification of its edges. To geometrically model each of these topological surfaces, some algebraic tools are required.

4.2. **Manifolds as the quotient by discrete groups.** The content here is applied both in dimension 2 and 3. The idea is to geometrically model a manifold by taking the quotient of some particular space by a special group acting on it.

A connected topological space is called *simply* if each closed curve can be continuously deformed into a point. Euclidean, hyperbolic, and spherical spaces are simply connected. The most famous problem related to such definition is Poincaré conjecture: the 3-sphere is the unique simple connected compact 3-manifold.

It is well know that for each connected manifold, there is a unique *simple connected universal covering* (Theorem 12.19 in [4]). Informally, a manifold $\widetilde{M}$ *covers* another manifold M if there is a map which "evenly covers" a neighborhood of each point in M. For example the torus is covered by the Euclidean space and the *projective space* (fundamental in computer graphics algorithms) is covered by the sphere. We take the opposite direction: given a simply connected space and a group acting on it, we consider its quotient. By previous observation we only need to consider simply connected manifolds.

Here we follow the notations and definitions of [3, 4]. Let G be a group endowed with a topology and M be a manifold, if the *action* $p \to g(p)$, where $p \in M$ and $g \in G$, is a continuous map, then G is called a *continuous group*. A subgroup Γ of the continuous group G is called a *discrete group* if there is a neighborhood U of the G's identity such that $U \cap \Gamma$ is the identity element. For example the group of translations in $\mathbb{E}^2$ spanned by $(x, y) \to (x \pm 1, y)$ and $(x, y) \to (x, y \pm 1)$ is a discrete group acting on $\mathbb{E}^2$. In particular, this group acts *freely*, since it leaves no fixed points.

Let M be a simply connected manifold and Γ be a discrete group acting on M, the *quotient* M/Γ is the set $\{\Gamma \cdot p| \ p \in M\}$ where $\Gamma \cdot p = \{g(p)| \ g \in \Gamma\}$ is the *orbit* of p. For example, the quotient of $\mathbb{E}^2$ by the group of translation cited above gives rise to the torus $\mathbb{T}^2$ presented in previous section. This provides a new description of a ray r leaving a point $p \in \mathbb{T}^2$ in a direction v: r is described by considering the fractional part of the coordinates of the ray $r(t) = p + t \cdot v$ in $\mathbb{E}^2$.

We are interested in cases where M is a *geometry* model, for example: Euclidean, hyperbolic, and spherical spaces. The quotient space M/Γ inherits the geometry of M. We say that M/Γ has the *geometric structure* modeled by M. For example, $\mathbb{T}^2$ has the geometric structure modeled by $\mathbb{E}^2$, in particular, for each $i, j \in \mathbb{Z}$ the unit square $[i, i+1] \times [j, j+1]$ in $\mathbb{E}^2$ is mapped isometrically to $\mathbb{T}^2$. Such squares *tessellate* $\mathbb{E}^2$: this is what you actually see in an immersive view of $\mathbb{T}^2$.

The *fundamental domain* Δ plays an important role in the above construction. Δ is the region of M which contains exactly one point for each of these orbits $\{g(p) | \ g \in \Gamma\}$. The unit square $[0,1] \times [0,1]$ is the fundamental domain of $\mathbb{T}^2$; note that it is not unique. No deformation is required if we consider $M = \mathbb{E}^2$.

Let M be a manifold and Γ be a discrete group acting on it, when is M/Γ a manifold? *Quotient manifold theorem* (Theorem 9.16 in [4]) provides an answer. This, informally, states that M/Γ is a manifold when the (*Lie*) group Γ acts smoothly, freely, (and *properly*) on M. For example, if M/Γ is a compact surface and $M = \mathbb{E}^2$, then the only two possibilities are torus or *Klein bottle* (see Section 6.2 of Martelli [5]). Klein bottle surface is obtained by identifying two of the parallel edges of a square and the other two in the reverse manner.

4.3. **Geometrical construction of surfaces.** We remind the well-known *geometrization* theorem of compact surfaces, which states that the geometry of any compact surface may be modelled by the Euclidean, hyperbolic, or spherical metric.

