Discrete Exterior Calculus and Applications

Lenka Ptáčková

VISGRAF Lab, Institute of Pure and Applied Mathematics

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Overview

The objective of DEC

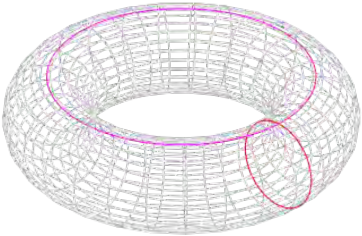
- Using geometric insight and exploring geometric meaning of quantities (in the continuous setting).
- Faithful discretization, consistency with the continuous world.
- Preservation of essential structures at the discrete level.
- Faster and simpler computations.
- The extension of the exterior calculus to discrete spaces including graphs and simplicial complexes.

Differences between DEC and other methods

- Finite difference and particle methods - discretization of local laws can fail to respect global structures and invariants.
- Finite element method - loss of fidelity following from a discretization process that does not preserve fundamental geometric and topological structures of the underlying continuous models.
- Discrete exterior calculus - stores and manipulate quantities at their geometrically meaningful locations, maintains the separation of the topological (metric-independent) and geometric (metric-dependent) components of quantities.

Related disciplines

- Differential geometry - studying problems in geometry using techniques of differential and integral calculus and algebra.
- Exterior calculus - geometry based calculus, the modern language of differential geometry and mathematical physics.
- Algebraic topology of simplicial and CW complexes - studies topological invariants, e.g., Betti numbers.



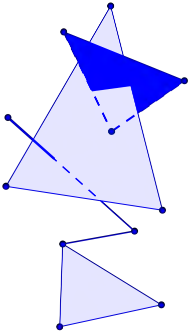
A simple torus has two non-contractible circles on its surface.

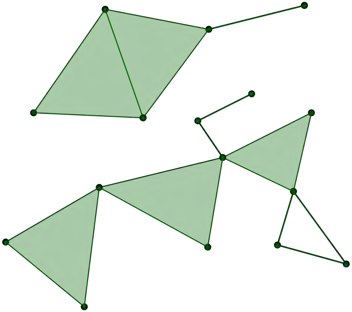
Image from

https://categoricalounge.wordpress.com /tag/homology/

Discrete differential geometry

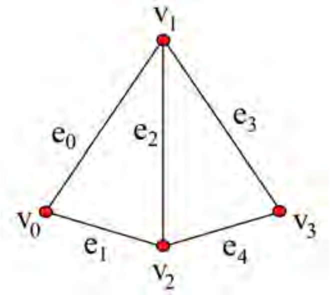
- Discrete versions of forms and manifolds formally identical to the continuous models.
- Forms represented as cochains and domains as chains of simplicial or CW complexes.





Definition

An *n*-dimensional **simplicial manifold** is an *n*-dimensional simplicial complex for which the geometric realization is homeomorphic to a topological manifold. That is, for each simplex, the union of all the incident *n*-simplices is homeomorphic to an *n*-dimensional ball, or half a ball if the simplex is on the boundary.



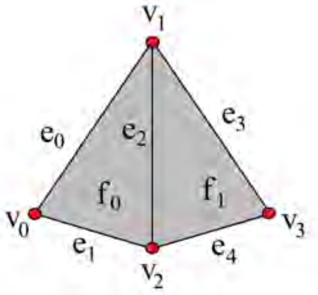


Image from [Desbrun et al., 2008].

Definition

A **p-chain** on a simplicial complex K is a function c from the set of oriented p-simplices of K to the integers, such that:

1. $c(\sigma) = -c(\bar{\sigma})$ if σ and $\bar{\sigma}$ are opposite orientations of the same simplex.
2. $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

We add p-chains by adding their values, the resulting group is denoted $C_p(K)$.

Definition

Let K be a simplicial complex and G an abelian group G, e.g. real numbers under addition. The p-dimensional **cochain** ω is the dual of a p-chain c_p in the sense that ω is a linear mapping that takes p-chains to G:

$$\omega : \; C_p(K) \to G, \;\; c_p \to \omega(c_p).$$

The group of p-dimensional cochains of K, with coefficients in G is denoted $C_p(K, G)$.