Theorem 4.1 (Geometrization of surfaces). *Any compact, topological surface admit a geometric structure modeled by the Euclidean, hyperbolic, or spherical space.*

For an example of a compact surface modeled by the hyperbolic geometry consider the bitorus S, which is topologically the connect sum of two torus: $S = \mathbb{T}^2 \# \mathbb{T}^2$. S can be presented as a regular polygon P with 8 sides $aba^{-1}b^{-1}cdc^{-1}d^{-1}$ as discussed in Subsection 4.1. Observe that all vertices of P are identified into a unique vertex v after gluing all coupled edges of P. Then, the 8 corners of P are glued together producing a topological disk. However, if we consider P with the Euclidean geometry, the angular sum around v will be 6π. To avoid such problem, consider that P is a regular polygon centered in the hyperbolic plane, with an appropriate scale, the angles of P will all be $\pi/4$. The identification $aba^{-1}b^{-1}cdc^{-1}d^{-1}$ of the edges of P produces the desired group action Γ in the hyperbolic plane $\mathbb{H}^2$ such that $\mathbb{H}^2/\Gamma$ is the bitorus. In terms of *tessellation*, the group Γ tessellates $\mathbb{H}^2$ with regular polygons with 8 sides.

To present a compact surface geometrically modeled by the spherical geometry consider the projective plane $\mathbb{RP}^2$. Topologically $\mathbb{RP}^2$ can be presented as a polygon P with two sides thought the following identification aa (see Subsection 4.1). Embedding P in the 2-sphere $\mathbb{S}^2$ as one of its hemisphere, the desired geometrical identification follows. The group generated by the identification aa of P in $\mathbb{S}^2$ tessellates the sphere into the south and north hemispheres. The group is basically made up of two elements: the identity and the antipodal map.

5. Geometric 3-manifolds

As for surfaces, there is a combinatorial procedure to built three dimensional manifolds from identifications of polyhedral faces.

5.1. **Topological construction of 3-manifolds.** Let a finite number of (convex) polyhedra be given. Endow such polyhedra with an appropriate pair-wise identification of its faces. Each couple of faces has the same number of edges and are mapped homeomorphically to each other. Such identification gives rise to a *polyhedral complex* K, which is not necessary a 3-manifold. However, K is a 3-manifold if and only if its Euler characteristic is equal to zero (Theorem 4.3 in [2]).

We now take the opposite approach. Let M be a compact 3-manifold, we represent M as a polytope P endowed with a pair-wise identification of its faces. The following algorithm is a try to mimic the surface case presented in Subsection 4.1.

- Let T be a triangulation of M. The existence of T is guaranteed by the well-know triangulation theorem: here M is decomposed into tetrahedra, triangles, edges, and vertices;
- Detaching every face identification in T, gives rise to a collection of tetrahedra which can be embedded in $\mathbb{E}^3$. Remember the pair-wise face gluing;
- Gluing in a topological way each possible coupled of tetrahedra in the collection without leaving $\mathbb{E}^3$ produces the desired polytope P. The faces in the boundary ∂P are pair-wise indentyfied;
- However the combinatorial problem of reducing P to a standard form, as in the surface case, remain open (see page 145 in [4]).

Despite the fact that there is no (yet) classification of compact 3-manifold in the sense which we present for compact surfaces, it is still possible to model each such manifold geometrically by just eight (Thurston's) geometries. This is the 3-dimensional case of Theorem 4.1: the *Thurston–Perelman geometrization theorem*. This paper focus only on Euclidean, hyperbolic, and spherical spaces.

5.2. **Geometrical construction of 3-manifolds.** *Thurston–Pereman geometrization theorem* states that every compact 3-manifold has a certain geometric structure that can be associated to it [7]. However, in this case, it is not possible to associate a single geometry to the whole manifold, the geometrization theorem states that we can decompose the manifolds in parts, each of these with geometric structure modeled by one of the eight Thurston's geometries. These include Euclidean, hyperbolic, and spherical spaces.

Theorem 5.1 (Geometrization)**.** *Any compact, topological 3-manifold can be constructed using just 8 geometry models.*

Briefly, the others five geometries are the product spaces $\mathbb{R} \times \mathbb{S}^2$ and $\mathbb{R} \times \mathbb{H}^2$, and the 3-dimensional *Lie group Nil*, *Sol*, and $\widetilde{SL_2}(\mathbb{R})$. We skip the details of such geometries, since our experimental result focus on classical manifolds modeled by $\mathbb{E}^3$, $\mathbb{H}^3$, or $\mathbb{S}^3$.

Example 5.2 (Flat torus)**.** Probably the most famous and easiest example of a compact 3-manifold is the *flat torus* $\mathbb{T}^3$. Topologically, it is obtained by gluing opposite faces of the unit cube $[0, 1] \times [0, 1] \times [0, 1] \subset \mathbb{E}^3$. It is easy to check that the neighborhood of each point in $\mathbb{T}^3$ is a 3-ball of the Euclidean space. Thus $\mathbb{T}^3$ is indeed a 3-manifold.

$\mathbb{T}^3$ admits a geometric structure modeled by $\mathbb{E}^3$ since it is also the quotient of the Euclidean space by the group of translation spanned by $(x, y, z) \to (x \pm 1, y, z)$, $(x, y, z) \to (x, y \pm 1, z)$, and $(x, y, z) \to (x, y, z \pm 1)$. Thus, the face $[0, 1] \times [0, 1] \times 0$ is identified to $[0, 1] \times [0, 1] \times 1$ by the translation map $(x, y, z) \to (x, y, z + 1)$. The remaining pairs of faces can be identified in an analogous way. The unit cube is the fundamental domain of $\mathbb{T}^3$.

A ray leaving a point $p \in \mathbb{T}^3$ in a direction v is parameterized as $r(t) = p + t \cdot v$ in $\mathbb{E}^3$. For each intersection between r and a face F of the unit cube, we update p by its correspondent point $p - n$ in the opposite face, where n is the unit vector normal to F. The ray direction v does not need to be updated.

Therefore, we have the ingredients for an immersive visualization of $\mathbb{T}^3$ using ray tracing. The scene can be set in the unit cube since it is the fundamental domain. The rays in $\mathbb{T}^3$ can return to the starting point, providing many copies of the scene. In fact, the immersive perception is $\mathbb{E}^3$ tessellated by unit cubes: each cube contains one copy of the scene.

The torus is not the unique compact oriented 3-manifold with geometry modelled by the Euclidean geometry. There are exactly five more, see Figure 1.

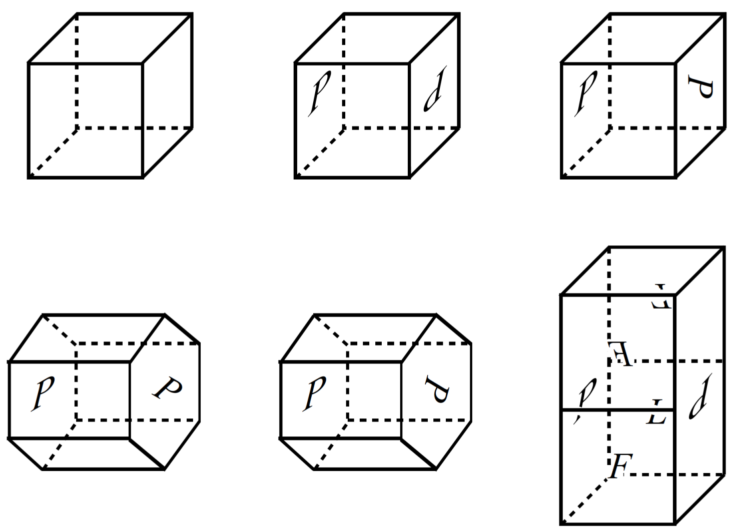

FIGURE 1. The six compact oriented manifolds with their geometry modeled by the flat space. The spaces are constructed through pair-wise identification of the faces in each of these polytopes. If a face has no label, it is glued to the opposite face in an obvious way. Otherwise, the faces are identified isometrically according to its label. Example and figure from page 378 of Martelli [5].

Example 5.3 (Seifert-Weber dodecahedral space)**.** To describe a compact 3-manifold with geometric structure modeled by the hyperbolic space consider a dodecahedron P. Identifying each pair of opposite faces in P with an addition clockwise rotation of $3\pi/10$ gives rise to a manifold know as *Seifert–Weber dodecahedral space* M.

Face pairing produces many identifications, for example, you can verify that edges are grouped into six groups of five. Thus, it is not possible to fit Euclidean geometry into such a manifold, since the regular Euclidean dodecahedron has a dihedral angle of approximately 116 degrees. The desired dodecahedron should have a dihedral angle of 72 degrees.

We use the hyperbolic geometry to model the geometry of M. Let the dodecahedron be centered at the origin of $\mathbb{H}^3$. The dihedral angle of the dodecahedral in the hyperbolic space is smaller than in the Euclidean case. In fact, with an appropriate scale, the dodecahedron admits a dihedral angle of 72 degree as desired.

Using Klein's model of $\mathbb{H}^3$, the rays are straight. So to compute a ray leaving a point $p \in M$ in a direction v, we use $r(t) = p + tv$. For each intersection between r and a dodecahedron face, we update p and v through the hyperbolic isometry that produces face pairing above. This isometry is quite distinct from Euclidean isometries (see [3]).