Fairing - general approach

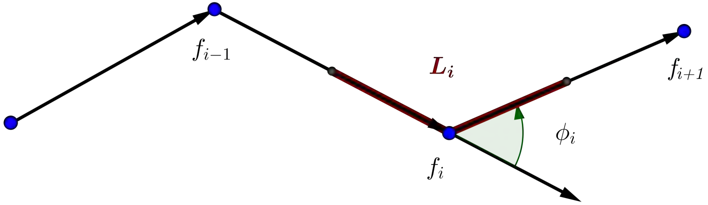
- Energy E measuring the smoothness of the manifold.
- E is a real valued function of:
  - immersion (vertex positions) f of the curve/surface, which leads to PDE, or
  - curvature, which leads to ODE.
- We reduce E via gradient descent.

Curvature flow on positions

A discrete curve f is an ordered set of vertices $f = (f_0, \ldots, f_n)$, $f_i \in \mathbb{R}^2$. We define the pointwise curvature κ at a vertex i as

$$\kappa_i = \frac{\phi_i}{L_i}, \tag{1}$$

where $L_i = \frac{1}{2}(|f_{i+1} - f_i| + |f_{i-1} - f_i|)$ and ϕ_i is the exterior angle at the corresponding vertex.



Curvature flow on positions

The curvature energy is given by

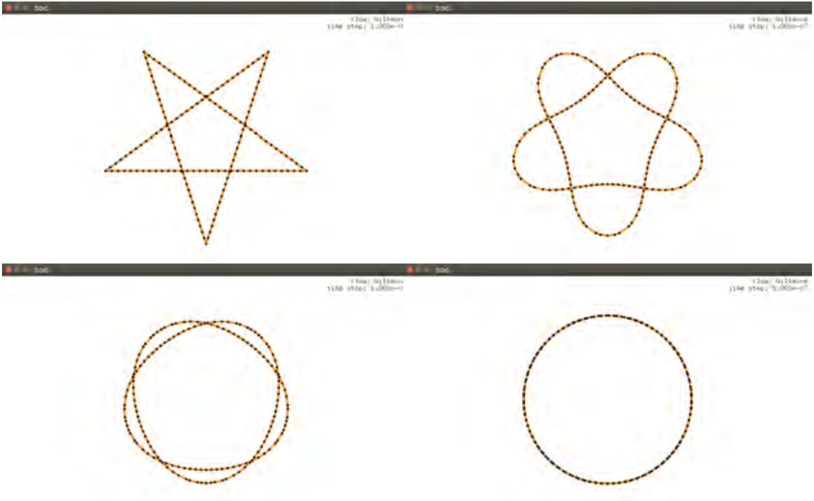
$$E(\gamma) = \sum_i \kappa_i^2 L_i = \sum_i \frac{\phi_i^2}{L_i}.$$

And the curvature flow is

$$\dot{\gamma} = -\nabla E(\gamma).$$

We integrate the flow using the forward Euler scheme, i.e.,

$$\gamma^t = \gamma^0 + t \cdot \dot{\gamma}.$$



Images generated by a program implemented by the author, its skeleton code can be found in the course notes of [Crane, Schroder, 2012], Homework 4.

Isometric curvature flow in curvature space

The curvature energy is now function of the curvature κ

$$E(\kappa) = \kappa^2 = \sum_i \kappa_i^2.$$

And the curvature flow becomes

$$\dot{\kappa} = -\nabla E(\kappa) = -2\kappa.$$

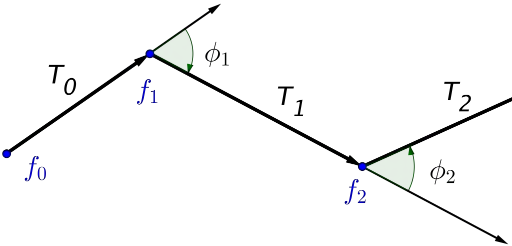
We integrate the flow using the forward Euler scheme again and obtain new vertex curvatures κ_i.