The immersive perception of M using ray tracing is a tessellation of $\mathbb{H}^3$ by dodecahedra with a dihedral angle of 72 degrees.

Example 5.4 (Poincaré dodecahedron space)**.** If the opposite faces of the dodecahedron are identified by a clockwise rotation of only $\pi/5$ we get *Poincaré dodecahedron space*, a manifold discovered by Poincaré. This manifold is also known as *Poincaré homological sphere* since its first homological group is trivial.

Again, the face pairing forces many identifications. The edges are grouped into ten groups of three edges. To model the geometry of such space the dihedral angle of the dodecahedron must be 120. It is not possible to model with Euclidean geometry. In this case, we use spherical geometry.

To find the desired dodecahedron we consider it embedded in the 3-sphere. If the dodecahedron is very small its dihedral angle is very close to the Euclidean dodecahedron. Then, with an appropriate scale, the dodecahedron dihedral angle equals to 120 degrees.

A ray passing through a point $p \in \mathbb{S}^3$ in the tangent direction v is parameterized by $r(t) = \cos tp + \sin tv$. If r intersects a face of the dodecarehedron we update p and v by the face transformation, which we discuss in more details below.

The immersive visualization of Poincar dodecahedral space is a tessellation of $\mathbb{S}^3$ by 120 dodecahedra. This is one of the 4-dimensional polytope, known as 120-cell.

6. Some Non–manifolds

Let M be a Euclidean, hyperbolic, or spherical space. The quotient M/Γ of M by a discrete group acting on it could be a non-manifold. In this case, M/Γ is called an *orbifold*. Informally, such spaces are modeled locally by quotients of M by discrete groups. We present two simple orbifold examples: the *mirrored cube*, and *mirrored dodecahedron*.

Example 6.1 (Mirrored cube)**.** The *mirrored cube* $\mathcal{Q}^3$ is an example of an orbifold with the geometric structure modeled by $\mathbb{E}^3$ through a special group of reflection Γ. Such group is generated by the reflections of the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$ in $\mathbb{E}^3$. The unit cube is the fundamental domain of $\mathcal{Q}^3$. Each time a ray r intersects a face of the fundamental domain of $\mathcal{Q}^3$ it is reflected, creating a polygonal curve in $\mathcal{Q}^3$: exactly what happened with the lights in a mirrored room. Such polygonal curve suspends to ray in $\mathbb{E}^3$, thus we see a tessellation of $\mathbb{E}^3$ by reflected unit cubes when immersed in $\mathcal{Q}^3$.

Example 6.2 (Mirrored dodecahedron)**.** For an example of an orbifold with a geometric structure modeled by the hyperbolic space, consider the dodecahedron embedded in $\mathbb{H}^3$. Let Γ be the group of reflections generated by the dodecahedral faces. With an appropriate scale, the dihedral angle of the dodecahedron reaches 90 degrees. The quotient $\mathbb{H}^3/\Gamma$ is the *mirrored dodecahedral space.* Γ tessellates $\mathbb{H}^3$ with dodecahedra, each edge has exactly 4 cells.

7. Implementation

This section presents the implementation details for an immersive visualization of 3-manifolds with geometry modeled by Euclidean, hyperbolic, or spherical.

7.1. **Euclidean geometry.** We explore seven spaces with their geometries modeled by Euclidean model: the six oriented manifolds presented in Figure 1, and the mirrored cube $\mathcal{Q}^3$. The fundamental domain these spaces are in $\mathbb{E}^3$, so there is no problem in setting the scene.

We restrict ourselves to torus, the ray tracing in the other (flat) manifolds follows similar ideas. When a ray crosses the cube boundary (fundamental domain) at a point p, we take the normal vector n at p. Then we cast another ray from the point $p - dn$ in the same direction; d is the diameter of the cube. This procedure is like lightning travels in this space.

The ray tracing of the mirrored cube is even simpler. Instead of update the point when the ray reaches the cube's boundary, we only have to update the ray direction v, by its reflection $v - 2n\langle v, n\rangle_{\mathbb{E}}$: the classical reflection formula.

7.2. **Hyperbolic geometry.** We approach two hyperbolic spaces with their geometry modeled by the hyperbolic space: the Siefert–Weber dodecahedron, and the mirrored hyperbolic dodecahedron. In both cases the fundamental domain is the dodecahedron, but with different scales.

For the Seifert–Weber dodecahedron we consider the regular dodecahedron D embedded in the Klein's model $\mathbb{K}^3$. To compute its right scale we remember that in this space the edges of D are glued into groups of five, so the dihedral angle of D should be $3\pi/5$. We find this scale by a continuous argument. If D is very small its angles are very close to the Euclidean case, but when its vertices belong to the unit sphere the dihedral angle is $2\pi/6$. As the dihedral angle is a function which depends continuously on the scale of D, the right scale follows from the intermediate value theorem. In practice, we let this scale be a global variable, then we choose the correct one.

As the rays are straight in Klein's model $\mathbb{K}^3$, we have no difficult to launch them. However, when they hit a face of the dodecahedron D we have to calculate the transformation that produces the Seifert–Weber dodecahedron space. This transformation is basically a composition of a translation and a rotation, which can be represented as matrices 4×4 — very similar to the classical linear algebra used in computer graphics algorithms. A good source for this is [6]. To use these 4×4 matrices, we use the hyperboloid model, which is embedded in $\mathbb{R}^4$.

Again, the mirrored case is simpler. Each time the ray intersects at a point p a face F of the dodecahedron D we update the ray direction v by its hyperbolic reflection $v - 2n\langle v, n\rangle_{\mathbb{H}}/\langle n, n\rangle_{\mathbb{H}}$. Where v is presented in homogeneous coordinates, and n is the homogeneous coordinates of the plane containing the face F. The only difference between Euclidean and hyperbolic geometry is the inner product. The dodecahedron scale should be sufficient to provide a dihedral angle of $\pi/2$, since each edge in the underlying tessellation of $\mathbb{H}^3$ is surrounded by four dodecahedra.

7.3. **Spherical geometry.** We provide an immersive view of Poincaré dodecahedral space, which is obtained topologically by identifying opposite faces of a dodecahedron with a rotation of $\pi/5$. Unlike the hyperbolic case, there is no spherical model with geodesics been straight lines. The rays in this model are arcs since the dodecahedron must be embedded in the 3-sphere — this is our main challenge.

To model Poincaré dodecahedron space $\mathbb{P}$ we use a special parameterization, which suspends the regular dodecahedron D centered in the origin of the Euclidean space to a spherical dodecahedron centered in the point $(0, 0, 0, 1)$ of the 3-sphere embedded in $\mathbb{E}^4$. This enables us to set the scene in the Euclidean space and suspends it to the sphere through the parameterization. When a ray hit a face we must update the hit point and its direction. The hit point can be updated directly in the Euclidean dodecahedron since the parameterization is, in particular, a bijection. Special parallel transport is used to update the direction.

We describe the above model in more details. The regular dodecahedron D centered in the origin of $\mathbb{E}^3$ suspends to 3-sphere through the map $\Phi : \mathbb{E}^3 \to \mathbb{E}^4$ given by $\Phi(x, y, z) = (x, y, z, 1)/|(x, y, z, 1)|_{\mathbb{E}}$. We denote $\Phi(D)$ by $D_{\mathbb{S}}$, the spherical dodecahedron. We list some properties of this model.

- The faces of $D_{\mathbb{S}}$ are contained in unit 2-spheres, which are the spherical analogous to the plane in the Euclidean space;
- A 2-sphere S in $\mathbb{S}^3$ is represented as the set $\{p \in \mathbb{S}^3 | \ \langle p, n_S\rangle_{\mathbb{E}} = 0\}$, where n_S belongs to $\mathbb{S}^3$;
- Let $p \in D$ and v a direction in p. The suspension of v is the differential of Φ at p applied to v — $d\Phi_p(v)$;
- Let r be a ray leaving $p \in \mathbb{S}^3$ in the tangent direction v — it is parameterized by $r(t) = \cos tp + \sin tv$. The intersections between r and a 2-sphere S are given by the solutions of $\langle \cos tp + \sin tv, n_S\rangle_{\mathbb{E}} = 0$ which is equivalent to $\tan t = -\langle p, n_S\rangle_{\mathbb{E}}/\langle v, n_S\rangle_{\mathbb{E}}$;

Now the scene can be configured in D and Φ pushes it to $D_{\mathbb{S}}$. We use the algorithms for stereo ray tracing built on top of Falcor [1] in the following way. Falcor RTX verifies intersections between each ray and the scene object in the Euclidean space. When there is no intersection the ray should hit a dodecahedral face in a point p since the dodecahedron is a convex body. The point p is updated together with the ray direction in p using an intrinsic computation in $\mathbb{S}^3$. This approach implies in some deformation since in the last bounce when the ray hits the scene objects, we consider a straight line in the domain of the map Φ.

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