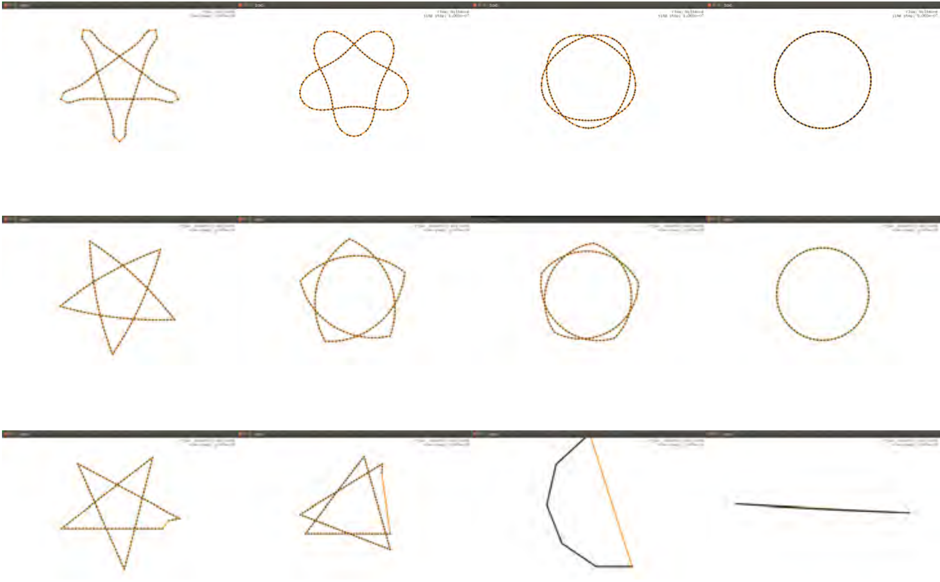
To recover the curve, we integrate curvatures to get tangents:

$$T_i = L_i(\cos\theta_i, \sin\theta_i), \text{ where } \theta_i = \sum_{k=0}^{i} \phi_k.$$

Then we integrate tangents to get the positions:

$$\gamma_i = \sum_{k=0}^{i} T_k.$$





Integrability constraints

- Closed loop f must satisfy:

$$\sum_i \kappa_i L_i = 2\pi k,$$

for some turning number $k \in \mathbb{Z}$. Which is equivalent to

$$T_1 = T_n \iff \sum_i \dot{\kappa}_i = 0.$$

- The endpoints must meet up, i.e., $f_0 = f_n$, which leads to:

$$\sum_i \dot{\kappa}_i f_i = 0.$$

- Overall, then, the change in curvature must avoid a three-dimensional subspace of directions:

$$\langle \dot{\kappa}, 1 \rangle = \langle \dot{\kappa}, f_x \rangle = \langle \dot{\kappa}, f_y \rangle = 0.$$

Implicit Mean Curvature Flow

On the surface $f : M \to \mathbb{R}^3$ we consider the flow

$$\dot{f} = 2HN = \triangle f,$$

that is, we move the points in the direction of normal with magnitude proportional to the mean curvature.
The Laplace operator $\triangle f$ reads:

$$(\triangle f)_i = \frac{1}{2} \sum_j (\cot \alpha_j + \cot \beta_j)(f_j - f_i). \tag{2}$$

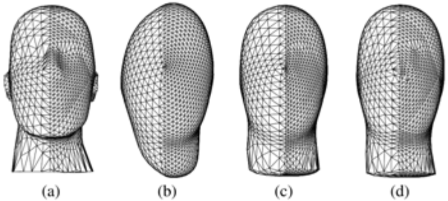
And we use the backward Euler scheme

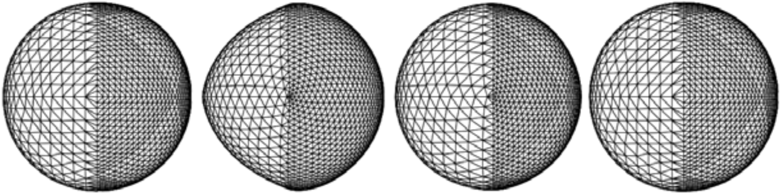
$$(I - t\triangle) f^t = f^0.$$

The matrix $A = (I - t\triangle)$ is highly sparse, therefore it is not too expensive to solve the linear system.

The quality of the resulting process highly depends on the approximation of the Laplace operator:

- linear approximation, so called **umbrella operator** expects the edges to be of equal length, which leads to distortion of the shape,
- **scale-dependent umbrella operator** almost keeps the original distribution of triangle sizes,
- **cotangent discretization of the Laplace operator** (equation (2)) achieves the best smoothing with respect to the shape, no drifting occurs.





(a) (b) (c) (d)

Smoothing of a mesh (a), (b) the umbrella operator, (c) the scale-dependent umbrella operator, (d) the cotangent discretization of the Laplace operator.

Images are from [Desbrun, Meyer, Schroder, Barr, 1999].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Conformal Curvature Flow

Keenan Crane in [Crane et al., 2013] suggests a curvature flow in curvature space that yields conformal smoothing of surfaces. Instead of using the potential energy $E(f)$ as a function of vertex positions, he uses Willmore energy $E_W(\mu)$ as a function of mean curvature half density:

$$E_W(\mu) = ||\mu||^2.$$

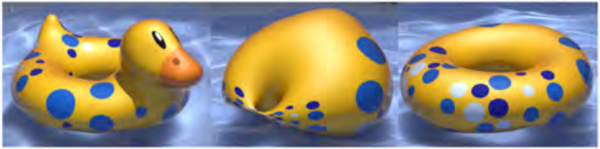
Gradient flow with respect to μ becomes $\dot{\mu} = -2\mu = -H$. Applying forward Euler scheme gives:

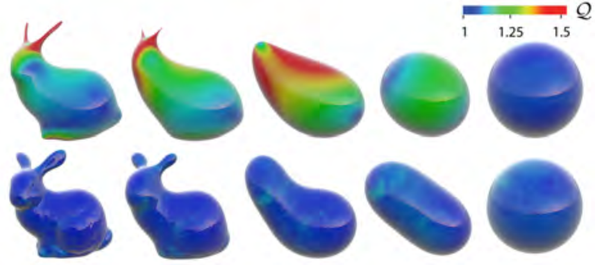
$$\mu^t = \mu^0 - 2tH,$$

where H is the pointwise mean curvature of the current mesh computed via the cotangent Laplacian $(\triangle f = 2HN)$.

Constraints

In order to obtain conformality and avoid distortion or cracks, the flow must satisfy several linear constraints, for details see [Crane et al., 2013].




Q
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1.25
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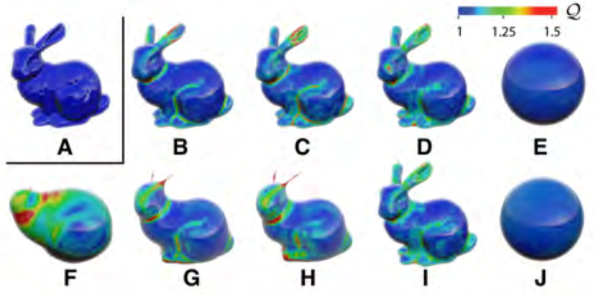
Q
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A
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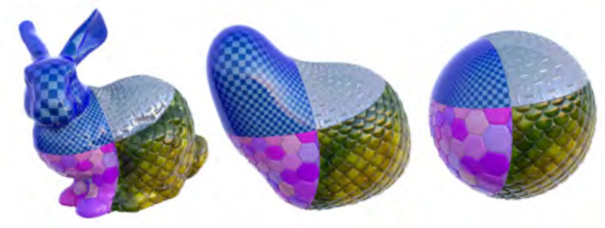
Summary

- Using geometric insight can significantly improve geometry processing.
- DEC offers operators consistent with their continuous counterparts.
- These new tools improve computations, which become faster end simpler.



The preceding set of images are from [Crane et al., 2013].

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